VIBRATIONS OF TWISTED BEAMS II*

BY

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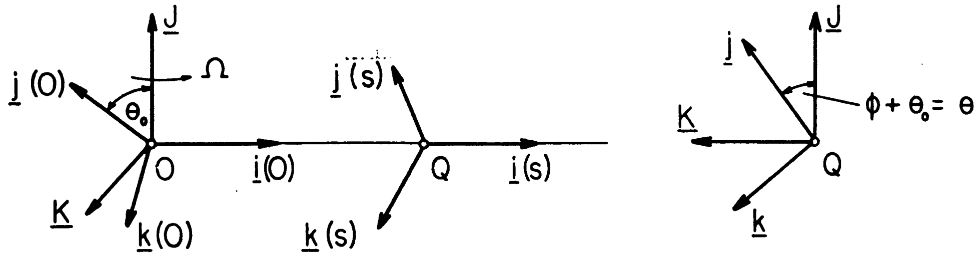
1. Introduction. The basic equations governing the transverse vibrations of a straight, twisted beam rotating about an axis, passing through one end and perpendicular to the undeflected central axis of the bar, have been discussed in an earlier paper [1].** It is the purpose of this note to explore some of the implications of these equations and to answer some unresolved questions raised in the previous work. We shall not be concerned with the problems of numerical calculations, for these have been studied extensively by others (see, for example, [2] and [3]). The results presented here are of a somewhat more qualitative nature.

The twisted beam is described in terms of a straight center line which is the locus of the centroids of the cross-sectional planes taken normal to the line. A cross-section is specified by means of the arc length s measured along the center line from a fixed origin 0 on the axis of rotation. Two triads of orthogonal unit vectors, as shown in the figure,



are used in the analysis. The first is a moving triad **i**, **j**, **k**, in which **i** is directed along the undeflected center line at a generic point Q, positive in the direction of increasing s; whereas, the vectors **j** and **k** have the directions of the principal axes of inertia of the cross section at Q. In general, the triad rotates about the **i** axis as the center line is traversed with $\mathbf{j}(s)$ at Q making an angle $\phi(s)$ with $\mathbf{j}(0)$. The second triad of unit vectors is a rotating, untwisted frame consisting of **I**, coincident with **i**; **J**, lying along the axis of rotation; and **K**, perpendicular to **I** and **J** forming a right-handed system. The vector $\mathbf{j}(0)$ forms the angle θ_0 with **J**.

If the displacement $\mathbf{u}(s, t)$ is assumed to have harmonic time dependence, $\mathbf{u}(s, t) = \mathbf{v}(s) \exp (i\lambda t)$, and $\mathbf{w}(s) = \mathbf{i} \times \mathbf{v}$, the equation of motion [see [1], Eq. (4.8)] is given by

$$\frac{d^2}{ds^2}\left\{\mathbf{B}^{-1}\cdot\frac{d^2\mathbf{w}}{ds^2}\right\} - \Omega^2 \frac{d}{ds}\left\{L(s)\frac{d\mathbf{w}}{ds}\right\} - m(s)A(s)\Omega^2\mathbf{\Theta}\cdot\mathbf{w} - m(s)A(s)\lambda^2\mathbf{w} = 0. \tag{1.1}$$

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**Numbers in square brackets refer to the list of References given at the end of the paper.

The mass per unit volume and cross-sectional area are denoted by $m(s)$ and $A(s)$ respectively, while Ω is the constant angular speed of rotation. When referred to the **i**, **j**, **k** triad, the dyadic $\mathbf{B}^{-1}$ is given by

$$\mathbf{B}^{-1} = EI_1\mathbf{jj} + EI_2\mathbf{kk},$$

where $E(s)$ denotes Young's modulus, and $I_1(s)$ and $I_2(s)$ are the centroidal moments of inertia about the **j** and **k** axes respectively. The dyadic $\boldsymbol{\Theta}(s)$ is defined in terms of $\theta(s) = \phi(s) + \theta_0$ as

$$\boldsymbol{\Theta}(s) = \cos^2\theta\mathbf{jj} - \sin\theta\cos\theta(\mathbf{jk} + \mathbf{kj}) + \sin^2\theta\mathbf{kk};$$

and

$$L(s) = \int_s^l m(\xi)A(\xi)\xi\, d\xi,$$

where l is the total length of the bar.

The beam is assumed to be elastically supported at the axis of rotation. Consequently, at $s = 0$

$$\mathbf{w} = \mathbf{w}' - \boldsymbol{\varepsilon}\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' = 0, \tag{1.2}$$

where primes and Roman numerals denote differentiation with respect to s and

$$\boldsymbol{\varepsilon} = \epsilon_1\mathbf{jj} + \epsilon_2\mathbf{kk}, \qquad \epsilon_1 \geq 0, \qquad \epsilon_2 \geq 0.$$

In the **i**, **j**, **k** coordinate system, ϵ_2 is the ratio of the bending moment about the **k** axis to the angle of inclination of the beam with respect to the (**i**, **k**)-plane. A similar interpretation holds for ϵ_1 . The end $s = l$ is assumed to be free; therefore,

$$\mathbf{B}^{-1}\cdot\mathbf{w}'' = (\mathbf{B}^{-1}\cdot\mathbf{w}'')' = 0 \tag{1.3}$$

there. It is the purpose of this study to examine some of the properties of the eigenvalue problem defined by the differential equation (1.1) and boundary conditions (1.2) and (1.3).

