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SOME APPLICATIONS OF DIFFERENTIAL GEOMETRY TO THE INTERPRETATION OF PHYSICAL PHENOMENA*

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1. Introduction. The motivation of the present investigation was the conviction that a differential geometric analysis of such physical phenomena as heat conduction in a homogeneous medium and electrical charge distribution on closed conductors should yield interesting quantitative geometric interpretations. Although the study has been somewhat rudimentary, the results have been, in the author's opinion, of sufficient interest to warrant extending them to their mathematical analogues in n-dimensional Euclidean space. For the sake of elegance of presentation, the three-dimensional results will appear as specializations of the n-dimensional theory.

Let ϕ, ψ denote single-valued point functions of class C^2 in a domain D of a Euclidean space R_n of n-dimensions. Let V_{n-1} denote the hypersurface on which ϕ is constant which passes through a given point P of D. Let the family of such hypersurfaces which pass through points of D be denoted by F. Let σ denote arc length of the orthogonal trajectory T of F at P, the positive and negative senses of σ along T being taken to be those in which ϕ decreases and increases, respectively. The positive function Ω defined by the relation:

$$\Omega = -\partial\phi/\partial\sigma \tag{1.1}$$

will be called the "gauge function" of F at P. A locus of points on V_{n-1} at which Ω has a constant value is an $n - 2$ dimensional variety V_{n-2} which will be called an "isogauge" of V_{n-1}. An isogauge of V_{n-1} with respect to F is, in general, an $n - 2$ dimensional variety. It can be $n - 1$ dimensional only if it coincides with V_{n-1}. In such a case, F is a family of concentric hyperspheres, each V_{n-1} being defined by a constant value of a differentiable function of the radius.

Let M denote the mean curvature of V_{n-1} at P, and let $\mathbf{p}$ and K_T denote the unit principal normal and curvature, respectively, of T at P. Let $\Delta\psi$, $\Delta_1\psi$ denote the Laplacians of ψ with respect to R_n and V_{n-1} respectively. The following theorems will be proved.

Theorem 1. *The relation which exists between the two Laplacians* $\Delta\psi$, $\Delta_1\psi$ *is given by*

$$\Delta\psi = \frac{\partial^2\psi}{\partial\sigma^2} - M\frac{\partial\psi}{\partial\sigma} + \Delta_1\psi - K_T\mathbf{p}\cdot\nabla\psi, \tag{1.2}$$

in which $\mathbf{p}\cdot\nabla\psi$ *is the projection of the gradient of* ψ *on the principal normal of* T *at* P.

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COROLLARY (1.1). *If ϕ is harmonic*

$$M = \frac{\partial}{\partial \sigma} (\ln \Omega), \tag{1.3}$$

in which $\Omega = -\partial\phi/\partial\sigma$, and M is the mean curvature of V_{n-1} at P (a result due to H. Weyl [5, p. 181]).

Theorem 2. *Let τ denote the orthogonal trajectory on V_{n-1} at P of the family of isogauges of V_{n-1}, and let s denote its arc length. The principal normal $\mathbf{p}$ of T at P is tangent to τ, and the curvature of T at P is given by the relation*

$$K_T = \frac{\partial}{\partial s} (\ln \Omega), \tag{1.4}$$

An application to steady-state heat flow may be described as follows. Let F be a family of isothermal surfaces with respect to heat flow in a homogeneous isotropic medium in ordinary space. The fundamental hypothesis in steady-state heat flow states that heat flows orthogonally across the isothermal surfaces at a rate proportional to $-\partial\phi/\partial\sigma$, in which ϕ denotes temperature. It follows that Corollary (1.1) and Theorem 2, in combination, yield the following interesting result.

Theorem 3. *In the case of a steady-state heat flow in a homogeneous isotropic medium the ratio of the rates of flow at any two points P_0, P_1 of the medium is given by the relation*

$$\Omega_0/\Omega_1 = \exp\left(-\int_{\sigma_0}^{\sigma} M\, d\sigma + \int_{s}^{s_1} K_T\, ds\right), \tag{1.5}$$

in which the first integral is along the arc of T from P_0 to the point P of intersection of T with the isothermal surface which passes through P_1 and the second is along the arc of τ from P to the intersection Q_1 of τ with the "isogauge" which passes through P_1.

It is known that on a closed charged conductor S_0 of class C^2 the density δ of charge at a point P_0 of S_0, if no other charges than that on the conductor are present, is proportional to $(-\partial\phi/\partial\sigma)_0$. From corollary (1.1) it follows that δ *is given by the relation*

$$\delta = k\Omega_1 \exp \int_{\sigma_0}^{\sigma_1} - M\, d\sigma, \tag{1.6}$$

in which Ω is the value of the gauge function at the point P_1 to which σ_1 corresponds, and the integration is from P_0 to P_1 along T.

It is known [4, p. 191] that electrical charge placed on an ellipsoidal conductor E_0 becomes distributed so that its density at a point P_0 of E_0 is proportional to the "supporting function" of E_0 at P_0, that is, the distance h_0 from the center of E_0 to the tangent plane to E_0 at P_0. The equipotential surfaces induced by the charged ellipsoid are confocal ellipsoids. Let E denote an equipotential ellipsoid whose potential is given by $\phi = c$. At points P_0, P of E_0, E, respectively, the gauge functions are given by

$$\Omega_0 = k_0 h_0, \qquad \Omega_c = k_c h, \tag{1.7}$$

in which k_0, k_c are constants, and Ω_0 is proportional to the density of charge of E_0 at P. On putting E_0, E in the roles of S_0, S for interpretation of (1.6) relative to confocal ellipsoids and making use of (1.7), the following relation is obtained.

$$mh_0/h = \exp\left(\int_\sigma^{\sigma_0} - M\,d\sigma\right), \tag{1.8}$$

in which $m = k_0/k_c$ is a constant which corresponds to the selection E_0 , E.

