ON AN INITIAL VALUE PROBLEM CONCERNING TAYLOR INSTABILITY OF INCOMPRESSIBLE FLUIDS*

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Abstract. By taking the effect of surface tension and viscosity into consideration, Bellman and Pennington [2] have generalized the original treatment of the problem of Taylor instability [1]. They claim, however, that a problem in which the motion starts from rest cannot be treated with their linearized formulation. In this paper, the initial value problem is treated and, although the algebra becomes more complicated, the linearized analysis suffices. In particular, the cut-off wave number as found in [2] is not modified.

1. Introduction. The problem of Taylor instability [1], asks about the instability of a small disturbance imposed on the interface separating two infinitely extended fluids when the system is undergoing a constant acceleration directed from the lighter towards the heavier fluid.

The original treatment was limited to incompressible nonviscous fluids. Surface tension effects at the interface were ignored. Referring to a simple sinusoidal initial disturbance of amplitude a and wave number k, the result showed that the time history of the surface disturbance with respect to the accelerated frame of reference is given by

$$\eta(x, t) = a \cosh (nt) \cos kx. \tag{1.1}$$

The growth factor "n" is related monotonically to the wave number "k" by the expression

$$n^2 = -\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} g^* k. \tag{1.2}$$

In the above, ρ_1, ρ_2 are the densities of the lighter and the heavier fluid respectively, while

$$g^* = g + g_1 \tag{1.3}$$

with g as the usual gravitational constant, and g_1, the imposed acceleration.

Bellman and Pennington [2], later generalized Taylor's original result by taking the effect of surface tension and viscosity into consideration. They assumed a standing wave type solution and obtained the following expression for the growth factor "n"

$$n^2 = -\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} g^* k - \frac{T}{\rho_2 + \rho_1} k^3, \tag{1.4}$$

where T is the surface tension of the interface. Equation (1.4) indicates the existence of a cut-off wave number k_c, i.e. a wave number such that no instability arises when $k > k_c$, where

$$k_c = \left[-(\rho_2 - \rho_1) \frac{g^*}{T} \right]^{1/2}. \tag{1.5}$$

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Their result also shows that viscosity tends to retard the growth of the interface. However, they note a dilemma, for according to their solution it is impossible to start a motion from rest. They attributed this seeming paradox to the linearization performed in their analysis.

The object of this paper is to show that this paradox is easily resolved. As a consequence, the time history of the interface is given by a more complicated expression than that of Eq. (1.1); in particular, it is no longer possible to define a simple growth factor. However, this algebraic complication, neither compromises the existence of the cut-off wave number nor changes its value as given by Eq. (1.5).

2. Formulation of the problem. For simplicity, we shall confine our attention to an air-liquid system, i.e. we may approximate the density ratio by $\rho_1/\rho_2 = 0$. The more general problem can be treated in precisely the same way. The flow field is taken as the half plane bounded above by the free surface $y = \eta(x, t)$. The medium is assumed to be incompressible but viscous; surface tension effects are to be included. The whole body of the fluid is assumed to undergo a constant vertical acceleration g_1 directed from air towards the liquid. Referring to a stream function ψ_y, the linearized momentum equation with respect to the accelerated frame of reference can be written as

$$\frac{\partial}{\partial y}\frac{\partial}{\partial t}[\nabla^2 \psi - \nu \nabla^2 \nabla^2 \psi] = 0 \tag{2.1}$$

and ψ itself must satisfy

$$\frac{\partial}{\partial t}\nabla^2 \psi = \nu \nabla^2 \nabla^2 \psi. \tag{2.2}$$

The coordinate axes are chosen in such a way that the x-axis is taken in the unperturbed plane of the free surface while the y-axis is taken normal to it.

With some amount of algebraic manipulation it can be shown that ψ must satisfy the following boundary conditions (they imply $\sigma_{yy} = \sigma_{xy} = 0$) at the free surface:

$$\begin{aligned} \nu(\psi_{xxx} - \psi_{xyy})_t - \psi_{xtt} + g^*\psi_{xy} &= -\frac{T}{\rho}\psi_{xxy}, \\ -\psi_{xxy} + \psi_{yyy} &= 0, \quad \text{at} \quad y = 0. \end{aligned} \tag{2.3}$$

The usual kinematic requirement at the free surface is

$$\eta_t = -\psi_{xy}. \tag{2.4}$$

With no loss in generality, we can restrict the analysis to a single Fourier component in x; thus we may write

$$\begin{aligned} \psi(x, y, t) &= \phi(y, t)e^{ikx}, \\ \eta(x, t) &= \sigma(t)e^{ikx}. \end{aligned} \tag{2.5}$$