2. Positive definiteness of the Rayleigh quotient. The Rayleigh quotient, $R(\mathbf{w}) = D(\mathbf{w})/H(\mathbf{w})$, which corresponds to the eigenvalue problem defined by Eqs. (1.1), (1.2), and (1.3), has been discussed in [1]. It was found there that

$$D(\mathbf{w}) = \int_0^l (\mathbf{w}''\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' + \Omega^2 L\mathbf{w}'\cdot\mathbf{w}' - mA\Omega^2\mathbf{w}\cdot\boldsymbol{\Theta}\cdot\mathbf{w})\, ds + \mathbf{w}'(0)\cdot\boldsymbol{\varepsilon}^{-1}\cdot\mathbf{w}'(0), \tag{2.1}$$

and

$$H(\mathbf{w}) = \int_0^l mA\mathbf{w}\cdot\mathbf{w}\, ds. \tag{2.2}$$

A straightforward calculation shows that each of the terms can be written as a sum of squares, but it is not obvious that $D(\mathbf{w})$ is itself positive definite since the last term is preceded by a minus sign. It is not reasonable, on physical grounds, to expect that this functional could ever be negative, for this would imply the possibility of purely imaginary frequencies. Nevertheless, it is necessary, from the mathematical point of view, to establish positive definiteness in order to apply the usual Rayleigh-Ritz procedures and comparison theorems.

The proof depends on theorems concerning M-definite and N-definite eigenvalue problems given by Kamke [4] and Kestens [5]. These results, although actually stated for scalar equations, can be extended to vector equations of the type considered here. The major change requires the use of the integral representation of the vector solution in terms of a Green's tensor rather than a scalar Green's function. The resulting algebraic structure is the same as the standard situation and the proofs can be carried through by appropriate changes in notation.

Let us assume that $D(\mathbf{w})$ is not positive definite and there is a value of Ω^2 for which the corresponding eigenvalue λ^2 is less than zero. Since λ^2 is a continuous function of Ω^2 and $\lambda^2 > 0$ for $\Omega^2 = 0$, there must be a value of $\Omega^2 > 0$ for which $\lambda^2 = 0$.

Let ω^2 be that value of Ω^2 for which $\lambda^2 = 0$. Thus ω^2 is an eigenvalue of the problem

$$(\mathbf{B}^{-1}\cdot\mathbf{w}'')'' - \omega^2(L\mathbf{w}')' - mA\omega^2\mathbf{\Theta}\cdot\mathbf{w} = 0, \qquad 0 < s < l, \tag{2.3}$$

$$\begin{cases} \mathbf{w} = \mathbf{w}' - \boldsymbol{\varepsilon}\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' = 0, & s = 0, \\ \mathbf{B}^{-1}\cdot\mathbf{w}'' = (\mathbf{B}^{-1}\cdot\mathbf{w}'')' = 0, & s = l. \end{cases} \tag{2.4}$$

The Rayleigh quotient R corresponding to this problem is

$$R = \frac{\int_0^l \mathbf{w}''\cdot\mathbf{B}^{-1}\cdot\mathbf{w}''\,ds + \mathbf{w}'(0)\cdot\boldsymbol{\varepsilon}^{-1}\cdot\mathbf{w}'(0)}{\int_0^l (mA\mathbf{w}\cdot\mathbf{\Theta}\cdot\mathbf{w} - L\mathbf{w}'\cdot\mathbf{w}')\,ds}. \tag{2.5}$$

Since the numerator is positive definite and the denominator is of undetermined sign, the problem is of the type designated by Kamke [4: II, III] as M-definite. He has shown that maximum-minimum theorems of the Courant type and comparison theorems hold for this problem. An upper bound can be found for the positive eigenvalues ω^2 through these theorems. Let

$$M = \max(EI_1, EI_2), \qquad 0 < s < l;$$

then direct computation shows that

$$\mathbf{w}''\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' \leq M\mathbf{w}''\cdot\mathbf{w}''.$$

In addition, if

$$\mu = \max mA, \qquad \nu = \min mA, \qquad 0 < s < l,$$

then

$$L(s) \leq \tfrac{1}{2}\mu(l^2 - s^2).$$

Furthermore if

$$\mathbf{w} = W_1\mathbf{J} + W_2\mathbf{K},$$

then

$$\mathbf{w}\cdot\mathbf{\Theta}\cdot\mathbf{w} = W_1^2,$$

and

$$\int_0^l mA\mathbf{w}\cdot\mathbf{\Theta}\cdot\mathbf{w}\,ds \geq \nu\int_0^l W_1^2\,ds.$$