The simple yet fundamental role of the "supporting function" in characterizing physical phenomena associated with confocal ellipsoids suggests its usefulness in a general study of isothermal surfaces. Such a study is presented in the present paper in which use is made of the supporting function together with an associated "natural" coordinate system. The representation employed may be described as follows. Let a one-parameter family of surfaces of class C^2 be defined by a functional relation,

$$h = h(\theta, \omega, \phi), \qquad \phi = \text{constant}, \tag{1.9}$$

in which relative to a point $P(\theta, \omega, \phi)$ the numbers h, θ, ω, ϕ have the following meanings. One surface S on which ϕ is constant passes through P. The numbers h, θ, ω are spherical coordinates, with respect to some fixed rectangular cartesian coordinate frame, of the foot of the perpendicular line from the origin to the tangent plane to S at P; that is to say, h is the supporting function of S at P. A curve described by a point P as ϕ varies, but θ and ω are held fixed, is called a ϕ-curve. A curve is a ϕ-curve relative to F if, and only if, at its points the tangent planes to F are parallel.

Let a system F of surfaces, not necessarily isothermal, be represented by an equation of form (1.9). At a point P of a surface S of the system, let γ denote the acute angle between the normal to S at P and the tangent to the ϕ-curve at P, and let K denote the curvature at P of the curve of section of S by that normal plane of S which contains the tangent to the ϕ-curve at P. The following relations will be established

$$h_3 \frac{\partial h_3}{\partial \sigma} = h_{33} + (K \tan^2 \gamma) h_3^2 , \tag{1.10}$$

$$h_3^2 \,\Delta\phi + M h_3 + \frac{\partial h_3}{\partial \sigma} = 0, \tag{1.11}$$

where

$$h_1 = \partial h/\partial\theta,\ h_2 = \partial h/\partial\omega,\ h_3 = \partial h/\partial\phi.$$

If ϕ is a harmonic function, relations (1.10), (1.11) reduce to

$$h_{33} + h_3^2(M + K \tan^2 \gamma) = 0, \qquad \frac{\partial h_3}{\partial \sigma} + M h_3 = 0. \tag{1.12}$$

The coefficients of Eqs. (1.12) *are geometric quantities whose values are independent of the choice of coordinates. The quantity* $-1/h_3$ will be shown to be *equal to the gauge function* Ω *of* F *at* P. It follows that the first equation of (1.12) is equivalent to the following first-order equation in ϕ and Ω

$$\frac{\partial \Omega}{\partial \phi} + M + K \tan^2 \gamma = 0. \tag{1.13}$$

This equation leads to the following integral equation which defines the difference between the gauge functions at two points P_0 , P, of a ϕ-curve

$$\Omega_1 - \Omega_0 = -\int_{\phi_0}^{\phi_1} (M + K \tan^2 \gamma)\, d\phi. \tag{1.14}$$

An interesting physical interpretation of (1.14) may be stated as follows.

Theorem 4. *In steady-state heat flow through a homogeneous isotropic medium, the difference between the rates of flow* ρ_0 , ρ_1 *at points* P_0 , P_1 , *respectively of a flow line is given by a relation of the form*

$$\rho_0 - \rho_1 = k \int_{\phi_0}^{\phi_1} (M + K \tan^2 \gamma)\, d\phi, \tag{1.15}$$

in which k is a positive constant and ϕ_0 , ϕ_1 *are temperatures at* P_0 , P_1 *respectively, and the integration is along the flow line.*

The results stated above may be specialized to obtain the analogous geometric interpretations of physical phenomena applicable to the systems of cylinders which constitute the integral surfaces of the Laplace equation in two variables.

2. The Laplacians Δ and Δ_1 of a function ψ with respect to the surfaces ϕ = constant. In a domain D of a Euclidean space R_n let ϕ, ψ denote single-valued point functions of class C^2, let the family F of hypersurfaces V_{n-1} (introduced in Sec. 1) be coordinate hypersurfaces $x^n = \phi = c$ whose orthogonal trajectories T are the x^n-curves of R_n . Let V_{n-1} be referred to an orthogonal system of coordinate curves. The fundamental metric then assumes the form[1]

$$ds^2 = g_{ii}(dx^i)^2 = g_{nn}(dx^n)^2 + g_{\alpha\alpha}(dx^\alpha)^2. \tag{2.1}$$

Theorem 1 will be proved and some specializations will be noted in this section.

