ON A GENERALIZATION OF A RESULT OF WINTNER*

By RICHARD BELLMAN (*The Rand Corporation*)

In a recent note [1] Wintner proved the following interesting result.

Theorem 1. *Consider the two equations*

$$\text{(a)} \quad u'' + f(t)u = 0, \qquad \text{(b)} \quad v'' + g(t)v = 0. \tag{1}$$

If there exist two linearly independent solutions of (1a), u_1 *and* u_2 , *such that*

$$\int^{\infty} (| u_1 |^2 + | u_2 |^2) \, | f - g | \, dt < \infty \tag{2}$$

then every solution of (1b) *can be written in the form*

$$v = c_1 u_1 + c_2 u_2 + o(| u_1 | + | u_2 |). \tag{3}$$

This is an extension of known stability results, see [2], to which it reduces if we assume that all solutions of (1a) are bounded as $t \to \infty$.

Let us now show that we can obtain a generalization of this result following the method used in our book [2], to establish the Hukuwara stability theorem, of which this will be an extension.

Theorem 2. *Consider the vector-matrix systems*

$$\text{(a)} \quad \frac{dx}{dt} = A(t)x, \qquad \text{(b)} \quad \frac{dy}{dt} = B(t)y. \tag{4}$$

Let $X(t)$ *be the solution of*

$$\frac{dX}{dt} = A(t)X, \qquad X(0) = I. \tag{5}$$

If

$$\int^{\infty} || B(t) - A(t) || \, || X(t) || \, || X^{-1}(t) || \, dt < \infty \tag{6}$$

then every solution of (4b) *may be written*

$$y = Xc + o(|| X ||) \tag{7}$$

as $t \to \infty$.

The norms of matrices and vectors are taken to be respectively $\sum_{i,j} | x_{ij} |$ and $\sum_i | x_i |$.

Proof. Write

$$\frac{dy}{dt} = A(t)y + [B(t) - A(t)]y. \tag{8}$$

Then, if $y(0) = b$, we have

$$y = X(t)b + \int_0^t X(t)X^{-1}(s)[B(s) - A(s)]y \, ds. \tag{9}$$

*Received February 6, 1958.

Hence

$$|| y || \leq || X(t) || \, || b || + \int_0^t || X(t) || \, || X^{-1}(s) || \, || B(s) - A(s) || \, || y || \, ds. \tag{10}$$

Thus, if we set

$$\begin{aligned} u(t) &= || X^{-1}(t) || \, || B(t) - A(t) || \, || y(t) ||, \\ v(t) &= || B(t) - A(t) || \, || X(t) || \, || X^{-1}(t) ||, \end{aligned} \tag{11}$$

we obtain the scalar inequality

$$u \leq c_1 v + v \int_0^t u \, ds. \tag{12}$$

This yields, as a consequence of the fundamental inequality, [2], or directly, the estimate

$$\int_0^t u \, ds \leq c_1 \int_0^t v(s) \exp\left[\int_s^t v \, dr\right] ds. \tag{13}$$

By assumption $\int^\infty v \, ds < \infty$. Hence the integral

$$\int_0^\infty X^{-1}(s)[B(s) - A(s)]y \, ds \tag{14}$$

converges. This means that we can write (9) in the form

$$y = X(t)b + X(t) \int_0^\infty X^{-1}(s)[B(s) - A(s)]y \, ds - X(t) \int_t^\infty X^{-1}(s)[B(s) - A(s)]y \, ds \tag{15}$$

which yields the stated result.

References

1. A. Wintner, *On linear perturbations*, Quart. Appl. Math. **XV**, 428-430 (1958)
2. R. Bellman, *Stability theory of differential equations*, McGraw-Hill, New York, 1954

BOUNDS ON THE ERROR IN THE UNIT STEP RESPONSE OF A NETWORK*

By PAUL CHIRLIAN (*New York University*)

Summary. Bounds have been placed upon the error, in the unit step response of a network, caused by a departure of the transfer function from its desired value. The theorems developed place bounds on the error when it is caused by:

1) A departure of the amplitude function from zero, for frequencies above a stated cut-off frequency.

2) Any deviation of the amplitude or phase function from its desired value.

These bounds are more readily evaluated than the actual errors, and thus prove useful in the design of networks.

*Received November 29, 1957. This paper is based on a portion of a thesis which has been accepted by the faculty of the Graduate Division, College of Engineering, New York University, in partial fulfillment of the requirements for the degree of Doctor of Engineering Science.