

Hence

$$\|y\| \leq \|X(t)\| \|b\| + \int_0^t \|X(t)\| \|X^{-1}(s)\| \|B(s) - A(s)\| \|y\| ds. \quad (10)$$

Thus, if we set

$$\begin{aligned} u(t) &= \|X^{-1}(t)\| \|B(t) - A(t)\| \|y(t)\|, \\ v(t) &= \|B(t) - A(t)\| \|X(t)\| \|X^{-1}(t)\|, \end{aligned} \quad (11)$$

we obtain the scalar inequality

$$u \leq c_1 v + v \int_0^t u ds. \quad (12)$$

This yields, as a consequence of the fundamental inequality, [2], or directly, the estimate

$$\int_0^t u ds \leq c_1 \int_0^t v(s) \exp \left[\int_s^t v dr \right] ds. \quad (13)$$

By assumption $\int_0^\infty v ds < \infty$. Hence the integral

$$\int_0^\infty X^{-1}(s)[B(s) - A(s)]y ds \quad (14)$$

converges. This means that we can write (9) in the form

$$y = X(t)b + X(t) \int_0^\infty X^{-1}(s)[B(s) - A(s)]y ds - X(t) \int_t^\infty X^{-1}(s)[B(s) - A(s)]y ds \quad (15)$$

which yields the stated result.

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BOUNDS ON THE ERROR IN THE UNIT STEP RESPONSE OF A NETWORK*

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Summary. Bounds have been placed upon the error, in the unit step response of a network, caused by a departure of the transfer function from its desired value. The theorems developed place bounds on the error when it is caused by:

- 1) A departure of the amplitude function from zero, for frequencies above a stated cut-off frequency.
- 2) Any deviation of the amplitude or phase function from its desired value.

These bounds are more readily evaluated than the actual errors, and thus prove useful in the design of networks.

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Introduction. In many analysis or synthesis procedures, the transfer function of a network only approximates the desired value. It is often necessary to know how this approximation affects the transient response. Existing methods require tedious analytic or graphic analysis to determine the deviation of the transient response. More readily evaluated bounds, which eliminate the need for this analysis would be useful. Such bounds have been developed in the theorems presented in the body of this paper.

The first considers the error, in the unit step response, caused by an assumption that the transfer function is zero above a given cut-off frequency.

The remaining theorems deal with the errors, in the unit step response, which result from deviation of either the amplitude or phase functions from their desired values.

Throughout this paper, the transfer function of the network in question will be written as $T(\omega)e^{j\theta(\omega)}$ where the amplitude function $T(\omega)$ is never negative and $T(0) > 0$. In addition the magnitude of the per cent error in the unit step response, referred to its final value will be written as $|\% \delta|$. That is, if $A(t)$ is the actual unit step response and $A_1(t)$ is the approximate unit step response, then

$$|\% \delta| = \frac{|A(t) - A_1(t)|}{T(0)} \max \times 100.$$

Bounds on the error in the unit step response caused by a band-limiting approximation.

It is often convenient to assume that the transfer function of a network is zero for all values of frequency above a given cut-off frequency ω_c . This assumption, called a band limiting approximation, introduces an error into the calculated value of the unit step response. It is desirable to bound this error so that the approximation can be justified. This bound can be obtained by evaluating an integral of the form

$$\int_{\omega_c}^{\infty} \frac{T(\omega)}{T(0)} \frac{1}{\omega} d\omega.$$

This evaluation may be simplified by replacing $T(\omega)/T(0)$ by a new, more readily integrated function $G(\omega)$, where $G(\omega) \geq T(\omega)/T(0)$ for all $\omega \geq \omega_c$ and $\int_{\omega_c}^{\infty} [G(\omega)/\omega] d\omega$ exists. The best bound on the error will be obtained when $G(\omega) = T(\omega)/T(0)$ and this procedure should always be considered as a first choice.

Theorem 1 will bound the error caused by the band limiting approximation.

Theorem 1. If $T(\omega)/T(0) \leq G(\omega)$ for $\omega > \omega_c$, then

$$|\% \delta| \leq \frac{100\%}{\pi} \int_{\omega_c}^{\infty} \frac{G(\omega)}{\omega} d\omega.$$

Proof. For any network the unit step response is given by

$$A(t) = \frac{1}{2}T(0) + \frac{1}{\pi} \int_0^{\infty} \frac{T(\omega)}{\omega} \sin [\omega t + \theta(\omega)] d\omega.$$

Thus in this case the error is given by

$$|\% \delta| = \frac{100\%}{\pi} \left| \int_{\omega_c}^{\infty} \frac{T(\omega)}{\omega T(0)} \sin [\omega t + \theta(\omega)] d\omega \right|.$$

The bound is then obtained by utilizing the fact that $T(\omega)/T(0) \leq G(\omega)$ and $|\sin [\omega t + \theta(\omega)]| \leq 1$. Theorem 1 will now be applied to a particular function which falls off as $1/\omega^n$, where $n > 0$. This is of use in many commonly encountered networks. If $G(\omega) = \epsilon(\omega_c/\omega)^n$ the following is obtained.

Corollary 1a. If $T(\omega)/T(0) \leq \epsilon(\omega_c/\omega)^n$ for $\omega > \omega_c$ and $n > 0$ then

$$|\% \delta| \leq \frac{100\epsilon\%_0}{n\pi}.$$

Some typical values given by this bound are

If $\epsilon = 0.1$ and $n = 1$ then $|\% \delta| \leq 3.18\%$

If $\epsilon = 0.707$ (3 db down) and $n = 3$, then $|\% \delta| \leq 7.5\%$.

Bounds on the error in the unit step response caused by an approximation of the amplitude or phase function. When an arbitrary transfer function is not realized exactly, but is only approximated, an error appears in the unit step response.

