ON TRANSFER FUNCTIONS AND TRANSIENTS*

BY

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Abstract. In the first part of this paper the concept of the positive real function is generalized so that it is applicable to transfer functions and the functions, satisfying this generalized concept, are arranged into classes. Some tests are then developed which may be used to determine whether a transfer function belongs to a particular class. It is also shown that if transfer functions have certain general forms then they will automatically be members of one of the classes. Finally, several properties of the phase functions for such system functions are developed.

The second part of the paper considers the impulse and step responses corresponding to these transfer functions. It is found that these transient responses are bounded and moreover the rise time and settling time of the step response are found to be greater than lower bounds which depend on the amount of the maximum overshoot (or undershoot). These results are also generalizations of the restrictions developed on the transient responses of positive real system functions and are considerably stronger. As the difference in degree between the numerator and denominator of the transfer function increases, the magnitudes of these lower bounds also increase.

Introduction. The question of what restrictions exist on the transient responses of various classes of networks has been treated in a series of papers [1–3]. In particular, bounds on the impulse and step responses have been given when the corresponding frequency responses are restricted to being of constant sign or monotonic in a semi-infinite interval. An example of this is that, if a rational system function, $Z(s)$, is positive real and has one more pole than zero, then the absolute value of the impulse response is never greater than $1/C$ which is the constant multiplier of the system function. These results lead in turn to lower bounds on the rise time or settling time of the step response of such networks when the overshoot and undershoot is specified.

Recently, one of these results has been improved by Ovseyevich [4]. The improvement is on the bounds developed on the impulse response of the positive real system functions in the interval $(0, T)$ when the bounds on that response in the interval (T, ∞) are known. A related problem has been considered by Cutteridge [5] who determines the optimum rise time of passive two-terminal networks under various restrictions. For instance, considering only the step responses which are monotonic or have only a small overshoot, the optimum rise time is determined when the system function has only two zeros and three poles.

This paper deals with a generalization of these results which is applicable to transfer functions. In particular, certain classes of functions are defined from the system functions having any number of poles in excess of zeros such that the corresponding transient responses are restricted in a manner similar to the restrictions existing on the transient responses of positive real functions. The strength of the restriction on the transient responses increases as the excess of poles over zeros increases.

In the first part of this paper, these classes of system functions are defined. The

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definition may be considered a generalization of the concept of positive real system functions which is suitable for application to transfer functions. Various tests are then developed which determine whether a system function is a member of any class. One immediate result is that, if the system function has poles only in the left hand complex frequency plane and no zeros anywhere except at infinity (that is, if the system function is the reciprocal of a Hurwitz polynomial), then it will be a member of one of the classes. Furthermore, certain properties of the phase functions for such system functions are also developed.

In the second section the restrictions on the transient responses for the defined classes of functions are derived. These constitute bounds on the impulse responses and the step responses. From these, lower bounds are developed on the rise time and settling time of the step response when the overshoot and undershoot are given and, conversely, the specification of the rise time or settling time fixes the lower bounds on the maximum overshoot or undershoot.

Part I. A generalization of the concept of positive real functions. The systems considered in this paper are lumped, linear, fixed, finite and stable systems so that the system functions[1] $Z(s)$ have the following rational form where s is the complex frequency variable $\sigma + j\omega$, the coefficients and the constant multiplier K are real numbers, and n and m are positive integers $(m > n)$.

$$Z(s) = K\frac{s^n + a_{n-1}s^{n-1} + \cdots + a_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_0} = K\frac{N(s)}{D(s)}. \tag{1}$$

The term "stable system" is taken to mean a system whose response will eventually become arbitrarily small once the input is removed. Thus the polynomial $D(s)$ is a Hurwitz polynomial all of whose roots have a negative (non-zero) real part.