The Laplacian of a function ψ with respect to R_n at P is given by the well-known formula

$$\Delta\psi = g^{ii}\left(\frac{\partial^2\psi}{\partial x^i\,\partial x^i} - \Gamma^h_{ii}\,\frac{\partial\psi}{\partial x^h}\right), \tag{2.2}$$

in which the functions g^{ii} are defined by relations of the form

$$g^{ii}g_{ik} = \delta^k_i = \begin{cases} 0, & k = i \\ 1, & k \neq i \end{cases} \tag{2.3}$$

and the functions Γ^h_{ij} are the Christoffel functions of the second kind defined by the relations

$$\Gamma^h_{ij} = g^{hk}[ij, k] = \frac{g^{hk}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right). \tag{2.4}$$

In view of the conditions of orthogonality

$$g_{ij} = 0, \qquad i \neq j \tag{2.5}$$

and Eqs. (2.3) and (2.4), Eq. (2.2) assumes the form

$$\Delta\psi = g^{nn}\,\frac{\partial^2\psi}{\partial x^{n^2}} - g^{nn}\Gamma^n_{nn}\,\frac{\partial\psi}{\partial x^n} - g^{\alpha\alpha}\Gamma^n_{\alpha\alpha}\,\frac{\partial\psi}{\partial x^n} - g^{nn}\Gamma^\gamma_{nn}\,\frac{\partial\psi}{\partial x^\gamma} + \Delta_1\psi, \tag{2.6}$$

[1]Greek indices will have the range 1,2,. . .,n-1. Latin indices (except n) will have the range 1,2,. . .,n. Repeated indices in a term denote the usual summation convention over the respective ranges.

in which $\Delta_1\psi$ is defined by

$$\Delta_1\psi = g^{\alpha\beta}\left(\frac{\partial^2\psi}{\partial x^\alpha\,\partial x^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial\psi}{\partial x^\gamma}\right) \tag{2.7}$$

and the Christoffel functions are given by the formulas

$$\begin{aligned} \Gamma^n_{nn} &= g^{nn}[nn, n] = \frac{g^{nn}}{2}\frac{\partial g_{nn}}{\partial x^n}, \\ \Gamma^n_{\alpha\beta} &= g^{nn}[\alpha\beta, n] = -\frac{g^{nn}}{2}\frac{\partial g_{\alpha\beta}}{\partial x^n}, \\ \Gamma^\gamma_{nn} &= g^{\gamma\omega}[nn, \omega] = -\frac{g^{\gamma\omega}}{2}\frac{\partial g_{nn}}{\partial x^\omega}. \end{aligned} \tag{2.8}$$

The Laplacian of ψ may, therefore, be written in the form

$$\Delta\psi = g^{nn}\frac{\partial^2\psi}{(\partial x^n)^2} + \frac{g^{nn}g^{\alpha\alpha}}{2}\frac{\partial g_{\alpha\alpha}}{\partial x^n}\frac{\partial\psi}{\partial x^n} - g^{nn}\Gamma^i_{nn}\frac{\partial\psi}{\partial x^i} + \Delta\psi. \tag{2.9}$$

The following relations will be established

$$\frac{g^{nn}g^{\alpha\alpha}}{2}\frac{\partial g_{\alpha\alpha}}{\partial x^n}\frac{\partial\psi}{\partial x^n} = M\frac{\partial\psi}{\partial\sigma}, \tag{2.10}$$

$$\frac{\partial^2\psi}{\partial\sigma^2} - K_T p^i\frac{\partial\psi}{\partial x^i} = g^{nn}\left(\frac{\partial^2\psi}{\partial x^{n^2}} - \Gamma^i_{nn}\frac{\partial\psi}{\partial x^i}\right), \tag{2.11}$$

in which p^i denotes the contravariant components of the unit principal normal of T at P.

Let ζ denote the unit normal to V_{n-1} at P, and let ζ^i denote its contravariant components. The following equations are well known [1, p. 32]

$$M = -\operatorname{div}\zeta = -\frac{1}{(g)^{1/2}}\frac{\partial}{\partial x^i}(\zeta^i g^{1/2}), \tag{2.12}$$

in which g denotes the determinant whose element in row i and column j is g_{ij}.

Equations (2.12) will be used to establish (2.10). Since ζ is the unit tangent to T at P, its contravariant components are defined by

$$\zeta^i = \frac{\partial x^i}{\partial x^n}\Bigg/\left(g_{ij}\frac{\partial x^i}{\partial x^n}\frac{\partial x^j}{\partial x^n}\right)^{1/2} = \delta^i_n / g^{1/2}_{nn}. \tag{2.13}$$

On substituting these values in (2.12) we find

$$-M = g^{-1/2}\frac{\partial}{\partial x^n}\left(\frac{g}{g_{nn}}\right)^{1/2} = \frac{1}{2g_{\alpha\alpha}(g_{nn})^{1/2}}\frac{\partial g_{\alpha\alpha}}{\partial x^n}. \tag{2.14}$$

Moreover, since $d\sigma^2 = g_{nn}(dx^n)^2$, we have

$$\frac{\partial\psi}{\partial\sigma} = \frac{\partial\psi}{\partial x^n}\frac{dx^n}{d\sigma} = g_{nn}^{-1/2}\frac{\partial\psi}{\partial x^n}. \tag{2.15}$$

Since $g^{ii} = 1/g_{ii}$ $(i = 1, 2, \cdots, n)$, Eq. (2.10) is an obvious consequence of (2.14) and (2.15).