Also, if $\alpha = \max (\epsilon_1^{-1}, \epsilon_2^{-1})$,
then

$$\mathbf{w}' \cdot \boldsymbol{\varepsilon}^{-1} \cdot \mathbf{w}' \leq \alpha \mathbf{w}' \cdot \mathbf{w}'.$$

Let ω_n^{*2} be the nth positive eigenvalue corresponding to the Rayleigh quotient

$$R^* = \frac{\displaystyle\int_0^l M\mathbf{w}'' \cdot \mathbf{w}'' \, ds + \alpha \mathbf{w}'(0) \cdot \mathbf{w}'(0)}{\displaystyle\nu \int_0^l W_1^2 \, ds - \tfrac{1}{2}\mu \int_0^l (l^2 - s^2)\mathbf{w}' \cdot \mathbf{w}' \, ds}. \tag{2.6}$$

We then obtain from Kamke's comparison theorems

$$\omega_n^2 \leq \omega_n^{*2}.$$

The differential equations and natural boundary conditions which arise from the Rayleigh quotient (2.6) can be found, by elementary variational techniques, to be

$$\begin{cases} MW_1^{\mathrm{IV}} - \tfrac{1}{2}\mu\omega^{*2}[(l^2 - s^2)W_1']' - \omega^{*2}\nu W_1 = 0, \\ W_1(0) = \alpha W_1'(0) - MW_1''(0) = 0, \\ W_1''(l) = W_1'''(l) = 0, \end{cases} \tag{2.7}$$

and

$$\begin{cases} MW_2^{\mathrm{IV}} - \tfrac{1}{2}\mu\omega^{*2}[(l^2 - s^2)W_2']' = 0, \\ W_2(0) = \alpha W_2'(0) - MW_2''(0) = 0, \\ W_2''(l) = W_2'''(l) = 0. \end{cases} \tag{2.8}$$

Let us first consider the simpler of the two eigenvalue problems, (2.8). Multiplying the differential equation by W_2, integrating from 0 to l, and applying partial integration together with the boundary conditions yield

$$-\tfrac{1}{2}\mu\omega^{*2} \int_0^l (l^2 - s^2)W_2'^2 \, ds = \alpha[W_2'(0)]^2 + \int_0^l MW_2''^2 \, ds.$$

Consequently, there are no positive eigenvalues ω^{*2} for problem (2.8); and the assumed solution must be found, if it exists, from problem (2.7).

Equations (2.7) can be analyzed by observing that they represent the transverse vibrations of a uniform bar rotating about an axis at $s = 0$ with a constant angular velocity $\omega^*(\mu)^{1/2}$. The bar is elastically restrained at $s = 0$ and free at $s = l$. The circular frequency of transverse vibration corresponds to $\omega^*(\nu)^{1/2}$. According to the Southwell [6] inequality, the smallest eigenvalue $\omega^{*2}\nu$ must satisfy

$$\omega^{*2}\nu \geq K + \omega^{*2}\mu,$$

where $K > 0$ (with one possible exception to be considered later). Consequently,

$$\omega^{*2}\nu > \omega^{*2}\mu. \tag{2.9}$$

However, $0 \leq \nu \leq \mu$, contradicting (2.9) unless $\omega^{*2} < 0$. Thus there are no eigenvalues $\omega^{*2} \geq 0$ and hence no eigenvalues $\omega^2 \geq 0$. Therefore in our original problem there is no value of $\Omega^2 \geq 0$ for which $\lambda^2 = 0$, and all of the eigenvalues of the problem defined by Eqs. (1.1), (1.2) and (1.3) are positive.

With this result we can now show that for any admissible vector $\mathbf{w}$, as defined in [1], p. 255,

$$F(\mathbf{w}) = \int_0^l mA\mathbf{w}\cdot\mathbf{\Theta}\cdot\mathbf{w}\,ds - \int_0^l L\mathbf{w}'\cdot\mathbf{w}'\,ds \leq 0, \tag{2.10}$$

and hence complete the proof that $D(\mathbf{w})$ is positive definite for any admissible vector. If (2.10) were false, there would be a vector $\mathbf{u}$ such that

$$\rho = \frac{\int_0^l \mathbf{u}''\cdot\mathbf{B}^{-1}\cdot\mathbf{u}''\,ds + \mathbf{u}'(0)\cdot\boldsymbol{\varepsilon}^{-1}\cdot\mathbf{u}'(0)}{\int_0^l mA\mathbf{u}\cdot\mathbf{\Theta}\cdot\mathbf{u}\,ds - \int_0^l L\mathbf{u}'\cdot\mathbf{u}'\,ds} > 0.$$

A result of Kamke's [4, III, p. 257] states that there must then be an eigenvalue ω^2 of Eqs. (2.3), (2.4), and (2.5) satisfying

$$0 \leq \omega^2 \leq \rho^2.$$

However, we have just shown that such an eigenvalue cannot exist and hence $F(\mathbf{w}) \leq 0$, proving that $D(\mathbf{w}) \geq 0$ for all admissible functions.

