Bibliography

1. J. Proudman, *On the motion of solids in a liquid possessing vorticity*, Proc. Roy. Soc. **A92**, 408-24 (1916)
2. Sir Geoffrey Taylor, *Experiments on the motion of solid bodies in rotating fluids*, Proc. Roy. Soc. **A104**, 213-18 (1923)
3. R. R. Long, *Steady motion around a symmetrical obstacle moving along the axis of a rotating fluid*, J. Met. **10**, 197-203 (1953)
4. G. W. Morgan, *A study of motions in a rotating liquid*, Proc. Roy. Soc. **A206**, 108-30 (1951)
5. Paul Gariél, *Recherches experimentales sur l'écoulement de couches superposées de fluides de densités differentes*, La Houille Blanche, No. 1, 56-64 (1949)
6. T. G. Cowling, *Magnetohydrodynamics*, Interscience, 1957

A GENERALIZATION OF LATTA'S METHOD FOR THE SOLUTION OF INTEGRAL EQUATIONS*

By MARVIN SHINBROT (*Lockheed Aircraft Corporation*)

1. Introduction. The integral equation

$$\varphi(x) = \int_a^b k(x - t)f(t)\,dt, \qquad a < x < b \tag{1.1}$$

with a and b finite arises in many applications. As examples, we mention the problem of optimization of filters having a finite memory [1] and the problem of diffraction through a slit [2, 3]. In the first of these problems, the kernel represents the autocorrelation of the inputs to the filter; in the second, $k(x) = H_0^{(1)}(|\,x\,|)$, the first Hankel function.

Its importance notwithstanding, no general solution of (1.1) is known; with but two exceptions, only special methods exist for application to special kernels. The first exception is the method developed not long ago by Latta [4, 5]. The second—which is now subsumed under Latta's method—applies to the conceptually trivial case when the kernel has a rational Fourier transform [1].

Latta's method can be applied in the circumstance that $k(x)$ satisfies a linear differential equation (of any order) with linear coefficients. This restriction is still very far from being moderate, of course, and since no general solution of (1.1) appears to be forthcoming, it appears worthwhile to see whether there are other integral equations having the form of (1.1) which can be reduced to one of Latta's type.

In a recent paper [6], Pearson considered (1.1) with

$$k(x) = p(x) \log |\,x\,| + q(x),$$

where p and q were polynomials. Although Pearson's method has no apparent connection with Latta's, the fact that $\log |\,x\,|$ is a Latta kernel leads one to consider kernels of the form

$$k(x) = p(x)j(x) + q(x), \tag{1.2}$$

where j satisfies a differential equation while, as in [6], p and q are polynomials. It is to consideration of this case that the present paper is devoted.

*Received October 23, 1957. This paper presents work done while the author was employed by the National Advisory Committee for Aeronautics.

One remark about (1.2). Although the restriction of p and q to polynomials seems rather severe, the situation is not as bad as it looks. The range of the argument $(x - t)$ of k in (1.1) is the *finite* interval $(a - b, b - a)$. Consequently, just so long as $k(x)$ is given by (1.2) with p and q continuous, the kernel can be successfully approximated by (1.2) with p and q polynomials. Thus, given, say, a singular kernel k, if the singularity alone is of Latta's type (*e.g.*, $\log | x |$ or $| x |^{\nu-1}$ or, for that matter, $H_0^{(1)}(| x |)$), the equation can be solved approximately by writing k in the form (1.2) where j contains the singularity and by approximating p and q by polynomials.

The paper begins with a brief discussion of the reduction of an equation with a kernel (1.2) to an equation with a Latta kernel. In the rest of the paper, three examples are worked out: the first two have logarithmic singularities; the last has the singularity $| x |^{\nu-1}$.

2. The idea of the method. Consider now (1.1) with $k(x)$ as in (1.2). It will be convenient to follow Latta and use operational notation—thus, we set

$$\int_a^b k(x - t)f(t)\,dt = \Lambda f$$

$$\int_a^b j(x - t)f(t)\,dt = \Gamma f,$$

where the kernel of Γf is the j of (1.2).

Now, the polynomial q of (1.2) is immaterial to the analysis; if $q(x) = \sum q_n x^n$, we have

$$\int_a^b q(x - t)f(t)\,dt = \sum q_n \int_a^b (x - t)^n f(t)\,dt$$
$$= \sum q_n^* x^n,$$

where the constants q_n^* depend on the moments $\int_a^b t^\nu f(t)\,dt$, $\nu = 1, \cdots, n$ of f. Thus, the term involving $q(x)$ can be incorporated into the function $\varphi(x)$ of (1.1); after the resulting equation has been solved for arbitrary values of the q_n^*, the solution can be substituted back into (1.1) to determine them. We may write, therefore, without loss of generality,

$$k(x) = p(x)j(x),$$

in place of (1.2).

