The resultant force vectors on faces BCD and ACD act at the centroids H and G of these faces and therefore contribute nothing to the sum of moments about axis GH. Denoting the mid-point of GH by Q, decompose the stress vector $\mathbf{S}_1$ acting at E into three components that are acting along QE, parallel to QH, and normal to the plane EGH (i.e., parallel to $\mathbf{n}_2$), respectively. Of these only the last component, which has the value $\mathbf{S}_1 \cdot \mathbf{n}_2$, produces a moment about axis GH. Similarly, only the component of $\mathbf{S}_2$, acting at F, in the direction of $\mathbf{n}_1$, produces a moment about axis GH. The resultant moment is given by

$$(\mathbf{S}_1 \cdot \mathbf{n}_2) A(EQ) - (\mathbf{S}_2 \cdot \mathbf{n}_1) A(FQ) = 0.$$

In the above equation A is the common area of triangles ABC and ABD. Since the moment arms EQ and FQ are equal by construction, it follows that

$$\mathbf{S}_1 \cdot \mathbf{n}_2 = \mathbf{S}_2 \cdot \mathbf{n}_1 .$$

UNSTEADY MOTION OF AN INFINITE LIQUID DUE TO THE UNIFORM ROTATION OF A SPHERE, $r = a$*

BY C. D. GHILDYAL (*Lucknow University*)

Introduction. The problem of steady motion of a viscous liquid due to the slow rotation of a sphere has already been discussed by various workers in the field of hydrodynamics [1]. Here we propose to discuss the unsteady flow of a liquid initially at rest due to the uniform rotation, Ω, of a sphere about the axis of z, under an external force $z\omega^2$ acting per unit mass of the liquid along the axis of rotation and the pressure at any point of the liquid is given by

$$p = \int \rho r \omega^2 \, dr + a \text{ constant}, \quad \text{(A)}$$

where $\omega \equiv \omega(r, t)$, $r^2 = x^2 + y^2 + z^2$, and velocity components at any point (x, y, z) of the liquid are assumed to be $u = -\omega y$, $v = \omega x$ and $w = 0$.

The result obtained is valid also for slow rotation of the sphere. In that case the external force does not exist and the pressure remains constant throughout the liquid.

The equations of motion of a viscous homogeneous incompressible liquid in this case reduce to

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= \nu\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{4}{r}\frac{\partial \omega}{\partial r}\right), \\ \frac{\partial p}{\partial r} &= \rho r \omega^2. \end{aligned} \quad \text{(B)}$$

A general solution of (B) is obtained by the method of Laplace transform under the boundary conditions

$$\begin{aligned} &\text{(i)} \quad \omega(a, t) = \Omega, \\ &\text{(ii)} \quad \omega(r, 0) = 0, \\ &\text{(iii)} \quad \lim_{r \to \infty} \omega(r, t) = 0. \end{aligned} \quad \text{(C)}$$

*Received August 26, 1959; revised manuscript received January 20, 1960.

1. Solution. Putting $\nu t = T$ and $\omega(r, T) = W(r, T)/r^{3/2}$, in (*B*) and (*C*), we get

$$\frac{\partial W}{\partial T} = \frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} - \frac{9W}{4r^2} \tag{1.1}$$

and

$$\begin{aligned} &\text{(i)} \quad W(a, T) = a^{3/2}\Omega, \\ &\text{(ii)} \quad W(r, 0) = 0, \\ &\text{(iii)} \quad \lim_{r\to\infty} W(r, T) = 0. \end{aligned} \tag{1.2}$$

Now we define the Laplace transform of $W(r, T)$ as

$$W^*(r, s) = \int_0^\infty W(r, T)e^{-sT}\, dT.$$

Thus Eq. (1.1) becomes

$$\frac{\partial^2 W^*}{\partial r^2} + \frac{1}{r}\frac{\partial W^*}{\partial r} - \left(s + \frac{9}{4r^2}\right)W^* = 0,$$

the solution of which is

$$W^*(r, s) = cI_{3/2}(rs^{1/2}) + DK_{3/2}(rs^{1/2}),$$

where $I_{3/2}$ and $K_{3/2}$ are modified Bessel functions of the order 3/2.

The boundary conditions now give

$$c = 0 \quad \text{and} \quad D = a^{3/2}\Omega/sK_{3/2}(as^{1/2}).$$

Therefore,

$$W^*(r, s) = a^{3/2}\Omega K_{3/2}(rs^{1/2})/sK_{3/2}(as^{1/2}).$$

From Laplace inversion formula

$$W(r, T) \equiv r^{3/2}\omega(r, T) = \frac{a^{3/2}\Omega}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K_{3/2}(rs^{1/2})e^{sT}}{sK_{3/2}(as^{1/2})}\, ds.$$

For the order $(n + 1/2)$, where n is an integer, the Bessel functions and modified Bessel functions both reduce to finite form. Thus we get

$$\begin{aligned} \omega(r, T) &= \frac{a^3\Omega}{2\pi r^3 i}\int_{c-i\infty}^{c+i\infty} \frac{(1 + rs^{1/2})}{s(1 + as^{1/2})} \exp\left[(a - r)s^{1/2}\right]e^{sT}\, ds, \\ &= \frac{a^3\Omega}{r^3}\left[I_1 + \left(\frac{r}{a} - 1\right)I_2\right], \end{aligned}$$

where I_1 and I_2 are the inverse Laplace transforms of

$$s^{-1}\exp\left[(a - r)s^{1/2}\right] \quad \text{and} \quad s^{-1/2}(a^{-1} + s^{1/2})^{-1}\exp\left[(a - r)s^{1/2}\right]$$