The components $K_T p^i$ of the first curvature vector of T at P are defined by the intrinsic derivative of ζ^i along T. Thus, we have

$$K_T p^i = \zeta^i_{,j}\zeta^j, \qquad \zeta^i = \delta^i_n \,/\, g_{nn}^{1/2}. \tag{2.16}$$

Evaluation of the intrinsic derivative yields

$$\begin{aligned} K_T p^i &= \zeta^i_{,n} \Big/ g_{nn}^{1/2} = \left(\frac{\partial \zeta^i}{\partial x^n} + \zeta^h \Gamma^i_{hn}\right) \Big/ g_{nn}^{1/2} \\ &= -\frac{\delta^i_n}{2g_{nn}^2}\frac{\partial g_{nn}}{\partial x^n} + \frac{\Gamma^i_{nn}}{g_{nn}}. \end{aligned} \tag{2.17}$$

In view of (2.8) it follows from (2.17) that $K_T p^i\ \partial\psi/\partial x^i$ is given by

$$K_T p^i \frac{\partial \psi}{\partial x^i} = K_T p^\gamma \frac{\partial \psi}{\partial x^\gamma} = g^{nn}\Gamma^\gamma_{nn}\frac{\partial \psi}{\partial x^\gamma}. \tag{2.18}$$

Again making use of (2.8), simple calculations yield the following equations

$$\begin{aligned} \frac{\partial^2 \psi}{\partial \sigma^2} &= g^{nn}\frac{\partial^2 \psi}{(\partial x^n)^2} + \frac{1}{2}\frac{\partial g^{nn}}{\partial x^n}\frac{\partial \psi}{\partial x^n} \\ &= g^{nn}\frac{\partial^2 \psi}{(\partial x^n)^2} - \frac{(g^{nn})^2}{2}\frac{\partial g_{nn}}{\partial x^n}\frac{\partial \psi}{\partial x^n} = g^{nn}\left(\frac{\partial^2 \psi}{(\partial x^n)^2} - \Gamma^n_{nn}\frac{\partial \psi}{\partial x^n}\right). \end{aligned} \tag{2.19}$$

By subtracting the members of (2.18) from the corresponding members of (2.19), Eq. (2.11) is obtained. Finally, in view of Eqs. (2.10) and (2.11), which are now established, Eq. (2.9) may be written in the form of the relation which constitutes Theorem 1 in which $\mathbf{p}\cdot\nabla\psi$ denotes $p^i\ \partial\psi/\partial x^i$.

In the special case in which $\psi = \phi$, since $p^n = 0$ and x^n is independent of $x^1, x^2, \cdots, x^{n-1}$, the scalars $\Delta_1\phi$ and $p^i\ \partial\phi/\partial x^i$ vanish. Therefore we have

$$\Delta\phi = \frac{\partial^2 \phi}{\partial \sigma^2} - M\frac{\partial \phi}{\partial \sigma}. \tag{2.20}$$

Corollary 1.1 is an obvious consequence of (2.20).

In case the system of hypersurfaces $\phi = x^n =$ constant are parallel, x^n may be taken to be the arc σ measured from $x^n = 0$ along a normal. It follows that

$$g_{nn} = 1 = g^{nn}, \qquad \Gamma^i_{nn} = 0, \qquad (i = 1, 2, \cdots, n).$$

In this case the relation between the Laplacians of ϕ becomes

$$\Delta\phi - \Delta_1\phi = \frac{\partial^2 \phi}{\partial \sigma^2} - M\frac{\partial \phi}{\partial \sigma}. \tag{2.21}$$

In particular, for a V_1 in a surface, relative to a system of curves geodesically parallel [1, p. 57] to V_1, (2.21) reduces to

$$\Delta\phi = \frac{\partial^2 \phi}{\partial \sigma^2} - K\frac{\partial \phi}{\partial \sigma} + \frac{\partial^2 \phi}{\partial s^2}, \tag{2.22}$$

in which K is the curvature of V_1 at P and s is arc length of V_1 [3].

3. Proof of Theorem 2. On substituting from (2.8) into (2.17), the components of the first curvature vector of T at P are found to be given by

$$K_T p^\alpha = g^{\alpha\omega} \frac{\partial(\ln \Omega)}{\partial x^\omega}, \qquad p^n = 0, \tag{3.1}$$

in which $(g_{nn})^{-1/2} = \Omega = -\partial\phi/\partial\sigma$. On forming the inner product of the members of (3.1) with $g_{\alpha\beta}\, dx_\beta/ds$ in which dx_β/ds denotes the components p_β the following relation is obtained

$$K_T = \partial(\ln \Omega)/\partial s, \tag{3.2}$$

in which s denotes the arc length of the orthogonal trajectory on V_{n-1} at P of the family of isogauges of V_{n-1}. This completes the proof of Theorem 2.

4. The Laplacian $\Delta\phi$ in terms of the supporting function. Let h denote the supporting function of a one-parameter family of surfaces of class C^2 defined by a functional relation of form (1.9). The quantities θ, ω, ϕ defined in association with (1.9) will serve here as curvilinear coordinates of the point P of any surface. To express the Laplacian $\Delta\phi$ in terms of the partial derivatives of h with respect to θ, ω, ϕ, the components of the fundamental metric tensors g_{ij} and g^{ij} must first be calculated. For this purpose we make use of the definitions of h, θ, ω, ϕ in terms of a fixed auxiliary rectangular cartesian coordinate system (x, y, z). Since (θ, ω, ϕ) are curvilinear coordinates of the contact point P of a tangent plane to a $\phi =$ constant surface S, the rectangular cartesian coordinates (x, y, z) of P satisfy the following three equations. The first of these equations is the equation of the tangent plane at P, and the next two are obtained from it by partial differentiation with respect to θ and ω, respectively

$$\begin{aligned} x \sin\omega \cos\theta + y \sin\omega \sin\theta + z \cos\omega &= h, \\ -x \sin\omega \sin\theta + y \sin\omega \cos\theta &= h_1, \\ x \cos\omega \cos\theta + y \cos\omega \sin\theta - z \sin\omega &= h_2, \end{aligned} \tag{4.1}$$

in which $h_2 = \partial h/\partial\omega$, $h_1 = \partial h/\partial\theta$.

