# QUARTERLY

## OF

# APPLIED MATHEMATICS



VOLUME XVIII APRIL · 1960 NUMBER 1

# QUARTERLY OF APPLIED MATHEMATICS

This periodical is published quarterly by Brown University, Providence 12, R. I. For its support, an operational fund is being set up to which industrial organizations may contribute. To date, contributions of the following industrial companies are gratefully acknowledged:

AVCO RESEARCH LABORATORY; EVERETT, MASS.,
BELL TELEPHONE LABORATORIES, INC.; NEW YORK, N. Y.,
THE BRISTOL COMPANY; WATERBURY, CONN.,
CALIFORNIA RESEARCH CORPORATION; SAN FRANCISCO, CAL.,
CURTISS WRIGHT CORPORATION; AIRPLANE DIVISION; BUFFALO, N. Y.,
EASTMAN KODAK COMPANY; ROCHESTER, N. Y.,
GENERAL ELECTRIC COMPANY; SCHENECTADY, N. Y.,
GULF RESEARCH AND DEVELOPMENT COMPANY; PITTSBURGH, PA.,
HUGHES AIRCRAFT COMPANY; CULVER CITY, CAL.,
INTERNATIONAL BUSINESS MACHINES CORPORATION; NEW YORK, N. Y.,
LEEDS & NORTHRUP COMPANY; PHILADELPHIA, PA.,
LOCKHEED AIRCRAFT CORPORATION; BURBANK, CAL.,
PRATT & WHITNEY, DIVISION NILES-BEMENT-POND COMPANY; WEST HARTFORD, CONN.,
THE RAMO-WOOLDRIDGE CORPORATION; LOS ANGELES, CAL.,
REPUBLIC AVIATION CORPORATION; FARMINGDALE, LONG ISLAND, N. Y.,
UNITED AIRCRAFT CORPORATION; EAST HARTFORD, CONN.,
WESTINGHOUSE ELECTRIC AND MANUFACTURING COMPANY; PITTSBURGH, PA.

The QUARTERLY prints original papers in applied mathematics which have an intimate connection with application in industry or practical science. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

Manuscripts submitted for publication in the QUARTERLY OF APPLIED MATHEMATICS should be sent to Professor W. Prager, Quarterly of Applied Mathematics, Brown University, Providence 12, R. I., either directly or through any one of the Editors or Collaborators. In accordance with their general policy, the Editors welcome particularly contributions which will be of interest both to mathematicians and to engineers. Authors will receive galley proofs only. The authors' institutions will be requested to pay a publication charge of $5.00 per page which, if honored, entitles them to 100 free reprints. Instructions will be sent with galley proofs.

The subscription price for the QUARTERLY is $8.00 per volume (April-January). Single issues can be purchased at $2.50, and back volumes at $8.00 per volume as far as they are available. Subscriptions and orders for single issues may be addressed to: Quarterly of Applied Mathematics, Box 1909, Brown University, Providence 12, R. I., or to Box 2-W, Richmond, Va.

---

Second-class postage paid at Providence, Rhode Island, and at Richmond, Virginia

---

WILLIAM BYRD PRESS, INC., RICHMOND, VIRGINIA

CONTENTS

# BOOK REVIEWS

*Dynamic programming.* By Richard Bellman. Princeton University Press, New Jersey, 1957. xxv + 340 pp. $6.75.

From a formal point of view, Dynamic Programming is the study of functional equations or of recursively defined systems of functional equations which involve a maximum operator. Its principal application is to the analysis of decisions over time or "multi-stage decision processes". Practically a single-handed creation of Richard Bellman, this novel technique has had a number of expositions in journal articles and shorter RAND publications. But the present book "Dynamic Programming", now in its second printing, stands out as the first systematic treatment at book length of this important and fruitful subject. It is an original and authoritative, attractive, sometimes brilliant, occasionally drawn out and repetitive, but throughout fascinating presentation of this lively and versatile mathematical tool.

The general plan of the book is as follows. The basic ideas are introduced in terms of a specific problem concerning the repeated allocation of a single resource, an example that may well represent the simplest non-trivial case of a Dynamic Program. Following a second more complicated example involving probability (see below) the general structure of Dynamic Programming is exhibited, existence and uniqueness theorems are developed, the sensitivity of solutions and such properties of the solutions as convexity and continuity are discussed. An important tool in these investigations is the method of successive approximation through iteration. It is shown in particular that a suitable choice of the initial approximation will always result in monotone convergence The remaining part of the book studies in detail various cases of a more advanced nature involving respectively, integral operators, systems of differential equations, partial differential equations, Min Max operators and Markov chains.