Set $p(x) = \sum p_n x^n$. We then have

$$\Lambda f = \sum_n p_n \sum_\nu (-1)^\nu \binom{n}{\nu} x^{n-\nu} \int_a^b j(x - t)t^\nu f(t)\,dt$$
$$= \sum_n p_n \sum_\nu (-1)^\nu \binom{n}{\nu} x^{n-\nu} \Gamma x^\nu f. \tag{2.1}$$

Now, Latta's condition that $j(x)$ must satisfy a differential equation allows one to relate Γxf (and, as a consequence, $\Gamma x^\nu f$) to Γf by a linear differential equation. Using this fact, one can derive from (2.1) a differential equation relating Γf and Λf. However, Λf is known by (1.1) to be equal to $\varphi(x)$. Solving this differential equation between Γf and Λf will thus result in an equation

$$\psi(x) = \Gamma f \tag{2.2}$$

which must be satisfied if (1.1) is. The new equation (2.2) can then be solved by Latta's method since its kernel, j, satisfies his assumptions.

This will now be illustrated by examples.

3. First example. For our first example, we set

$$k(x) = x \log | x |. \tag{3.1}$$

This kernel itself seems to be of Latta's type, since it satisfies a differential equation ($xk'' - 1 = 0$) of the appropriate kind. However, $k(x)$ is non-singular; consequently, we may expect that no solution will exist unless the function φ of (1.1) is properly chosen. Since this fact violates a second assumption made in [4] and [5]—that there is a unique solution—Latta's method cannot be applied directly. We have chosen (3.1) as our first example in spite of the apparently complicating factor of possible non-existence of a solution because of the simplicity of the corresponding relation between Λf and Γf.

To begin, consider (3.1). With the notation of (1.2), we have $j(x) = \log | x |$, so that

$$(x - t)j'(x - t) - 1 = 0$$

($' = d/dx$). As discussed in [5], this implies

$$\Gamma' x f = x\Gamma' f - \mu_0 , \tag{3.2}$$

where μ_0 is the zero order moment of f. Now, obviously,

$$\Lambda f = x\Gamma f - \Gamma x f.$$

Therefore,

$$\begin{aligned} \Lambda' f &= x\Gamma' f + \Gamma f - \Gamma' x f \\ &= \Gamma f + \mu_0 , \end{aligned}$$

using (3.2). Thus, if f satisfies (1.1) with $k(x)$ as in (3.1), it must also satisfy (2.2), where

$$\psi = \varphi' - \mu_0 . \tag{3.3}$$

Equation (2.2) can be solved by Latta's method if $\varphi(x)$ is a polynomial or an exponential polynomial (φ is restricted to this class of functions in [4] and [5]) or, for general φ, by Carleman's formula ([7]; see also [6]). As an example, we choose $\varphi(x) = x$, $b = -a = 1$. By (2.2) and (3.3), we then have to solve

$$\Gamma f = 1 - \mu_0 . \tag{3.4}$$

Since $1 - \mu_0$ is a constant, the solution, according to Carleman's formula, say, is

$$f = \frac{\mu_0 - 1}{\pi \log 2} (1 - x^2)^{-1/2}. \tag{3.5}$$

Integrating, we can find μ_0 from the equation

$$\begin{aligned} \mu_0 &= \frac{\mu_0 - 1}{\pi \log 2} \int_{-1}^{1} (1 - x^2)^{-1/2} \, dx \\ &= \frac{\mu_0 - 1}{\log 2}. \end{aligned}$$

Indeed,

$$\mu_0 = \frac{1}{1 - \log 2},$$

and

$$f = \frac{(1 - x^2)^{-1/2}}{\pi(1 - \log 2)}.$$

That this really is a solution can be verified by substitution into the integral equation.

4. Second example. Now, let

$$k(x) = (1 + \alpha x) \log |x|. \tag{4.1}$$

Equation (1.1) with $k(x)$ as in (4.1) and $\varphi(x) = 1$ was solved for small α by Pearson in [6].

Here again we have $j(x) = \log |x|$, so that (3.2) continues to hold. Also,

$$\Lambda f = (1 + \alpha x)\Gamma f - \alpha \Gamma x f,$$

so that

$$\begin{aligned} \Lambda' f &= (1 + \alpha x)\Gamma' f + \alpha \Gamma f - \alpha \Gamma' x f \\ &= \Gamma' f + \alpha \Gamma f + \alpha \mu_0 , \end{aligned}$$

using (3.2). Thus, in this case, f can only satisfy (1.1) if it also satisfies (2.2), where

$$\psi' + \alpha\psi = \varphi' - \alpha\mu_0 . \tag{4.2}$$

It remains, then, to solve (2.2), where ψ is some solution of (4.2).