We may also define

$$\begin{aligned} f(y, s) &= \int_0^\infty e^{-st}\phi(y, t)\, dt \\ \mu(s) &= \int_0^\infty e^{-st}\sigma(t)\, dt \end{aligned} \tag{2.6}$$

and obtain in place of Eqs. (2.2) and (2.3) the following equations

$$\left[\frac{d^2}{dy^2} - k^2\left(1 + \frac{s}{\nu k^2}\right)\right]\left[\frac{d^2}{dy^2} - k^2\right]f = -\frac{1}{\nu}\left[\frac{d^2}{dy^2} - k^2\right]\phi(y, 0), \tag{2.7}$$

$$\text{for} \quad -\infty < y < 0,$$

and

$$k^2 f_y + f_{yyy} = 0$$

$$s[k^2 f + f_{yy}] + s^2 f + \left(g^* + \frac{T}{\rho} k^2\right) f_y \tag{2.8}$$

$$= s\phi(0, 0) - \frac{i}{k}\left(g^* + \frac{T}{\rho} k^2\right)\sigma(0), \quad \text{at} \quad y = 0.$$

From these expressions one observes that the problem is determined theoretically once the initial surface disturbance and the initial velocity field throughout the domain are prescribed.

3. Illustrative example. To illustrate our discussion, let us take

$$\psi(x, y, 0) = 0 \tag{3.1}$$

and

$$\eta(x, 0) = a \cos kx, \tag{3.2}$$

i.e. the motion is started from rest with an initial disturbance imposed at the free surface in the form of a sinusoidal wave having amplitude a and wave number k.

Using the foregoing method, it can be verified that η_t is given by

$$\eta_t = -a\left\{\nu k^2 \beta \sum_{j=1}^{4} A_j \alpha_j \exp\left[(\alpha_j^2 - 1)k^2 t\right] \operatorname{erfc}\left[-\alpha_j(\nu k^2 t)^{1/2}\right]\right\} \cos kx \tag{3.3}$$

where α_j are the roots of the quartic

$$P(\tau) = \tau^4 + 2\tau^2 - 4\tau + (1 + \beta) \tag{3.4}$$

with

$$\tau^2 = 1 + \frac{s}{\nu k^2}$$

the quantity A_j is given by

$$A_j = \left(\frac{dP}{d\tau}\right)^{-1}_{\tau = \alpha_j} \tag{3.5}$$

and the parameter β is given by

$$\beta = \frac{T}{\nu^2 \rho k_c}\left[\left(\frac{k_c}{k}\right) - \left(\frac{k_c}{k}\right)^3\right]. \tag{3.6}$$

Here, k_c is the cut-off wave number corresponding to $\rho_1/\rho_2 = 0$, i.e.

$$k_c = \left\{-\rho \frac{g^*}{T}\right\}^{1/2}. \tag{3.7}$$

Equation (3.4) indicates that for each j the Re $(\alpha_j^2) < 1$ when $| k/k_c | \geq 1$, but this inequality does not hold when $| k/k_c | < 1$. Thus we are in agreement with the principal result obtained in [2] but note that the model of that paper has broader validity than was claimed.

References

1. Sir Geoffrey Taylor, *The instability of liquid surfaces when accelerated in a direction perpendicular to their plane, I*, Proc. Roy. Soc. (London) **A201**, 192-196 (1950)
2. B. Bellman and R. H. Pennington, *Effect of surface tension and viscosity on Taylor instability*, Quart. Appl. Math. **12**, 151-162 (1954)

Comments on the paper

WAVES PRODUCED BY A PULSATING SOURCE TRAVELLING BENEATH A FREE SURFACE*

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There appears to be an error in the analysis performed in the subject paper by Dr. Tan [1], which has led to erroneous conclusions in regard to the character of the resulting wave pattern and the asymptotic conditions at infinity that are used to arrive at a unique solution. Other treatments of this same problem have been carried out by Becker [2], Kaplan [3], and Wu [4], and they have arrived at different conclusions from those of Tan. Their results show two harmonic wave trains on the downstream side of the source when $\tau > \frac{1}{4}$, corresponding to the roots K_3 and K_4 , in agreement with the results of Tan. However, when $0 < \tau < \frac{1}{4}$, Tan indicated that four harmonic wave trains were on the downstream side, with no disturbance found to propagate to infinity upstream. This led to his conclusion that the imposition of an asymptotic upstream condition of "vanishing disturbance at infinity" is sufficient to render his solution unique. The results of Becker, Kaplan and Wu were all in agreement, and showed that when $0 < \tau < \frac{1}{4}$ there were three harmonic wave trains on the downstream side, and one harmonic wave train on the upstream side. It can be shown that the wave in the upstream side corresponds to the root K_2 and that the error in Tan's analysis was not recognizing the fact that, in the notation of [1],

$$\lim_{\beta \to 0} Im K_2 = 0- \qquad (0 < \tau < \tfrac{1}{4})$$

not $0+$, as he indicated.

If this error were corrected, and the resulting analysis carried out in the same manner as done in the paper, the results would have been in agreement with that of the other authors. The imposition of the requirement of no disturbance upstream would not be valid, but the use of the technique of Rayleigh's artifice in order to secure uniqueness would not be affected, in spite of the artificial nature of this factor. A more realistic

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