The only exception to the preceding argument occurs when the constant K in the Southwell inequality vanishes and $\mu = \nu$. This takes place when the bar is simply supported, $\mathbf{w}(0) = \mathbf{w}''(0) = 0$, and the maximum and minimum values of mA are equal, i.e., $mA =$ constant. In this case,

$$\mathbf{w} = s\mathbf{J}$$

is non-trivial and yields an eigenvalue $\lambda^2 = 0$ for any value of Ω^2. Noting the relation between $\mathbf{w}$ and the displacement $\mathbf{u}$, we see that this solution corresponds to the beam's remaining straight, always lying in the plane of rotation, and making a constant angle with the $\mathbf{i}$-direction. However, the non-negative character of $D(\mathbf{w})$ is preserved if we regard this situation as the limiting case which occurs when ϵ_1^{-1}, ϵ_2^{-1} approach zero.

3. Difference equations and continuity conditions. The preceding section, together with the results contained in [1], shows that the lowest eigenvalue of the twisted, rotating beam is described by the minimum principle

$$\lambda^2 = \min_{W_1} D(\mathbf{w})/H(\mathbf{w}),$$

where the class W_1 of admissible vectors $\mathbf{w}$ consists of all vectors $\mathbf{w}$ having continuous third derivatives, piecewise continuous fourth derivatives, and satisfying $\mathbf{w}(0) = 0$. In many instances, useful comparison results could be obtained if these continuity requirements were lightened. The proof of such a minimum principle, however, would normally require all the restrictions given above. Fortunately, Courant [7] has shown, by a much more delicate analysis, that these restrictions for the admissible functions can be reduced without disturbing any of the requisite properties of the minimizing function. His proof goes much deeper than that usually given since it includes an existence theorem as well as an analysis of the convergence of a related set of difference equations. Although his paper is concerned with the standard, second-order, Sturm-Liouville problem, he states that the results will carry over for higher order, self-adjoint problems.

The extension to the present vector problem is straightforward, requiring only a few changes in small details, and none in the basic method of proof. The algebra and notation are, however, considerably more awkward.

We shall therefore give only a brief discussion of the following theorem. Let the class W of admissible vectors contain those vectors $\mathbf{w}$ which are continuous, have continuous first derivatives, piecewise continuous second derivatives, and satisfy $\mathbf{w} = 0$ at $s = 0$. Then among the vectors in W, there exists a vector, $\mathbf{w}^{(1)}$, which renders $D(\mathbf{w})$ a minimum subject to $H(\mathbf{w}) = 1$. This vector is the first eigenvector of the eigenvalue problem (1.1), (1.2) and (1.3); the first eigenvalue is $\lambda^{(1)^2} = D(\mathbf{w})^{(1)})$. If

$$H(\mathbf{w}, \mathbf{u}) = \int_0^l mA\mathbf{w}\cdot\mathbf{u}\, ds,$$

then, more generally, there exists a vector $\mathbf{w}^{(k)}$, belonging to W, which minimizes $D(\mathbf{w})$ subject to $H(\mathbf{w}) = 1$, $H(\mathbf{w}), \mathbf{w}^{(i)}) = 0$, $i = 1, \cdots, k - 1$. This vector is the kth eigenvector of the system and $\lambda^{(k)^2} = D(\mathbf{w}^{(k)})$ is the kth eigenvalue.

The integrals $D(\mathbf{w})$ and $H(\mathbf{w})$ are replaced by finite sums obtained by dividing the interval $0 \leq s \leq l$ into n equal parts of length c_n. Let $\mathbf{w} = w_1\mathbf{j} + w_2\mathbf{k}$ and $\tau = d\phi/ds$. Furthermore, let $w_{1,0}, \cdots, w_{1,\nu}, \cdots, w_{1,n}$ represent the values of w_1 at the points of division and similarly for w_2. Then $w_{1,0} = w_1(0)$ and $w_{1,n} = w_1(l)$. The ordinary difference operators will be denoted by Δ, Δ^2, $\cdots$, i.e.,

$$\Delta w_{1,\nu} = \frac{1}{c_n}[w_{1,\nu+1} - w_{1,\nu}],$$

$$\Delta^2 w_{1,\nu} = \frac{1}{c_n^2}[w_{1,\nu+1} - 2w_{1,\nu} + w_{1,\nu-1}].$$