The following set of equations, which has the same solution as that of (4.1) and whose coefficients of x, y, z are functions solely of θ, is obtained by combining Eqs. (4.1) in the ways indicated by the forms of the right members.

$$\begin{aligned} x \cos\theta + y \sin\theta &= h \sin\omega + h_2 \cos\omega, \\ -x \sin\theta + y \cos\theta &= h_1 \csc\omega, \\ z &= h \cos\omega - h_2 \sin\omega. \end{aligned} \tag{4.2}$$

The solution of (4.2) is given by

$$\begin{aligned} x &= (h \sin\omega + h_2 \cos\omega) \cos\theta - h_1 \csc\omega \sin\theta, \\ y &= (h \sin\omega + h_2 \cos\omega) \sin\theta + h_1 \csc\omega \cos\theta, \\ z &= h \cos\omega - h_2 \sin\omega. \end{aligned} \tag{4.3}$$

On calculating the g's by making use of the relations

$$g_{ij} = \delta_{hk} \frac{\partial x^h}{\partial u^i} \frac{\partial x^k}{\partial u^j}, \qquad i, j = 1, 2, 3 \tag{4.4}$$

in which $x^1 = x, x^2 = y, x^3 = z, u^1 = \theta, u^2 = \omega, u^3 = \phi$, the following values are obtained

$$\begin{aligned} g_{11} &= e^2 \csc^2 \omega + f^2, & g_{12} &= f(g + e \csc^2 \omega), \\ g_{22} &= g^2 + f^2 \csc^2 \omega, & g_{13} &= -(fh_{23} + eh_{13} \csc^2 \omega), \\ g_{32} &= -(gh_{23} + fh_{13} \csc^2 \omega), & g_{33} &= h_3^2 + h_{23}^2 + h_{13}^2 \csc^2 \omega, \\ G &= | g_{ij} | = h_3^2 H^2 \csc^2 \omega, & H^2 &= (eg - f^2)^2, \end{aligned} \tag{4.5}$$

in which e, f, g are defined by the equations

$$e = -(h \sin^2 \omega + h_2 \cos \omega \sin \omega + h_{11}),$$

$$f = h_1 \cot \omega - h_{12}, \qquad g = -(h + h_{22}),$$

the subscripts 1, 2, 3 of h denoting partial derivatives with respect to θ, ω, ϕ, respectively.

The functions e, f, g are actually the coefficients of the second fundamental form of S, as may be seen by direct calculations of these coefficients. Let $\mathbf{x}$, $\mathbf{x}_\alpha$, $\mathbf{x}_{\alpha\beta}$ denote the vectors whose ith components are x^i, $\partial x^i/\partial u^\alpha$, $\partial^2 x^i/\partial u^\alpha \partial u^\beta$ respectively. From the definition of the second fundamental form of S together with some orthogonality relations [6, pp. 92–94] it follows that the coefficients of the form are given by the set of determinant relations

$$d_{\alpha\beta} = (\mathbf{x}_{\alpha\beta}\mathbf{x}_1\mathbf{x}_2)/| g_{\alpha\beta} |^{1/2}, \qquad \alpha, \beta = 1, 2 \tag{4.6}$$

in which the g's are defined in (4.5).

The following expressions for the components of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_{11}$, $\mathbf{x}_{12}$, $\mathbf{x}_{22}$ are obtained by differentiating Eqs. (4.3).

$$\begin{aligned} x_1^1 &= e \csc \omega \sin \theta - f \cos \omega \cos \theta, \\ x_2^1 &= f \csc \omega \sin \theta - g \cos \omega \cos \theta, \\ x_1^2 &= -e \csc \omega \cos \theta - f \cos \omega \sin \theta, \\ x_2^2 &= -g \cos \omega \sin \theta - f \csc \omega \cos \theta, \\ x_1^3 &= f \sin \omega, \\ x_2^3 &= g \sin \omega, \\ x_{11}^1 &= (e \csc \omega - f_1 \cos \omega) \cos \theta + (e_1 \csc \omega + f \cos \omega) \sin \theta, \\ x_{11}^2 &= (-e_1 \csc \omega - f \cos \omega) \cos \theta + (e \csc \omega - f_1 \cos \omega) \sin \theta, \\ x_{11}^3 &= f_1 \sin \omega, \\ x_{21}^1 &= (f \csc \omega - g_1 \cos \omega) \cos \theta + (f_1 \csc \omega + g \cos \omega) \sin \theta, \\ x_{21}^2 &= (-g \cos \omega - f_1 \csc \omega) \cos \theta + (f \csc \omega - g_1 \cos \omega) \sin \theta, \\ x_{21}^3 &= g_1 \sin \omega, \\ x_{22}^1 &= (g \sin \omega - g_2 \cos \omega) \cos \theta + (f_2 \csc \omega - f \csc \omega \cot \omega) \sin \theta, \\ x_{22}^2 &= (f \csc \omega \cot \omega - f_2 \csc \omega) \cos \theta + (g \sin \omega - g_2 \cos \omega) \sin \theta, \\ x_{22}^3 &= g_2 \sin \omega + g \cos \omega. \end{aligned} \tag{4.7}$$