Apart from the opening problem which has little intrinsic interest, the applications developed at length are essentially of three kinds. The first of these is illustrated by the so-called gold mining problem,–obviously a military mission problem in disguise–where the object is the repeated assignment of a given piece of equipment to alternative uses subject to constant (but different) probabilities of ruin and to diminishing returns. This problem is studied in both a discrete and a continuous version (chs. II and VIII) for two and for more alternatives, and its very simple and reasonable solution is given in explicit form.

The second kind of problem is illustrated by a "bottleneck problem" (chs. VI, VII) known in the economic literature as a "dynamic input-output model", which involves the allocation of products as inputs to both current production and the expansion of productive capacity. As in the more conventional linear programming treatment of this model, advantage is taken of the existence of dual variables or "efficiency prices" to obtain a sufficient characterization of the solution in terms of profitability conditions. However, the treatment is bogged down by a somewhat unfortunate choice of notation and never gets to the point of discovering in its general form the economically important principle of balanced growth. It is even doubtful whether the trial and error approach given here can be dignified by the name of Dynamic Programming.

Both the goldmining and the interindustry example concern closed systems. Of much greater potential usefulness in economic applications seem to be cases of open systems, exemplified by the third main application in the book, the so-called inventory problem (ch. V). This is a classical piece of operations research concerning the reordering of stocks by a firm facing a stochastic demand. Only the simplest case (linear ordering cost) is presented in full but several extensions are outlined. Recent work on the more interesting case which involves fixed ordering costs and especially some results by Scarf proving the optimality of the so-called $s$, $S$ policy under very weak assumptions, have now rendered the statement obsolete (p. 153) that "at the present time practically no solutions of the corresponding functional equations exist and very little seems to be known concerning the character of the optimal policies arising from processes of this more realistic type."

As a demonstration of the theoretical power of Dynamic Programming, the book includes three interesting applications to mathematics proper: to the calculus of variations with special reference to cases involving inequalities as constraints (ch. IX); to sequential games, in particular to so-called games of survival (ch. X); and to the study of Markov chains (ch. XI). In ch. IX a set of partial differential equations is obtained representing the more conventional Euler-Lagrange equations in an inte-

(*Continued on p. 30*)

# BOOK REVIEWS

(*Continued from p. 14*)

grated form. The method is then applied to the determination of the eigen-values for the differential equation of the vibrating string.

Among the most interesting and useful parts of the book are the collections of exercises and research problems at the end of each chapter which give an indication of the truly enormous range of potential applications, including such disparate items as a comparison of the arithmetic and geometric mean, Fibonaccian search, the design of missiles and the crossing of a desert by jeep.

The presentation is keyed to the level of ordinary calculus and is kept informal to the point where important proofs are left to the reader or are deferred to a second volume which is to be on a more technical level. These further developments will be looked forward to with keen expectation. Meanwhile it is a pleasure to recommend to anyone whose job is the application of mathematical tools in any context whatsoever the study of this most stimulating and rewarding book.

MARTIN J. BECKMANN

*Theory of relativity*. By W. Pauli. Translated from the German by G. Field. Pergamon Press, New York, London, Paris, Los Angeles, 1958. xiv + 241 pp. $6.00.

Pauli's "Theory of Relativity" is a remarkable book. It was written in 1920 when the general theory of relativity, the special theory of relativity and the author were respectively four, fifteen, and twenty-one years old. Now, almost thirty years later, it is still one of the best books on the subject. The point of view remains fresh; Pauli's comments and criticisms almost invariably remain relevant.

One of the striking features of the book is the description of the historical developments preceding the formulations of the special and general theories of relativity, the discussion of rival theories and of the relevant experiments. Pauli's book covers the field of relativity theory more extensively than most texts. Although written in an encyclopedic style which omits many derivations and proofs, enough details are given to enable the mature reader to supply the missing steps, with only occasional use of some of the many literature references.

"Relativitätstheorie" was written as an article for the Encyklopädie der Mathematischen Wissenschaften (Vol. 5, Part 2; No. 19), appearing there in 1921, and shortly afterwards was reprinted as a book. The present volume consists of an excellent translation by G. Field of the original text of "Relativitätstheorie," totalling 206 pages, and of 26 pages of supplementary notes in which Pauli reviews selected developments in relativity theory which occurred from 1920 to about 1955.