We now consider Pearson's example, setting $\varphi(x) = 1$, $b = -a = 1$, and solving (1.1) for small α. Equation (4.2) gives

$$\begin{aligned} \psi(x) &= -\mu_0 + Ae^{-\alpha x} \\ &= (A - \mu_0) - A\alpha x + O(\alpha^2), \end{aligned}$$

where A is a constant.

Define $f_0(x)$ by the equation

$$\Gamma f_0 = 1, \tag{4.3}$$

so that

$$f_0(x) = -\frac{1}{\pi \log 2}(1 - x^2)^{-1/2}$$

(see (3.4, 5)). Of course,

$$\begin{aligned} \Gamma' x f_0 &= -\int_{-1}^{1} f_0(x)\, dx \\ &= \frac{1}{\log 2}, \end{aligned}$$

by (3.2), so that

$$\Gamma x f_0 = \frac{x}{\log 2},$$

since xf_0 is an odd function.

We wish to consider

$$\begin{aligned}\Gamma f &= \psi \\ &= (A - \mu_0) - A\alpha x + O(\alpha^2) \\ &= (A - \mu_0)\Gamma f_0 - (A\alpha \log 2)\Gamma x f_0 + O(\alpha^2).\end{aligned} \tag{4.4}$$

Since the equation $\Gamma f = \psi$ has a unique solution [7] whose singularities can only be of the form $(1 - x^2)^{-1/2}$ at worst [5], it follows that

$$\begin{aligned}f &= (A - \mu_0)f_0 - (A\alpha \log 2)xf_0 + O[\alpha^2(1 - x^2)^{-1/2}] \\ &= (1 - x^2)^{-1/2}\left[\frac{\mu_0 - A}{\pi \log 2} + \frac{A\alpha}{\pi} x + O(\alpha^2)\right].\end{aligned} \tag{4.5}$$

The constant A can now be evaluated by integrating (4.5). We obtain

$$\mu_0 \equiv \int_{-1}^{1} f\, dx = \frac{\mu_0 - A}{\log 2} + O(\alpha^2),$$

so that

$$A = \mu_0(1 - \log 2) + O(\alpha^2)$$

and

$$f = (1 - x^2)^{-1/2}\left[\frac{\mu_0}{\pi} + \frac{\mu_0\alpha}{\pi}(1 - \log 2)x + O(\alpha^2)\right]. \tag{4.6}$$

To evaluate μ_0, recall that we must have

$$\begin{aligned}1 &= \Lambda f \\ &= (1 + \alpha x)\Gamma f - \alpha \Gamma x f.\end{aligned} \tag{4.7}$$

Now, (4.6) is equivalent to

$$f = -(\mu_0 \log 2)f_0 - (\mu_0\alpha \log 2)(1 - \log 2)xf_0 + O[\alpha^2(1 - x^2)^{-1/2}].$$

Hence,

$$\begin{aligned}\Gamma f &= -(\mu_0 \log 2)\Gamma f_0 - (\mu_0\alpha \log 2)(1 - \log 2)\Gamma x f_0 + O(\alpha^2) \\ &= -\mu_0 \log 2 - \mu_0\alpha(1 - \log 2)x + O(\alpha^2)\end{aligned}$$

by (4.3) and (4.4). Also,

$$\begin{aligned}\Gamma x f &= -(\mu_0 \log 2)\Gamma x f_0 + O(\alpha) \\ &= -\mu_0 x + O(\alpha).\end{aligned}$$

Therefore, (4.7) results in

$$\begin{aligned}1 &= (1 + \alpha x)[-\mu_0 \log 2 - \mu_0\alpha(1 - \log 2)x] + \mu_0\alpha x + O(\alpha^2) \\ &= -\mu_0 \log 2 + O(\alpha^2),\end{aligned}$$

so that

$$\mu_0 = -\frac{1}{\log 2} + O(\alpha^2),$$

giving finally,

$$f = (1 - x^2)^{-1/2}\left[-\frac{1}{\pi \log 2} - \frac{\alpha}{\pi}\left(\frac{1}{\log 2} - 1\right)x + O(\alpha^2)\right],$$

which agrees with Pearson's result.