Let

$$\begin{aligned}
P_\nu(\mathbf{w}) &= \Delta^2 w_{1,\nu} - w_{1,\nu}\tau_\nu^2 - 2\tau_\nu\Delta w_{2,\nu} - w_{2,\nu}\Delta\tau_\nu\,,\\
Q_\nu(\mathbf{w}) &= \Delta^2 w_{2,\nu} - w_{2,\nu}\tau_\nu^2 + 2\tau_\nu\Delta w_{1,\nu} + w_{1,\nu}\Delta\tau_\nu\,,\\
R_\nu(\mathbf{w}) &= \Delta w_{1,\nu} - w_{2,\nu}\tau_\nu\,,\\
N_\nu(\mathbf{w}) &= \Delta w_{2,\nu} - w_{1,\nu}\tau_\nu\,,\\
G_\nu(\mathbf{w}) &= w_{1,\nu}^2\cos^2\theta_\nu - 2w_{1,\nu}w_{2,\nu}\sin\theta_\nu\cos\theta_\nu + w_{2,\nu}^2\sin^2\theta_\nu\,,\\
&= [M_\nu(\mathbf{w})]^2,\\
T(\mathbf{w}) &= \epsilon_1^{-1}[R_0(\mathbf{w})]^2 + \epsilon_2^{-1}[N_0(\mathbf{w})]^2,\\
J_\nu(\mathbf{w}) &= w_{1,\nu}^2 + w_{2,\nu}^2\,,\\
L_\nu(s) &= \sum_{k=\nu}^{n} m_k A_k k c_n\,,\\
D_n(\mathbf{w}) &= c_n\sum_{\nu=0}^{n-1}\{EI_{1,\nu}P_\nu^2(\mathbf{w}) + EI_{2,\nu}Q_\nu^2(\mathbf{w})\\
&\qquad + \Omega^2 L_\nu[R_\nu^2(\mathbf{w}) + N_\nu^2(\mathbf{w})] - m_{\nu+1}A_{\nu+1}\Omega^2 G_{\nu+1}(\mathbf{w})\} + T(\mathbf{w}),
\end{aligned}$$

and

$$H_n(\mathbf{w}) = c_n \sum_{\nu=0}^{n} m_\nu A_\nu J_\nu(\mathbf{w}).$$

Our original minimum problem can now be replaced by an ordinary algebraic eigenvalue problem by asking for the vector $\mathbf{w}_\nu$ which makes $D_n(\mathbf{w})/H_n(\mathbf{w})$ a minimum. This yields, for example, a lowest eigenvalue $\lambda_n^{(1)^*}$ and a corresponding eigenvector $\mathbf{w}_\nu^{(n)}$. The eigenvector $\mathbf{w}_\nu^{(n)}$ is, of course, defined only at discrete points along $0 \leq s \leq l$. However, we may extend $\mathbf{w}_\nu^{(n)}$ to a vector $\mathbf{w}^{(n)}(s)$ defined over the entire interval by means of interpolation. Unlike the second order case, parabolic rather than linear interpolation is used.

It can then be shown that, as n becomes large, a subsequence of the $\lambda_n^{(1)^*}$ converges to a limit $\lambda^{(1)^*}$ and a corresponding subsequence of $\mathbf{w}^{(n)}(s)$ converges to a limiting vector $\mathbf{w}(s)$. Furthermore, $\mathbf{w}(s)$ has all the requisite continuity properties for the solution of the original differential equation and is an eigenvector with $\lambda^{(1)^*}$ the corresponding eigenvalue. The proof for the higher eigenvalues follows similar lines.

As noted before, this modification of Courant's proof yields an existence theorem, shows that the continuity requirements on the class of admissible vectors can be weakened, and finally proves convergence (in the sense of subsequences) for the difference equations which result from the minimization of $D_n(\mathbf{w})/H_n(\mathbf{w})$. These equations are the algebraic Euler equations corresponding to

$$\lambda_n^{(1)^*} = \min D_n(\mathbf{w})/H_n(\mathbf{w}),$$

and are found to be

$$\begin{aligned}\Delta^2[EI_{1,\nu}P_\nu(\mathbf{w}_\nu^{(n)})] &- EI_{1,\nu+1}\tau_{\nu+1}^2P_{\nu+1}(\mathbf{w}_\nu^{(n)}) \\ &- 2\Delta[EI_{2,\nu}\tau_\nu Q_\nu(\mathbf{w}_\nu^{(n)})] + EI_{2,\nu+1}\Delta\tau_{\nu+1}Q_{\nu+1}(\mathbf{w}_\nu^{(n)}) - \Omega^2\Delta[L_\nu R_\nu(\mathbf{w}_\nu^{(n)})] \\ &+ \Omega^2 L_{\nu+1}\tau_{\nu+1}N_{\nu+1}(\mathbf{w}_\nu^{(n)}) - m_{\nu+1}A_{\nu+1}\Omega^2\cos\theta_{\nu+1}M_{\nu+1}(\mathbf{w}_\nu^{(n)}) - \lambda_n^{(1)^*}m_{\nu+1}A_{\nu+1}w_{1,\nu+1}^{(n)} = 0,\end{aligned}$$