The general theory of relativity has had a peculiar position in modern theoretical physics. Gravitational forces are extremely weak. They can be ignored in most of atomic and nuclear physics and, on the astronomical scale, Newtonian gravitational theory, which is not even Lorentz invariant, agrees so well with observation that the few minute deviations predicted by general relativity are very difficult to measure. As a result, physicists have paid little attention to general relativity between 1920 and 1955, and this is reflected by the fact that Pauli can do reasonable justice to developments during this period in some 20 pages.

Since the mid-fifties, there has been renewed interest in gravitation. This is partly due to a general interest in field theories, spurred on by the successes and limitations of quantumelectrodynamics, and partly to the possibilities of achieving greater accuracy in astronomical observation when telescopes can be lifted above our atmosphere or when suitably chosen motions of artificial celestial bodies can be studied. The resurgence of interest in general relativity theory is reflected by the fact that in the few cases where Pauli's information is outdated, this is due to very recent work, such as that of Bass and Pirani (Phil. Mag. *46*, 850; 1955) on the relativity of centrifugal force, and that of Robinson and Bondi on gravitational waves (Nature *179*, 1072; 1957).

ALFRED SCHILD

(*Continued on p. 36*)

# BOOK REVIEWS

(*Continued from p. 30*)

*Handbook of supersonic aerodynamics. Section 7—Three-dimensional airfoils.* Produced and edited by the Aerodynamics Handbook Staff of the Johns Hopkins University Applied Physics Laboratory, Maryland, 1957. For sale by the Supt. of Documents, U. S. Government Printing Office, Washington 25, D. C. 86 pp. $1.50.

Section 7 of series first reviewed here in Vol. XVII, No. 3, p. 298. Results on those parts of lift, drag and pitching moment of thin wings at very small incidence which can be obtained from the Linearised Theory of steady potential flow are collected from many sources and displayed in the form of graphs.

R. E. MEYER

*Applications of finite groups.* By J. S. Lomont. Academic Press, New York, London, 1959. xi + 346 pp. $11.00.

This book is concerned with applications of group theory to problems of physics, and with the parts of group theory itself (chiefly representation theory) which are needed in these applications. Because of the style and the arrangement of the material, the book stands somewhere midway between a textbook or treatise in the usual sense and a handbook, in which a great number of definitions and theorems are presented without much in the way of motivation, and with almost nothing in the way of proof. Seemingly, the usefulness of the book will be limited, for the most part, to persons who already have a good general knowledge of the field, and who are in search of technical information about particular details.

The applications discussed include applications to crystallography, thermodynamics, wave-guide theory, and to nuclear, atomic, and molecular theories. As is suggested above, most of the remaining material centers around the representation problem for finite groups. This part of the discussion goes quite deep, and includes much that is certainly not widely known. There is a general bibliography containing over two hundred entries; and the book terminates with three appendices, one of which gives a twenty-five page summary of the properties of the Lorentz groups.

It would be easy for numerous errors to lurk undetected in the dense mass of material given here. It is only proper to state, however, that the reviewer discovered only very few errors of any kind, and none of any consequence. On the other hand, an occasional poorly thought-out sentence suggests that the book was perhaps written in excessive haste. If the author had reflected more deeply on the question of what his book was intended to accomplish, and on the question of why group theory affords useful tools not found in classical analysis, he might have presented the subject in a more readable form, and appealed to a larger public. It would have been of great value, also, if he had given more specific references to places where proofs of the less familiar results can be found.

L. A. MACCOLL

*Numerical methods for nuclear reactor calculations.* By G. I. Marchuk. Translated from Russian. Consultants Bureau, Inc., New York, and Chapman & Hall, Ltd., London, 1959. 295 pp. $60.00.

This is the first book to discuss comprehensively and profoundly the many interesting problems for the numerical analyst arising in the practical calculation of nuclear chainreactor systems. This is done to the exclusion of the physical background, so that questions of the validity of the various approximations are neglected, but the author disclaims any intention of providing a comprehensive treatise on nuclear reactor theory. Instead, he derives the Boltzmann equation for neutron transport, presents approximations useful for its simplification, and deduces the finite difference equations corresponding

(*Continued on p. 60*)

# BOOK REVIEWS

(*Continued from p. 36*)

to the differential or integral equations of the problem. Amongst others, the diffusion and the diffusion-age approximations are discussed in detail. The difference equations are then carefully discussed and solved, and there is a chapter on pertubation theory.

The book should be of great interest not only to specialists in nuclear reactor calculations but to anyone interested in the numerical solution of partial differential equations. Unfortunately, the price of $60.00 will take the book outside the range of most.

WALTER FREIBERGER

*Elementary matrix algebra.* By Franz E. Hohn. The Macmillan Co., New York, 1958. xi + 305 pp. $7.50.