The special classes of functions that are of interest here will be defined after a few preliminary remarks. Let

$$Z_q(s) = (-j)^q \int_\infty^s ds_{q-1} \int_\infty^{s_{q-1}} ds_{q-2} \cdots \int_\infty^{s_1} Z(s_0)\, ds_0 , \tag{2}$$

where the real parts of the complex variables $s_0, s_1, \cdots, s_{q-1}, s$ are all non-negative and $q \leq m - n - 1$. Letting $s_q = \sigma_q + j\omega_q$, the real and imaginary parts, $R_q(\omega)$ and $I_q(\omega)$, of $Z_q(j\omega)$ may be obtained from the real and imaginary parts, $R(\omega)$ and $I(\omega)$, of $Z(j\omega)$ by (3) and (4).

$$Z_q(j\omega) = R_q(\omega) + jI_q(\omega)$$

$$R_q(\omega) = \int_{-\infty}^{\omega} d\omega_{q-1} \int_{-\infty}^{\omega_{q-1}} d\omega_{q-2} \cdots \int_{-\infty}^{\omega_1} R(\omega_0)\, d\omega_0 \tag{3}$$

$$I_q(\omega) = \int_{-\infty}^{\omega} d\omega_{q-1} \int_{-\infty}^{\omega_{q-1}} d\omega_{q-2} \cdots \int_{-\infty}^{\omega_1} I(\omega_0)\, d\omega_0 . \tag{4}$$

It is evident from the following argument that these successive integrations may be performed $m - n - 1$ times. Since $Z(s)$ is analytic for $\sigma \geq 0$, the integral of $Z(s)$ between two given points will yield the same result for all paths between these points which do not enter the left half s plane. Furthermore, the inverse power series expansion of $Z(s)$,

[1]These system functions may be impedances, admittances or transfer functions that are ratios of currents or voltages.

which holds for $|s| \geq M$, is

$$Z(s) = \sum_{\mu=m-n}^{\infty} \frac{K_\mu}{s^\mu}, \tag{5}$$

where $K_{m-n} = K$ and M is a positive number greater than the distance from the origin to the pole of $Z(s)$ farthest away from the origin. Since this series converges uniformly for $|s| \geq M$, it may be integrated term by term yielding a series which also converges uniformly in the same region. This process may be continued q times where $q \leq m - n - 1$. Thus the successive integrations of $Z(s)$ given by (2) yield unique functions $Z_q(s)$ which are all analytic for $\sigma_q \geq 0$.

Now the aforementioned classes of functions $Z(s)$ given by (1) are defined as follows.

Definition. $Z(s)$ *will be called a class k function, where* $k = m - n$, *if one of the following inequalities holds for* $-\infty < \omega < +\infty$. *For* $k = 2\nu + 1$ $(\nu = 0, 1, 2, \cdots)$,

$$(-1)^\nu R_{k-1}(\omega) \geq 0 \tag{6}$$

and, for $k = 2\nu$ $(\nu = 1, 2, 3, \cdots)$,

$$(-1)^{\nu+1} I_{k-1}(\omega) \geq 0. \tag{7}$$

The class k functions have the interesting property that they are related to the positive real functions in the region where they are defined (that is, in the right half s plane and on the imaginary axis).

Theorem 1. *If the system function* $Z(s)$ *is a class k function, then* $(-j)^{m-n-1} Z_{m-n-1}(s)$ *is a positive real function for* $\sigma \geq 0$ *and the constant multiplier K is positive.*

Proof. The inequalities (6) and (7) state that the real part of $(-j)^{m-n-1} Z_{m-n-1}(j\omega)$ is non-negative for all ω. Moreover $Z_{m-n-1}(s)$ is analytic for $\sigma \geq 0$ since $Z(s)$ is analytic in this region. Thus by the minimax theorem, the real part of $(-j)^{m-n-1} Z_{m-n-1}(s)$ is non-negative for $\sigma \geq 0$. Therefore to prove the first part of the theorem it need only be shown that $(-j)^{m-n-1} Z_{m-n-1}(s)$ is real for $s = \sigma \geq 0$. But

$$(-j)^{m-n-1} Z_{m-n-1}(\sigma) = (-1)^{m-n-1} \int_\infty^\sigma d\sigma_{m-n-2} \int_\infty^{\sigma_{m-n-2}} d\sigma_{m-n-3} \cdots \int_\infty^{\sigma_1} Z(\sigma_0)\, d\sigma_0$$

and the right hand side of this expression is a real function of a real variable.