and

$$\begin{aligned}\Delta^2[EI_{2,\nu}Q_\nu(\mathbf{w}_\nu^{(n)})] &- EI_{2,\nu+1}\tau_{\nu+1}^2Q_{\nu+1}(\mathbf{w}_\nu^{(n)}) \\ &+ 2\Delta[EI_{1,\nu}\tau_\nu P_\nu(\mathbf{w}_\nu^{(n)})] - EI_{1,\nu+1}\Delta\tau_{\nu+1}P_{\nu+1}(\mathbf{w}_\nu^{(n)}) - \Omega^2\Delta[L_\nu N_\nu(\mathbf{w}_\nu^{(n)})] \\ &- \Omega^2 L_{\nu+1}\tau_{\nu+1}R_{\nu+1}(\mathbf{w}_\nu^{(n)}) + m_{\nu+1}A_{\nu+1}\Omega^2\sin\theta_{\nu+1}M_{\nu+1}(\mathbf{w}_\nu^{(n)}) - \lambda_n^{(1)^*}m_{\nu+1}A_{\nu+1}w_{2,\nu+1}^{(n)} = 0,\end{aligned}$$

with the natural boundary conditions

$$\begin{aligned} P_n(\mathbf{w}_\nu^{(n)}) &= Q_n(\mathbf{w}_\nu^{(n)}) = 0, \\ \Delta[EI_{1,n}P_n(\mathbf{w}_\nu^{(n)})] &= \Delta[EI_{2,n}Q_n(\mathbf{w}_\nu^{(n)})] = 0, \\ \Delta w_{1,0}^{(n)} - \epsilon_1 EI_{1,0}P_0(\mathbf{w}_\nu^{(n)}) &= 0, \\ \Delta w_{2,0}^{(n)} - \epsilon_2 EI_{2,0}Q_0(\mathbf{w}_\nu^{(n)}) &= 0. \end{aligned}$$

As a direct consequence of these results, we obtain Courant's maximum-minimum principle. Let $\mathbf{v}^{(i)}$, $i = 1, \cdots, n-1$, be a set of independent vectors and let W be the class of vectors $\mathbf{w}$ which have a continuous first derivative, a piecewise continuous second derivative, and satisfy

$$H(\mathbf{w}, \mathbf{v}^{(i)}) = 0, \qquad i = 1, \cdots, n-1.$$

If

$$M(\mathbf{v}^{(i)}) = glb_W D(\mathbf{w})/H(\mathbf{w}),$$

where glb_W stands for the greatest lower bound with respect to all vectors $\mathbf{w}$ in W, then

$$\lambda^{(n)2} = \max_{\mathbf{v}^{(i)}} M(\mathbf{v}^{(i)}).$$

4. Some applications of the minimum principle. Two simple applications of the comparison theorems implied by the Courant maximum-minimum principle will be mentioned here. Until now we have considered the case in which the fixed end of the beam occurs at the axis of rotation $s = 0$. Let us now assume that the beam is mounted on a finite hub of radius d so that the flexible portion is in the range $d \leq s \leq l$. Then the differential equation (1.1) holds over the range $d < s < l$ rather than $0 < s < l$ and the boundary conditions (1.2) are applied at $s = d$.

Consider two beams identical save for the fact that one has a hub radius d_1, whereas the other has a hub radius d_2 ($d_1 \leq d_2$). In the first case s has the range $d_1 < s < l$, whereas $d_2 < s < l$ in the second. We shall now establish the inequality

$$\lambda^{(n)2}(d_1) \leq \lambda^{(n)2}(d_2), \tag{4.1}$$

a result which is physically reasonable.

We shall define $D_1(\mathbf{w})$ in the same fashion as $D(\mathbf{w})$ in Eq. (2.1) except that the range of integration now runs from d_1 to l and the boundary term is evaluated at $s = d_1$. We define $H_1(\mathbf{w})$, $D_2(\mathbf{w})$ and $H_2(\mathbf{w})$ correspondingly. Furthermore, let $W(d_2)$ contain those vectors $\mathbf{w}$ which, in the interval $d_2 \leq s \leq l$, have a continuous first derivative, a piecewise continuous second derivative, and satisfy $\mathbf{w}(d_2) = 0$. We shall normalize these vectors by requiring that $H_2(\mathbf{w}) = 1$. The maximum-minimum principle thus states that if

$$M_2(\mathbf{v}^{(i)}) = glb_{W(d_2)} D_2(\mathbf{w}),$$

under the condition $H_2(\mathbf{w}, \mathbf{v}^{(i)}) = 0$, $i = 1, \cdots, n-1$, then

$$\lambda^{(n)2}(d_2) = \max_{\mathbf{v}^{(i)}} M_2(\mathbf{v}^{(i)}).$$

Similarly, let $W(d_1)$ contain all vectors $\mathbf{w}$ in $d_1 \leq s \leq l$ which have continuous first derivatives, piecewise continuous second derivatives, and satisfy $\mathbf{w}(d_1) = 0$. If

$$M_1(\mathbf{v}^{(i)}) = glb_{W(d_1)} D_1(\mathbf{w})/H_1(\mathbf{w})$$

under the condition $H_1(\mathbf{w}, \mathbf{v}^{(i)}) = 0$, $i = 1, \cdots, n-1$, then