This book fulfills a long-felt need for an introductory textbook on modern matrix algebra which is suitable for students in all fields of applied mathematics. In particular, it provides admirably the theoretical background for a course in the numerical methods of linear algebra. It is exceptionally well organized and written with utmost clarity. There is a wealth of worked and unworked examples. The contents cover: rules of matrix algebra; determinants; notions of inverse, rank, equivalence; linear dependence; vector spaces; linear, unitary and orthogonal transformations; characteristic equation of a matrix; bilinear, quadratic and hermitian forms.

WALTER FREIBERGER

*Mathematics for communication engineers.* By S. J. Cotton. The Macmillan Co., New York, 1959. x + 245 pp. $7.50.

In the preface it is stated that the book is written to bridge "the gap between the level of mathematics reached on many (undergraduate) courses, . . . , and that required to read intelligently many of the articles in professional journals or to tackle the many problems facing engineers-particularly communications engineers." There is a definite need for such a book. The reviewer believes this book will be quite useful to practicing electrical engineers needing a lucid and fairly precise discussion of much of the mathematics usually studied by seniors or first year graduate students in electrical engineering. Because of the wide range of subjects considered, and its high level of presentation, the book would probably be valuable as a reference for other readers.

A summary of the topics included may give an idea of the scope of the book; review of algebra and differential calculus, determinants, Maclaurin and Taylor series including a short discussion of convergency and approximation, trigonometric functions, integral calculus with schemes for evaluating certain types of integrals, a summary of differential equations emphasizing the solution of linear equations, a brief treatment of transmission lines and filters, theory of functions of a complex variable including Cauchy's integral formula, contour integration and conformal mapping, Fourier series and the Fourier integral, the Laplace transform and Heaviside's expansion formula, Vector Analysis and the derivation of Maxwell's equation and, in a problem, the derivation of the wave equation for perfect dielectrics, a brief mention of Legendre functions, Bessel functions and the Gamma function including the useful expansion of sin ($z$ sin $\theta$) in an infinite series of Bessel functions, and finally, a chapter on probability and statistics containing Poisson's approximation to the binomial law but omitting the normal distribution. The author has not attempted a complete discussion of each of the above subjects but has managed to include most of the areas which have been found useful up to now in electrical engineering. Because of this selection, the book may not be entirely satisfactory as a textbook for a course in mathematics. The mathematics presented is at least as sound as that given in the better, recent engineering or engineering mathematics textbooks; probably as important to the practicing

(*Continued on p. 70*)

# BOOK REVIEWS

(*Continued from p. 60*)

engineer, however, is that the author has included sufficient sensible examples and comments to give motivation to the mathematical exposition.

Probably every reader of a book of this sort would make his own selections of material to be covered and the reviewer feels that a discussion of the normal distribution function which is essential to an understanding of statistical communication problems and of the unit impulse which is probably more used in this country than the unit step would be valuable to many readers. In general, this book will be most helpful to practicing engineers desiring to increase their mathematical competence and to advanced students needing a convenient reference covering most of the mathematics required in graduate engineering courses. The applied mathematician may also find the sections interpreting the mathematics in terms of engineering problems interesting.

B. HAZELTINE

*Operational mathematics*. By Ruel V. Churchill. Second edition. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1958. ix + 337 pp. $7.00.

Applied mathematicians will not need to be reminded that R. V. Churchill's "Modern Operational Mathematics in Engineering", first published in 1944, was the first really accessible book on operational methods and the underlying theory of the Laplace transform to become available. The present book is a thorough revision of that earlier work. The chapter headings remain unaltered but much new material has been added especially to the last chapter. In the first two chapters, for instance, some of the operational rules are derived by new methods, and in the third chapter problems involving servo-mechanisms and additional ones on circuit theory are introduced. The central six chapters of the book remain virtually unaltered. It is in the last chapter that the major additions are made—the introduction of "generalized" Fourier transforms. Anyone who knows Professor Churchill's writings will not need to be advised of the lucid manner in which he presents his material. The publishers have taken the opportunity to produce the new edition in a much more attractive way than the original and the result is a book which is a delight to use. It can be unreservedly recommended to anyone beginning the study of the theory of the Laplace transform and the operational calculus based on it and to anyone entrusted with its teaching.

I. N. SNEDDON

*An introduction to plasticity*. By William Prager. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959. viii + 148 pp. $9.50.