5. Final example. For our final example, we choose

$$k(x) = (1 + \alpha x^2) \mid x \mid^{\nu-1} + 1, \qquad 0 < \nu < 1;$$

this will illustrate a kernel with a non-logarithmic j and a non-zero q [see (1.2)]. The integral equation is

$$\varphi(x) = \int_a^b [1 + \alpha(x - t)^2] \mid x - t \mid^{\nu-1} f(t)\, dt + \int_a^b f(t)\, dt. \tag{5.1}$$

The last term in this equation is a constant, μ_0, and so if we set

$$\varphi_* = \varphi - \mu_0, \tag{5.2}$$

we have

$$\varphi_*(x) = \int_a^b k_*(x - t)f(t)\, dt$$
$$\equiv \Lambda_* f,$$

where

$$k_*(x) = (1 + \alpha x^2) \mid x \mid^{\nu-1}.$$

Now, if $j(x) = \mid x \mid^{\nu-1}$,

$$xj' + (1 - \nu)j = 0,$$

and so if $\Gamma f = \int_a^b j(x - t)f(t)\, dt$,

$$\Gamma' x f = x\Gamma' f + (1 - \nu)\Gamma f. \tag{5.3}$$

Also,

$$\Lambda_* f = (1 + \alpha x^2)\Gamma f - 2\alpha x\Gamma x f + \alpha\Gamma x^2 f. \tag{5.4}$$

We obtain from (5.3) that

$$\Gamma'' x^2 f = x^2\Gamma'' f + 2(2 - \nu)x\Gamma' f + (1 - \nu)(2 - \nu)\Gamma f; \tag{5.5}$$

therefore, differentiating (5.4) and using (5.3) and (5.5), we obtain

$$\Lambda_*'' f = \Gamma'' f + \alpha\nu(1 + \nu)\Gamma f.$$

Thus, in order to solve (5.1), we must first consider the equation

$$\Gamma f = \psi, \tag{5.6}$$

where ψ is a solution of

$$\psi'' + \alpha\nu(1 + \nu)\psi = \varphi_*'' .$$

Equation (5.6) can be solved by Latta's method if ψ (and, therefore, φ) is an exponential polynomial. The details of the computations, though complicated, are not impossible, but it would serve no purpose to complete them here.

References

1. J. Halcombe Laning, Jr. and Richard H. Battin, *Random processes in automatic control*, McGraw-Hill, 1956
2. Harold Levine, *Diffraction by an infinite slit*, Stanford Univ. Appl. Math. and Stat. Lab. Tech. Rept. no. 61 (1957)
3. C. J. Tranter, *A further note on dual integral equations and an application to the diffraction of electromagnetic waves*, Quart. J. Mech. and Appl. Math. VII, part 3, 317-325 (1954)
4. Gordon Latta, *The solution of a class of integral equations reducible to ordinary differential equations*, Stanford Univ. Appl. Math. and Stat. Lab. Tech. Rept., no. 32 (1955)
5. G. E. Latta, *The solution of a class of integral equations*, J. Ratl. Mech. and Anal. 5, no 5, 821-833 (1956)
6. Carl E. Pearson, *On the finite strip problem*, Quart. Appl. Math. XV, no 2, 202-208 (1957)
7. T. Carlemann, *Über die Abelsche Integralgleichung mit konstanten Integrationsgrenzen*, Math. Z. 15 111-120 (1922)

ON THE FIRST STABILITY INTERVAL OF THE HILL EQUATION*

By C. R. PUTNAM (*Purdue University*)

Let λ denote a real parameter and let $f = f(t)$ be a real-valued, continuous periodic function of period 1. It is known (Liapounoff) that the Hill equation

$$x'' + (\lambda + f(t))x = 0 \qquad [' = d/dt, f(t + 1) = f(t)] \tag{1}$$

is stable for $\lambda = 0$, so that every solution of the equation $x'' + f(t)x = 0$ is bounded, whenever

$$f \geq 0, \qquad f \not\equiv 0 \quad \text{and} \quad \int_0^1 f\,dt \leq 4; \tag{2}$$

see, e.g., [1], [6]. Moreover the constant 4 of (2) is the best possible in the sense that (2) is not sufficient for the stability of $x'' + fx = 0$ if the 4 is replaced by $4 + \epsilon$ (ϵ, a positive constant) [3]. If λ_0 and λ_1 denote respectively the left and right end-points of the first stability interval of (1) then the first two conditions of (2) imply $\lambda_0 < 0$ while all conditions together imply $\lambda_1 > 0$ (and so $\lambda = 0$ is interior to the first interval of stability of (1)). Actually the inequality $\lambda_1 > 0$ is implied by the single condition

$$\int_0^1 f^+\,dt \leq 4, \quad \text{where} \quad f^+(t) = \max\,[0, f(t)] \tag{3}$$

(see [6]); moreover, the estimate

$$\lambda_1 > 4 - \int_0^1 f^+\,dt = 4\left(1 - \frac{1}{4}\int_0^1 f^+\,dt\right) \tag{4}$$

*Received October 28, 1957.