$$\lambda^{(n)2}(d_1) = \max_{\mathbf{v}^{(i)}} M_1(\mathbf{v}^{(i)}).$$

Let W^* be the set of vectors $\mathbf{w}^*$ defined over $d_1 \leq s \leq l$ which are found by continuing each vector belonging to $W(d_2)$ into $d_1 \leq s \leq d_2$ under the following restrictions. When $s = d_1$, $\mathbf{w}^* = 0$; $\mathbf{w}^*$ has a continuous first derivative and a piecewise continuous second derivative in $d_1 \leq s \leq l$; and $D_1(\mathbf{w}^*) - D_2(\mathbf{w}^*) < \rho$, where ρ is an arbitrarily small number. Finally, if for a given set of vectors $\mathbf{v}^{(i)}$, $i = 1, \cdots, n-1$, a vector $\mathbf{w}$ in $W(d_2)$ satisfies $H_2(\mathbf{w}, \mathbf{v}^{(i)}) = 0$, then the corresponding continuation $\mathbf{w}^*$ satisfies

$$\int_{d_1}^{d_2} mA\mathbf{w}^* \cdot \mathbf{v}^{(i)}\, ds = 0, \qquad i = 1, \cdots, n-1;$$

i.e., $H_1(\mathbf{w}^*, \mathbf{v}^{(i)}) = 0$. Thus, W^* is included in $W(d_1)$.

With these definitions, we see that for any set of vectors $\mathbf{v}^{(i)}$ for which $H_1(\mathbf{w}, \mathbf{v}^{(i)}) = 0$,

$$glb_{W(d_1)} \frac{D_1(\mathbf{w})}{H_1(\mathbf{w})} \leq glb_{W^*} \frac{D_1(\mathbf{w}^*)}{H_1(\mathbf{w}^*)}.$$

In addition, $H_1(\mathbf{w}^*) \geq H_2(\mathbf{w}^*) = 1$. Therefore, if $H_2(\mathbf{w}, \mathbf{v}^{(i)}) = 0$, $i = 1, \cdots, n-1$,

$$glb_{W^*} \frac{D_1(\mathbf{w}^*)}{H_1(\mathbf{w}^*)} < glb_{W^*} D_2(\mathbf{w}^*) + \rho = glb_{W(d_2)} D_2(\mathbf{w}) + \rho.$$

This result may also be written as

$$M_1(\mathbf{v}^{(i)}) < M_2(\mathbf{v}^{(i)}) + \rho.$$

Maximizing over $\mathbf{v}^{(i)}$ and noting that ρ is arbitrary, we have (4.1). Inequality (4.1) agrees with the quantitative results found for the uniform beam by Boyce [8] for zero angle of inclination to the plane of rotation and by Schilhansl [9] for a general angle of inclination.

As a second example, we shall see how some information concerning the effect of twist can be obtained simply from the maximum-minimum principle. Consider a non-rotating, twisted beam, clamped at the end $s = 0$. The governing equations are Eq. (1.1) with $\Omega = 0$, Eq. (1.2) with $\varepsilon = 0$, and Eq. (1.3). We shall also assume that $EI_1 \geq EI_2$ for $0 \leq s \leq l$. The corresponding Rayleigh quotient R is

$$R = \int_0^l \mathbf{w}''\cdot\mathbf{B}^{-1}\cdot\mathbf{w}''\,ds \Big/ \int_0^l mA\mathbf{w}\cdot\mathbf{w}\,ds.$$

From the inequality $EI_1 \geq EI_2$, we see by direct computation that

$$EI_2\mathbf{w}''\cdot\mathbf{w}'' \leq \mathbf{w}''\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' \leq EI_1\mathbf{w}''\cdot\mathbf{w}''.$$

This may also be written in terms of the unit vectors $\mathbf{J}$ and $\mathbf{K}$ as

$$EI_2(W_1''^2 + W_2''^2) \leq \mathbf{w}''\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' \leq EI_1(W_1''^2 + W_2''^2).$$

Consequently, if $\lambda_i^{(n)^2}$, $i = 1, 2$, represents the nth eigenvalue corresponding to the Rayleigh quotient R_i defined by

$$R_i = \int_0^l EI_i(W_1''^2 + W_2''^2)\,ds \Big/ \int_0^l mA(W_1^2 + W_2^2)\,ds,$$

we have from the comparison theorem

$$\lambda_2^{(n)^2} \leq \lambda^{(n)^2} \leq \lambda_1^{(n)^2}.$$

The scalar differential equations which correspond to the Rayleigh quotients R_i are

$$\left.\begin{aligned} &(EI_1W_k'')'' - mA\lambda_1^2W_k = 0, \\ &W_k(0) = W_k'(0) = W_k''(l) = W_k'''(l) = 0, \end{aligned}\right\} k = 1, 2 \tag{4.2}$$

and similarly for R_2 with the subscript 1 replaced by 2.