The three-rowed determinants which appear in (4.6) may now be written in the form

$$(\mathbf{x}_{\alpha\beta}\mathbf{x}_1\mathbf{x}_2) = \begin{vmatrix} x^1_{\alpha\beta} & e \csc\omega \sin\theta - f\cos\omega\cos\theta & f\csc\omega\sin\theta - g\cos\omega\cos\theta \\ x^2_{\alpha\beta} & -e\csc\omega\cos\theta - f\cos\omega\sin\theta & -g\cos\omega\sin\theta - f\csc\omega\cos\theta \\ x^3_{\alpha\beta} & f\sin\omega & g\sin\omega \end{vmatrix}, \quad (4.8)$$

in which $x^i_{\alpha\beta}$ $(i = 1, 2, 3)$ are given by (4.7).

After transforming this determinant by elementary transformations on the rows (the transformations being indicated by the transformed elements of the first column), the following form is assumed

$$(\mathbf{x}_{\alpha\beta}\mathbf{x}_1\mathbf{x}_2) \quad (4.9)$$

$$= \begin{vmatrix} x^1_{\alpha\beta}+x^3_{\alpha\beta}\cot\omega\cos\theta+(x^2_{\alpha\beta}+x^3_{\alpha\beta}\cot\omega\sin\theta)\tan\theta & 0 & 0 \\ x^2_{\alpha\beta}+x^3_{\alpha\beta}\cot\omega\sin\theta & -e\csc\omega\cos\theta & -f\csc\omega\cos\theta \\ x^3_{\alpha\beta} & f\sin\omega & g\sin\omega \end{vmatrix}.$$

In view of Eqs. (4.5), (4.7), and (4.9), simple calculations yield the values e, f, g for the coefficients d_{11}, d_{12}, d_{22} of the second fundamental form, respectively.

The components of the contravariant metric tensor, calculated according to the definition

$$g^{ij} = (\text{cofactor of } g_{ij})/G,$$

are found to have the following values

$$\begin{aligned} g^{11} &= [f^2 + g^2\sin^2\omega + (gh_{13} - fh_{23})^2/h_3^2]/H^2, \\ g^{12} &= [(eh_{23} - fh_{13})(gh_{13} - fh_{23})/h_3^2 - f(e + g\sin^2\omega)]/H^2, \\ g^{13} &= (gh_{13} - fh_{23})/Hh_3^2, \qquad g^{22} = [e^2 + f^2\sin^2\omega + (fh_{13} - eh_{23})^2/h_3^2]/H^2, \\ g^{23} &= (eh_{23} - fh_{13})/Hh_3^2, \qquad g^{33} = 1/h_3^2. \end{aligned} \quad (4.10)$$

Since the absolute value of $\partial\phi/\partial\sigma$ is equal to the length of the gradient of ϕ, where the components of the gradient of ϕ are δ^3_j, we have the relations

$$\left(\frac{\partial\phi}{\partial\sigma}\right)^2 = (\nabla\phi)^2 = g^{ij}\delta^3_i\delta^3_j = g^{33} = 1/h_3^2. \quad (4.11)$$

For a system of closed surfaces ϕ = constant, according to the convention adopted for the sign of σ, the direction of increasing σ is outward. Therefore, if the origin is an interior point of the system, h increases as σ increases, and it follows that

$$\partial\phi/\partial\sigma = 1/h_3 = -\Omega. \quad (4.12)$$

The components η^i of the unit normal to the surface $S(\phi = c)$ at P are given by

$$\eta^i = h_3 g^{i3}. \quad (4.13)$$

From (4.12) and (4.13) it follows that

$$\frac{\partial^2\phi}{\partial\sigma^2} = -\frac{1}{h_3^2}\frac{\partial h_3}{\partial\sigma} = -\frac{1}{h_3}\left(\frac{\partial h_3}{\partial u^i}\right)(g^{i3}), \quad (4.14)$$

in which $u^1 = \theta$, $u^2 = \omega$, $u^3 = \phi$. On substituting the values of the g's as given by (4.10) into (4.14) and combining the result with (4.12) to evaluate the expression for $\Delta\phi$ given by (2.20) one readily obtains the relation

$$-h_3^2\,\Delta\phi = Mh_3^2 + h_{33} + (eh_{23}^2 - 2fh_{13}h_{23} + gh_{13}^2)/H. \tag{4.15}$$

In view of (4.14), Eq. (2.20) may now be written in the form

$$h_3^2\,\Delta\phi + Mh_3 + \frac{\partial h_3}{\partial\sigma} = 0 \tag{4.16}$$

which appears in Sec. 1 as (1.11).