A reference to the *Tables of integral transforms*, edited by Erdélyi and associates [2] gives

$$I_1 = 1 - \operatorname{erf}\left(\frac{r - a}{2T^{1/2}}\right) \quad \text{and} \quad I_2 = \left[1 - \operatorname{erf}\left(\frac{r - a}{2T^{1/2}} + \frac{T^{1/2}}{a}\right)\right]\exp\left(\frac{r - a}{a} + \frac{T}{a^2}\right),$$

where

$$\text{erf}\,(u) = \frac{2}{\pi^{1/2}} \int_0^u \exp(-x^2)\,dx, \quad \text{and} \quad r \geq a.$$

We thus have,

$$\omega(r, T) = \frac{a^3\Omega}{r^3}\left\{\left(\frac{r}{a} - 1\right)\left[1 - \text{erf}\left(\frac{r-a}{2T^{1/2}} + \frac{T^{1/2}}{a}\right)\right]\exp\left(\frac{r-a}{a} + \frac{T}{a^2}\right) + 1 - \text{erf}\left(\frac{r-a}{2T^{1/2}}\right)\right\} \qquad (1.3)$$

The value of $\omega(r, T)$ is given by the expression (1.3) and satisfies the equations of motion as well as the boundary conditions. The transient part in (1.3), say K, can be written as

$$K = \frac{2a^2(r-a)\Omega}{r^3}\left\{1 - P\left[\frac{r-a}{(2T)^{1/2}} + \frac{(2T)^{1/2}}{a}\right]\right\}\exp\left(\frac{T}{a^2} + \frac{r}{a} - 1\right) + \frac{a^3\Omega}{r^3}\left\{1 - P\left[\frac{r-a}{(2T)^{1/2}}\right]\right\}, \qquad (1.4)$$

where $P(x)$ is the probability integral

$$P(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}u^2)\,du.$$

For large values of r, $K \propto 1/r^3$. As $r \to \infty$, K approaches the value zero quite rapidly. K also approaches zero as $T \to \infty$.

Now let a be the unit of length and $2T = 1$. Equation (1.4) then reduces to

$$K = \frac{\Omega}{r^3}\{2(r-1)[1 - P(r)]\exp(r - \tfrac{1}{2}) + 1 - 2P(r-1)\}.$$

The following table[1] gives the values of K/Ω for different values of r when $2T = 1$.

Relation between r and K/Ω.

r	K/Ω	r	K/Ω	r	K/Ω	r	K/Ω	r	K/Ω
1.00	−0.0000	1.60	−0.0620	2.40	−0.0495	3.60	−0.0210	6.00	−0.0046
1.10	−0.0227	1.70	−0.0628	2.60	−0.0437	3.80	−0.0180		
1.20	−0.0338	1.80	−0.0624	2.80	−0.0380	4.00	−0.0154		
1.30	−0.0485	1.90	−0.0615	3.00	−0.0324	4.50	−0.0109		
1.40	−0.0553	2.00	−0.0598	3.20	−0.0282	5.00	−0.0079		
1.50	−0.0596	2.20	−0.0551	3.40	−0.0241	5.50	−0.0061		

The table shows that unsteadiness spreads from the surface of the sphere into the interior of the liquid and attains its maximum value for $r = 1.70$, approximately. Thereafter it gradually decreases and becomes inappreciable at a finite distance from the

[1]The values of $P(x)$ are taken from *Biometrika tables for statisticians*, vol. 1, ed. by E. S. Pearson and H. O. Hartley, 1956, pp. 104–106.

sphere. The region in the neighbourhood of $r = 1.70$ is consequently the region of maximum unsteadiness.

Acknowledgment. I am grateful to Dr. Ram Ballabh for supervising this work. I am also grateful to the referee for pointing out an error in the earlier version of the paper.

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DYNAMIC PROGRAMMING APPROACH TO OPTIMAL INVENTORY PROCESSES WITH DELAY IN DELIVERY*

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Summary. The usual dynamic programming approach to inventory processes with delays in delivery leads to functions of many variables. This multi-dimensionality prevents the straightforward utilization of digital computers.

Using a type of transformation previously applied in the study of engineering control processes, we show that a class of inventory processes with time lags can be treated in terms of sequences of functions of one variable, regardless of the length of the delay.

1. Introduction. The problem of determining ordering policies which minimize the cost of operating supply depots and stockrooms is one which has attracted a great deal of attention in industrial and military circles in recent years. An analytic approach to these questions by way of functional equation techniques was inaugurated by Arrow, Harris, and Marschak, in a now classic paper, [1]. These investigations were extended by Dvoretzky, Kiefer, and Wolfowitz, [7], and Bellman, Glicksberg, and Gross, [6]; see also [2], and the books by Whitin, [8], and Arrow, Karlin and Scarf, [9].

Although this approach can be used to obtain analytic and computational solutions of a variety of processes in which there is no delay between an order for an additional supply of items and the delivery of these items, this method runs into dimensionality difficulties when time lags of more than a stage or two occur. If there is a delay of d stages in filling an order, the state of the system at any time is characterized not only by the present stock level, but also by the quantities on order which will arrive one, two, $\cdots$, d stages in the future.

It thus appears that functions of d variables necessarily arise when point of regeneration methods are employed to treat these processes.

In several papers devoted to the study of control processes arising in the engineering world [3], [4], we have shown that in some fortunate situations certain preliminary trans-

*Received February 3, 1960.