We note that Eqs. (4.2) represent the ordinary equations of transverse vibration of an untwisted beam of flexural rigidity EI_1 and mass per unit length mA. It is important to note, however, that the beam is capable of transverse vibration in two directions, W_1 or W_2, and hence each eigenvalue is a double one (one corresponding to $W_1 \equiv 0$

and one to $W_2 \equiv 0$). Consequently, if we think of $\lambda_1^{(n)*}$ or $\lambda_2^{(n)*}$ as the nth eigenvalue of an untwisted beam vibrating in only one transverse direction, the appropriate inequality becomes

$$\left.\begin{aligned} \lambda_2^{(k)*} &\leq \lambda^{(2k)*} \leq \lambda_1^{(k)*} \\ \lambda_2^{(k+1)*} &\leq \lambda^{(2k+1)*} \leq \lambda_1^{(k+1)*} \end{aligned}\right\} k = 1, 2, \cdots \tag{4.3}$$

Equations (4.3) thus give bounds for the frequencies of a twisted beam in terms of those of an untwisted beam.

5. The 1-*p* resonance problem. The techniques used in Sect. 2 can be applied to yield more insight into the practical problem of 1-p resonance. This question, which is of interest to the aircraft propeller designer, is to determine whether there is a frequency of rotation which coincides with the lowest natural frequency of the twisted blade. If such a frequency occurs within the operating range of the propeller, undesirable resonance phenomena can take place. It is hoped that the following remarks will shed some light on the mechanism responsible for 1-p resonance.

Lo and Renbarger [10] have set up the equations for a uniform, rotating bar whose plane of bending is inclined at an angle γ to the plane of rotation. The governing differential equation, in non-dimensional form, is

$$w^{\mathrm{IV}} - \tfrac{1}{2}\alpha^2[(1 - x^2)w']' - (\beta^2 - \alpha^2 \sin^2 \gamma)w = 0, \qquad 0 < x < 1.$$

Here α is the non-dimensional rotational frequency and β a similarly defined frequency of vibration. Boyce [11] has shown for the clamped bar that if $\alpha^2 \leq 12.36$, there is no angle of inclination γ for which 1-p resonance will take place. On the other hand, if $\alpha^2 \geq 15.2$, there will always be a value of γ for which one can have 1-p resonance. Similar results hold for other end conditions. In other words, if the beam rotates at a speed above a fixed limit, inclination to the plane of rotation will produce 1-p resonance.

Now let us consider whether it is possible to have 1-p resonance in an untwisted beam vibrating perpendicularly to the plane of rotation. In particular, we ask whether we can find variable EI and mA distributions which will permit this type of resonance.

Equations (1.1), (1.2), and (1.3) reduce in this case to

$$\begin{cases} (EIw'')'' - \Omega^2[L(s)w']' = mA\lambda^2 w, & 0 < s < l; \\ w = w' - \epsilon EIw'' = 0, & s = 0; \\ EIw'' = (EIw'')' = 0, & s = l. \end{cases} \tag{5.1}$$

For 1-p resonance $\lambda^2 = \Omega^2$, and the differential equation becomes

$$(EIw'')'' = \Omega^2[L(s)w']' + mA\Omega^2 w$$

with the boundary conditions of Eqs. (5.1). This defines an eigenvalue problem for Ω^2, and the corresponding Rayleigh quotient is

$$\Omega^2 = \frac{\displaystyle\int_0^l EIw''^2\, ds + \epsilon^{-1}[w'(0)]^2}{\displaystyle\int_0^l mAw^2\, ds - \int_0^l L(s)w'^2\, ds}.$$

If there is a positive eigenvalue Ω^2, 1-p resonance will take place.

As in Sec. 2, let

$$M = \max_s EI, \qquad \nu = \min_s mA, \qquad \mu = \max_s mA.$$

Again, the extended comparison theorems state that the positive eigenvalues Ω^2 are bounded above by the positive eigenvalues ω^2 which correspond to the Rayleigh principle

$$\omega^2 = \frac{M \int_0^l w''^2 \, ds + \epsilon^{-1}[w'(0)]^2}{\nu \int_0^l w^2 \, ds - \frac{1}{2}\mu \int_0^l (l^2 - s^2)w'^2 \, ds}.$$

The resulting eigenvalue problem is

$$\begin{cases} Mw^{\mathrm{IV}} + \frac{1}{2}\omega^2\mu[(l^2 - s^2)w']' = \omega^2 \nu w, \\ w(0) = \epsilon^{-1}w'(0) - Mw''(0) = w''(l) = w'''(l) = 0. \end{cases}$$

However, this system is identical with Eq. (2.7). We have previously found that there are no positive eigenvalues in this case unless the beam is simply supported and the mass distribution is uniform. It is easily seen from Eqs. (5.1) that there is always an eigenvalue $\lambda^2 = \Omega^2$ in this case and 1-p resonance takes place. Furthermore, this result remains true for the simply-supported, twisted beam with uniform mass distribution as is seen from the solution

$$\mathbf{w} = s\mathbf{K}.$$

Our analysis has thus shown that, except for the simply-supported bar of uniform mass distribution, there is no EI or mA distribution which will produce 1-p resonance. In other words, if the section is not inclined to the plane of rotation, this phenomenon cannot take place.

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