Finally the series expansion of $Z(s)$ given by (5) may be integrated term by term q times according to (2) for $q \leq m - n - 1$. The resulting expression for $Z_q(s)$, which holds for $|s| \geq M$, is

$$Z_q(s) = j^q \sum_{\mu=m-n}^{\infty} \frac{K_\mu}{(\mu - 1)(\mu - 2) \cdots (\mu - q)} \cdot \frac{1}{s^{\mu-q}}. \tag{8}$$

Thus the first term of the inverse power series expansion of $(-j)^{m-n-1} Z_{m-n-1}(s)$ is

$$\frac{K_{m-n}}{(m - n - 1)!s},$$

where $K_{m-n} = K$. Moreover for s real and sufficiently large, this first term becomes the dominant term. Therefore K must be positive. This completes the proof.

A property of the functions $Z_q(s)$ which will be used subsequently is the fact that their real and imaginary parts are either odd or even. This is stated by Lemma 2. To prove this however, Lemma 1 will be needed.

Lemma 1. *Let $h(\omega)$ be even (or odd) and integrable for $-\infty < \omega < +\infty$; let $q(\omega) = \int_{-\infty}^{\omega} h(u)\,du$; and let $q(\infty) = 0$. Then $q(\omega)$ is odd (or even).*

Proof.

$$q(\infty) = \int_{-\infty}^{\omega} h(u)\,du + \int_{\omega}^{\infty} h(u)\,du = 0.$$

Therefore,

$$q(\omega) = \int_{-\infty}^{\omega} h(u)\,du = \int_{\infty}^{\omega} h(u)\,du.$$

Replacing u in the last integral by $-v$,

$$q(\omega) = -\int_{-\infty}^{-\omega} h(-v)\,dv.$$

But for $h(\omega)$ even, $h(-v)$ equals $h(v)$ so that $q(\omega)$ equals $-q(-\omega)$. (For $h(\omega)$ odd, $h(-v)$ equals $-h(v)$ so that $q(\omega)$ equals $q(-\omega)$.)

Lemma 2. *For q even and less than $m - n$, the real and imaginary parts of $Z_q(s)$ are even and odd, respectively; for q odd and less than $m - n$, the real and imaginary parts of $Z_q(s)$ are odd and even, respectively.*

Proof. Consider a closed path of integration in the right half s plane for the integral $\oint Z_{q-1}(s)\,ds$ which is composed of a straight line segment parallel to the imaginary axis and to the right of it by the distance $c \geq 0$ and a segment of a circle C_1 whose center is at the origin.

$$\oint Z_{q-1}(s)\,ds = \int_{c-jd}^{c+jd} Z_{q-1}(s)\,ds + \int_{C_1} Z_{q-1}(s)\,ds.$$

This integral is zero since $Z_{q-1}(s)$ is analytic for $\sigma \geq 0$. Moreover as the distance from the origin to the circular segment increases without limit, the integral along this circular segment will vanish so long as $q \leq m - n - 1$, for then $Z_{q-1}(s)$ is $o(1/s)$ as $s \to \infty$. Thus for $c \geq 0$,

$$\int_{c-j\infty}^{c+j\infty} Z_{q-1}(s)\,ds = 0.$$

Hence the real and imaginary parts of $Z(s)$ satisfy the hypothesis of Lemma 1. Therefore the real and imaginary parts of $Z_1(s)$ are odd and even, respectively. Furthermore they also satisfy the hypothesis of Lemma 1 so long as $m - n$ is greater than 2. This application of Lemma 1 may be made $m - n - 1$ times to obtain Lemma 2.

While the first theorem of Part II applies to all class k functions, the other theorems and corollaries hold only for certain class k functions. In particular, Theorems 10 and 11 have been proved only for those functions in class $m - n$ whose real or imaginary parts, given by the left hand side of either (6) or (7), are not only positive but are, moreover, monotonic decreasing for positive ω. This condition is stated by the following expressions where $k = m - n$. When $k = 2\nu + 1$ ($\nu = 0, 1, 2, \cdots$),

$$(-1)^{\nu-1} R_{k-2}(\omega) \geq 0 \quad \text{for} \quad \omega \geq 0 \tag{9}$$

and, when $k = 2\nu$ ($\nu = 1, 2, \cdots$),

$$(-1)^{\nu} I_{k-2}(\omega) \geq 0 \quad \text{for} \quad \omega \geq 0. \tag{10}$$

Actually, any system function of the form of Eq. (1) which satisfies one of these inequalities will be automatically a class k function. For the function given by the left hand side of (9) or (10) is an odd function so that one more integration according to (3) or (4) will yield one of the inequalities of (6) or (7).