The derivation of (1.10) proceeds as follows. Consider a point P' on the orthogonal trajectory T of S at P whose coordinates are $(\theta + d\theta, \omega + d\omega, \phi + d\phi)$. As P' varies along T tending toward P the following limiting relations are approached

$$d\omega/d\theta = g^{23}/g^{13}, \qquad d\phi/d\omega = g^{33}/g^{23}, \tag{4.17}$$

since the contravariant components of the gradient of ϕ are given by g^{i3}. Let Q denote the point of intersection of S with the ϕ-curve passing through P'. The coordinates of Q are $(\theta + d\theta, \omega + d\omega, \phi)$. It is now clear that the limit of the direction of QP as P' and Q tend to P is given by the first equation of (4.17). It is clear from the definitions of P' and Q relative to P that the limit of the direction of QP is the direction at P of the line of intersection of the tangent plane to S at P with the normal plane to S which contains the tangent to the ϕ-curve at P. The normal curvature of S at P which corresponds to this direction will be denoted by K. It follows that K is given by the relation

$$K = (e(g^{13})^2 + 2fg^{13}g^{23} + g(g^{23})^2)/g_{\alpha\beta}g^{\alpha 3}g^{\beta 3}. \tag{4.18}$$

On substituting from (4.4) into (4.12) we find

$$K = (eh_{23}^2 - 2fh_{13}h_{23} + gh_{13}^2)/H[h_{23}^2 + h_{13}^2 \csc^2 \omega]. \tag{4.19}$$

Let γ denote the acute angle between the normal to S at P and the tangent to the ϕ-curve at P. The covariant components ζ_i of the unit normal are given by

$$\zeta_i = \delta_i^3/(g^{33})^{1/2}. \tag{4.20}$$

The contravariant components ξ^i of the unit tangent to the ϕ-curve at P are defined by

$$\xi^i = \delta_3^i/(g_{33})^{1/2}. \tag{4.21}$$

It follows that $\cos\gamma$ is given by the relation

$$\cos\gamma = \zeta_i\xi^i = 1/(g^{33}g_{33})^{1/2} = -h_3/(h_3^2 + h_{23}^2 + h_{13}^2 \csc^2 \omega)^{1/2}. \tag{4.22}$$

Therefore $\tan^2\gamma$ is given by the relation

$$\tan^2\gamma = (h_{23}^2 + h_{13}^2 \csc^2 \omega)/h_3^2\ . \tag{4.23}$$

In view of this and (4.19), relation (4.15) may be written in the form

$$-h_3^3\,\Delta\phi = Mh_3^2 + h_{33} + h_3^2 K \tan^2\gamma \tag{4.24}$$

which, in view of (4.16), is equivalent to (1.10).

If ϕ is a harmonic function, Eqs. (4.16) and (4.24) assume the forms

$$\frac{\partial h_3}{\partial \sigma} + Mh_3 = 0, \tag{4.25}$$

$$h_{33} + (M + K \tan^2 \gamma)h_3^2 = 0, \tag{4.26}$$

respectively.

In view of (4.12), Eq. (4.26) can be written in the form

$$\frac{\partial \Omega}{\partial \phi} + M + K \tan^2 \gamma = 0. \tag{4.27}$$

Equation (1.14) and Theorem 4 (stated in Sec. 1) now follow immediately.

5. The equation of Laplace in two variables. Consider a family F of $\phi =$ constant cylinders whose generators are parallel to a fixed axis, say the z-axis. The supporting function of a contact point P of a tangent plane to the $\phi =$ constant cylinder is independent of the position of P on the generator. The only admissible value of ω is, therefore, $\pi/2$ and the number triple $(\theta, \pi/2, \phi)$ determines a generator rather than a point of the generator. In order to apply results of Sec. 4 to this case it will be convenient to determine a point P of a generator by specifying its z coordinate. Thus we augment the original curvilinear coordinates θ, ϕ by the coordinate $u^2 = z$.

The auxiliary rectangular coordinates of P satisfy the set of equations

$$\begin{aligned} x \cos \theta + y \sin \theta &= h, \\ -x \sin \theta + y \cos \theta &= h_1 , \end{aligned} \tag{5.1}$$

in which $h = h(\theta, \phi)$, $h_1 = \partial h/\partial \theta$, $h_2 = \partial h/\partial z$, $h_3 = \partial h/\partial \phi$. The solution (x, y) of (5.1) is given by

$$\begin{aligned} x &= h \cos \theta - h_1 \sin \theta, \\ y &= h \sin \theta + h_1 \cos \theta. \end{aligned} \tag{5.2}$$

The g's of the metric tensor in terms of the coordinates $u^1 = \theta$, $u^2 = z$, $u^3 = \phi$, may be easily calculated. For this purpose, the following values of the partial derivatives are needed.

$$\begin{aligned} &\frac{\partial x}{\partial u^1} = -(h + h_{11}) \sin \theta, \qquad \frac{\partial y}{\partial u^1} = (h + h_{11}) \cos \theta, \qquad \frac{\partial z}{\partial u^1} = \theta, \\ &\frac{\partial x}{\partial u^2} = 0, \qquad \frac{\partial y}{\partial u^2} = 0, \qquad \frac{\partial z}{\partial u^2} = 1, \\ &\frac{\partial x}{\partial u^3} = h_3 \cos \theta - h_{13} \sin \theta, \qquad \frac{\partial y}{\partial u^3} = h_3 \sin \theta + h_{13} \cos \theta, \qquad \frac{\partial z}{\partial u^3} = 0. \end{aligned} \tag{5.3}$$