The following theorems place bounds upon this error. The first of these deals with an approximation of the amplitude function. In this case the transfer function of the desired network is $T(\omega)e^{j\theta(\omega)}$. Often this transfer function is approximated by the transfer function $[T(\omega) + T_*(\omega)] \exp[j\theta(\omega)]$. The amplitude error function $T_*(\omega)$ is a real function of ω which may be positive and negative. Then if $\int_0^\infty |T_*(\omega)|/\omega d\omega$ exists, $|\% \delta|$ can be bounded.

Theorem 2¹. If the amplitude error function $T_*(\omega)$ exists, then

$$|\% \delta| \leq \frac{100\%_0}{\pi} \int_0^\infty \left| \frac{T_*(\omega)}{T(0)} \right| \frac{1}{\omega} d\omega.$$

Proof. This theorem is proven by formally writing the expression for the error and then utilizing the fact that $|\sin[\omega t + \theta(\omega)]| \leq 1$.

If a function $M(\omega)$ can be found such that $|T_*(\omega)/T(0)| \leq M(\omega)$ for all ω and the term $\int_0^\infty [M(\omega)/\omega] d\omega$ is more readily evaluated than the integral of Theorem 2, then it may prove convenient to replace $|T_*(\omega)/T(0)|$ by $M(\omega)$ in this theorem; the bound developed in this case will be weaker than the one originally presented. However, in some cases, it may be expedient to use the simpler integral to obtain a rough estimate of the error.

The next two theorems will place a bound upon the error in the unit step response produced by an approximation of the phase function. The transfer function of the desired network is $T(\omega) \exp[j\theta(\omega)]$. Many times this transfer function is approximated by $T(\omega) \exp[j\{\theta(\omega) + \phi(\omega)\}]$. The phase error function $\phi(\omega)$ is a real function of ω which may be positive and negative. If the integrals

$$\int_0^\infty \frac{T(\omega)}{T(0)} \left| \frac{\phi(\omega)}{\omega} \right| d\omega \quad \text{and} \quad \int_0^\infty \frac{T(\omega)}{T(0)} \frac{[\phi(\omega)]^2}{\omega} d\omega$$

exist then $|\% \delta|$ may be bounded.

Theorem 3. If the phase error function $\phi(\omega)$ exists, then

$$\begin{aligned} |\% \delta| &\leq \frac{100\%_0}{\pi} \int_0^\infty \frac{T(\omega)}{T(0)} \frac{1}{\omega} \left\{ 2 \sin^2 \left[\frac{\phi(\omega)}{2} \right] + |\sin \phi(\omega)| \right\} d\omega \\ |\% \delta| &\leq \frac{100\%_0}{\pi} \int_0^\infty \frac{T(\omega)}{T(0)} \frac{1}{\omega} \left\{ \frac{[\phi(\omega)]^2}{2} + |\phi(\omega)| \right\} d\omega. \end{aligned}$$

¹The proof of this theorem is similar to one given by A. H. Zemanian, *An approximate method of evaluating integral transforms*, J. Appl. Phys. 25, 262-266 (Feb. 1954)

Proof. The first bound is obtained by formally evaluating the error and then utilizing the fact that $|\sin [\omega t + \theta(\omega)]| \leq 1$ and $|\cos [\omega t + \theta(\omega)]| \leq 1$. The second equation is obtained from the first by realizing that $|\sin x| \leq |x|$. The first of these equations will provide a smaller bound than the second. However, the second equation is more readily evaluated and if $\phi(\omega)$ is small, in the range of integration, then the results obtained from this second equation can be quite satisfactory.

If the normalized amplitude function $T(\omega)/T(0)$ is itself bounded, the results of Theorem 3 are readily extended to obtain a bound which is more easily evaluated.

Corollary 3a. If $T(\omega)/T(0) \leq M$ then

$$|\% \delta| \leq \frac{100 M \%}{\pi} \int_0^\infty \frac{1}{\omega} \left\{ 2 \sin^2 \left[\frac{\phi(\omega)}{2} \right] + |\sin \phi(\omega)| \right\} d\omega,$$

and

$$|\% \delta| \leq \frac{100 M \%}{\pi} \int_0^\infty \frac{1}{\omega} \left\{ \frac{[\phi(\omega)]^2}{2} + |\phi(\omega)| \right\} d\omega.$$

When networks whose amplitude functions are such that Theorems 1 or 1a indicate that a negligible error is introduced by neglecting the response at frequencies greater than ω_c , then the upper limit of integration can be changed from ∞ to ω_c in the equation of Theorems 3 and 3a. This technique should especially be used when the effects of phase correction is to be determined since extremely large phase errors usually occur at frequencies above ω_c . These phase errors usually have little bearing on the transient response.

Conclusion. The theorems which have been developed here place bounds on the error, in the unit step response, caused by a deviation of the transfer function from its desired value.

In transient analysis procedures, it is often convenient to neglect the response of a network above a given cut-off frequency. Theorems 1 and 1a bound the error produced by such procedure. In addition, knowledge of results obtained in Theorems 1 and 1a can simplify the design of a network, since these results indicate a frequency above which the response is no longer important. Theorems 2, 3, and 3a bound the error caused by a deviation of either the amplitude or phase function from its desired value. Deviation of this type will occur in synthesis procedures, where the desired transfer function cannot be exactly realized with a finite number of realizable elements.

When a network is designed to produce a desired unit step response, the error of the transfer function, in itself, has no direct significance. However, this is often the known error. Theorems 2, 3, or 3a can then be employed to relate the error in either the amplitude or phase function, to the error in the unit step response.

The use of these theorems, therefore, can provide valuable information to the network designer.

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