The contents of this book are based on a series of lectures delivered by the author at the Federal Polytechnic Institute of Zurich in 1954. The Swiss edition of the book (in German) was published in 1955 and the present volume under review represents more than a mere translation of the previous edition. Not only is this book broader in scope but it contains further exposition of the topics previously treated, a typical example being a more detailed treatment of the principle of virtual velocities. The references are brought up to date, an index added, and a large number of problems, totaling about 60, are included at the end of each chapter, thus making the book more suitable as a text. The material covered is very clearly presented, and the examples are carefully chosen.

As in the previous edition, the book contains four chapters entitled "Mechanical Behavior of Plastic Solids", "The Mechanical Behavior of Plastic Structures". "Limit Analysis and Design", and "Finite Plastic Deformation". Chapters 3 and 4 contain several additional topics of recent origin not included in the previous edition, for example, a discussion of uniqueness, a discussion of two-parameter states of loading, as well as recent developments in connection with estimate of deflection at load-carrying capacity, and the theory of locking materials.

By limiting the scope of the book to selected topics from the theory of perfectly plastic solids, the

(*Continued on p. 78*)

# BOOK REVIEWS

(*Continued from p. 70*)

author has been able to lead the reader to the forefront of the current state of research on these topics without requiring prior familiarity with the subject. This has been achieved by presenting the basic theory in a condensed manner in the first 18 pages of Chapter 1 through efficient use of mechanical models. Those desiring a detailed account of the development of the subject based on the many experimental investigations and theoretical contributions, beginning with the work of Tresca in 1868, should refer themselves to the several other available books in plasticity.

The chapter on limit analysis and design, comprising half of the book, (63 pages) is the most valuable part. Fifty references (mostly from works of the last decade) are cited, covering the fundamental theorems with application to frames, circular plates, and cylindrical shells.

The reviewer is confident that the author has achieved his hope, stated in the Preface, that the "book will usefully supplement the traditional texts on strength of materials and theory of structures which are primarily concerned with the elastic behavior".

P. M. NAGHDI

*Principles of optics.* By Max Born and Emil Wolf. Pergamon Press, New York, London, Paris, Los Angeles, 1959. xxvi + 808 pp. $17.50.

"Principles of Optics" is a new book; it is not simply a translation and revision of the famous "Optik" of 1933. This treatise is a model of completeness and thoroughness in classical optics, both geometrical and physical. It is a tremendous piece of work, and so well done that it is certain to be an essential part of the library of anyone concerned with optical phenomena.

The book contains more than 800 pages with 14 chapters and 9 appendices. The first two chapters are an excellent review of electromagnetic theory with special attention to the reflection refraction and transmission of plane waves. Elementary dispersion theory is also dealt with at this stage. The third chapter, Foundations of Geometrical Optics, shows how geometrical optics is obtained from Maxwell's equations as a limiting case for short wavelengths—the eikonal equation. The mathematics of geometrical or ray optics is introduced. Chapter 4 deals in detail with the geometrical theory of optical imaging using the various characteristic functions of Hamilton's methods. Chapter 5 is concerned with the geometrical theory of aberrations. Chapter 6 is entitled Image Forming Instruments and deals with the human eye, the camera, the refracting telescope, the reflecting telescope, and the microscope. Chapter 7 is a chapter of more than 100 pages on Elements of the Theory of Interference and Interferometers. Chapters 8, 9, 10, and 12 are devoted mainly to diffraction theory. Chapter 8, Elements of Diffraction Theory, is the best discussion of classical diffraction theory that the reviewer has seen in any book. Chapter 9, The Diffraction Theory of Aberrations, Chapter 10, Interference and Diffraction with Partially Coherent Light—this is one of the more sophisticated sections of Optics, and it is dealt with rather thoroughly here. The theory of partial coherence deals with correlation functions and time averaged intensities—the quantities one measures in experiment. This section of the book should be of interest to mathematicians as well as physicists. The same can be said of Chapter 11 entitled Regions Diffraction Theory. This chapter is confined mainly to the half-plane problem of Sommerfeld. It is here that the reviewer feels that the authors should have discussed the newer integral equation and variational techniques that lead to solutions of diffraction problems in closed form. This could have been done spacewise at the expense of several of the appendices, especially Appendix I on the calculus of variations, as well as Chapter 12 entitled Diffraction of Light by Ultrasonic Waves.

The remaining part of the book is devoted to excellent discussions of Optics of Metals (Chapter 13) and Optics of Crystals (Chapter 14). The discussion of the Optics of Metals in Chapter 13 is the classical version and one might have asked for a brief discussion of the importance of quantum mechanics in this area. Altogether the reviewer feels that this is by far the best advanced text in Optics now available.

ROHN TRUELL