Since the g's are defined in terms of these partial derivatives by a set of equations of form (4.2) in which $x^1 = x$, $x^2 = y$, $x^3 = z$, the following values are readily obtained

$$\begin{aligned} g_{13} &= (h + h_{11})h_{13} , \qquad g_{23} = 0, \qquad g_{33} = h_3^2 + h_{13}^2 , \\ g_{11} &= (h + h_{11})^2, \qquad g_{12} = 0, \qquad g_{22} = 1. \end{aligned} \tag{5.4}$$

The quantity $(h + h_{11})^2$ will be shown to be the square of the radius of curvature of the curve C of normal section of a $\phi =$ constant cylinder. For this purpose, let α

denote the angle of inclination of the tangent to C at P with respect to the x-axis. Then $\alpha = \theta + \pi/2$. From the definition of curvature K of C, we have

$$K = d\alpha/ds = d\theta/ds,$$

where differentiation is with respect to arc length of C at P. On differentiating (5.2) with respect to s we find

$$\frac{dx}{ds} = -(h + h_{11}) \sin \theta \frac{d\theta}{ds}, \qquad \frac{dy}{ds} = (h + h_{11}) \cos \theta \frac{d\theta}{ds}. \tag{5.5}$$

Squaring (5.5), and adding yields

$$1 = (h + h_{11})^2\left(\frac{d\theta}{ds}\right)^2 = K^2(h + h_{11})^2.$$

This is what was to be shown, since r^2 is by definition $1/K^2$.

It will be shown that the necessary and sufficient condition that ϕ be a solution of the equation of Laplace is that the function h be a solution of the equation

$$r \frac{\partial p}{\partial \phi} = p^2 + q^2, \tag{5.6}$$

in which r is the radius of curvature of C at P, and p and q are defined by the relations

$$p = \frac{\partial h}{\partial \phi}, \qquad q = \frac{\partial p}{\partial \theta}.$$

If the origin of rectangular coordinates is an interior point of the system of $\phi =$ constant cylinders, Eq. (4.12) holds. Recalling that the positive sense of σ is that in which ϕ decreases, and that the gauge function diminishes as P moves along a flow line (σ increasing) we find

$$\frac{\partial \Omega}{\partial \sigma} = -\frac{\partial^2 \phi}{\partial \sigma^2} = -\frac{\partial}{\partial \sigma}(h_3^{-1}) < 0, \qquad \Omega > 0. \tag{5.7}$$

A relation of form (4.14) holds for a system of $\phi =$ constant surfaces, independently of the choice of the coordinates u^1, u^2, and consequently, in the present coordinates $u^1 = \theta$, $u^2 = z$. Thus

$$-h_3^2 \frac{\partial^2 \phi}{\partial \sigma^2} = \frac{\partial h_3}{\partial \sigma} = h_3(h_{31}g^{13} + h_{33}g^{33}). \tag{5.8}$$

But in view of (4.16), $\partial h_3/\partial \sigma$ may be replaced by $-Mh_3$ if and only if ϕ is harmonic. On making this replacement and dividing by h_3, Eq. (5.8) assumes the form

$$-h_3 \frac{\partial^2 \phi}{\partial \sigma^2} = -M = h_{31}g^{13} + h_{33}g^{33}. \tag{5.9}$$

From (5.7), M is found to be negative. The mean curvature M which is the sum of the principal curvatures of the cylinder $\phi =$ constant at P, is given by the relation

$$M = -1/r, \tag{5.10}$$

since one of the principal curvatures is the curvature of C at P and the other is zero, the curvature of the generator at P.

The contravariant components g^{ii} are defined by the relations

$$g^{ii} = (\text{cofactor of } g_{ii})/|\, g_{ii} \,|, \qquad i, j = 1, 2, 3.$$

In view of (5.4) the determinant of the g's is given by the equation

$$g_{ij} = \begin{vmatrix} r^2 & 0 & rh_{13} \\ 0 & 1 & 0 \\ rh_{13} & 0 & h_3^2 + h_{13}^2 \end{vmatrix}.$$

It follows that g^{13}, g^{33} are given by the relations

$$g^{13} = -h_{13}/rh_3^2\,, \qquad g^{33} = 1/h_3^2\,. \tag{5.11}$$

On substituting from (5.10) and (5.11) into (5.9) and clearing, the resulting equation is

$$rh_{33} = (h_3^2 + h_{13}^2), \tag{5.12}$$

which is (5.6), since $h_3 = \partial h/\partial\phi = p$, $h_{13} = \partial p/\partial\theta$, $h_{33} = \partial p/\partial\phi$.

The theorems stated in the introduction can be easily specialized for the two dimensional case. The statements of these two dimensional theorems will be omitted here.

Additional results which are not specializations of the theorems hereto proved, but whose proofs involve only routine calculations will be presented here (without proof).

Theorem 5.1. *The ratio of the gauge functions at points P_0, P of a ϕ-curve is given by the integral*

$$\Omega_0/\Omega = \exp \int_{s_0}^{s} (K^2 + K_T^2)^{1/2}\, ds \tag{5.13}$$

in which the integration is with respect to arc length of the ϕ-curve from P_0 to P, and K, K_T are the curvatures at P of the ϕ = constant curve (isothermal curve) at P and the orthogonal trajectory of the ϕ-curves at P, respectively.

Theorem 5.2. *If the dènsity of charge at each point P of the curve C_0 of a clossd conductor is proportional to a constant power of the supporting function h of P, then the exponent of the power is unity and the curve C_0 is an ellipse, and the equipotential curves induced by the distribution of charge on C_0 are ellipses confocal with C_0.*

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