Finally, it should be noted that a slightly different definition of the class k functions leads to a simpler form for Theorem 1. If the factor $(-j)^q$ in the right hand side of Eq. (2) is replaced by $(-1)^q$, the class k functions may be defined by the condition that the real part of $Z_{k-1}(j\omega)$ is non-negative for all ω. It may then be shown in this case that $Z_{k-1}(s)$ is a positive real function. However, the proofs of some of the theorems are simpler if the former definition is used.

The subclass k functions. The question of determining whether a particular $Z(s)$ is a member of any class without performing the required integrations remains. Several tests have been devised which are applicable to certain functions in a given class but not to all. These functions comprise the subclass k.

Definition. $Z(s)$ will be called a subclass k function, where $k = m - n$, if, for k odd, $R(\omega)$ has $k - 1$ changes of sign for $-\infty < \omega < +\infty$ and $R(0)$ is positive and, for k even, $I(\omega)$ has $k - 1$ changes of sign for $-\infty < \omega < \infty$ and $dI/d\omega$ at $\omega = 0$ is negative.

If a system function satisfies the conditions of the second definition then it will also satisfy those of the first definition and one of the inequalities of (9) or (10). Therefore, all the results of part II hold for all subclass k functions.

Theorem 2. All members of subclass k are members of class k. Moreover all subclass k functions satisfy one of the inequalities given by (9) *or* (10).

Proof. First consider the case where $m - n = 2\nu + 1$ ($\nu = 0, 1, 2, \cdots$). The function $R_q(\omega)$ must have a smaller number of changes of sign in the interval $-\infty < \omega \leq \omega_i$, where ω_i is any zero of $R_{q-1}(\omega)$, than does $R_{q-1}(\omega)$. Moreover, invoking Lemma 2, it may be seen that $R_q(\omega)$ is odd or even when $R_{q-1}(\omega)$ is even or odd, respectively, so that $R_q(\omega)$ has less changes of sign in the interval $\omega_i \leq \omega < \infty$ than does $R_{q-1}(\omega)$. Thus each integration removes at least one change of sign from the finite ω axis. Moreover $R(\omega)$ has $m - n - 1$ changes of sign and $R_{m-n-1}(\infty)$ equals zero, so that each integration must remove only one change of sign. (Otherwise $R_{m-n-1}(\infty)$ would not equal zero.) Therefore $R_{m-n-2}(\omega)$ has only one change of sign which is at the origin and $R_{m-n-1}(\omega)$ has none. Since $R(0)$ is positive, $(-1)^{\nu-1} R_{m-n-2}(\omega) \geq 0$ for $0 \leq \omega < \infty$ and $(-1)^{\nu} R_{m-n-1}(\omega) > 0$ for $-\infty < \omega < +\infty$. This proves the theorem when $m - n$ is odd.

The same argument given in the preceding paragraph may be applied, when $m - n = 2\nu$ ($\nu = 1, 2, 3, \cdots$), to obtain the remaining portion of this theorem. In this case, the fact that dI/dt at $t = 0$ is a negative quantity implies that $(-1)^{\nu} I_{m-n-2}(\omega) \geq 0$ for $0 \leq \omega < \infty$ and that $(-1)^{\nu+1} I_{2\nu-1}(\omega) > 0$ for all finite ω. This completes the proof.

The argument given in this proof yields a lower bound on the number of sign changes in the real or imaginary parts of a system function for real frequencies even though it may not be a class k function. This fact will be needed subsequently.

Lemma 3. Any system function having the form of Eq. (1) *must have at least $m - n - 1$ changes of sign for $R(\omega)$ if $m - n$ is odd and for $I(\omega)$ if $m - n$ is even.*

Proof. Otherwise $R_{m-n-1}(\infty)$ and $I_{m-n-1}(\infty)$ would not be equal to zero as they must.

A system function having a positive constant multiplier and no zeros in the finite complex frequency plane (that is, a function which is a reciprocal of a Hurwitz polynomial) will automatically be a subclass k function as stated by the next theorem. An example of a network whose transfer function is of this type is the RC ladder network which is considered by a number of authors [6–9]. The same form of transfer function holds for more generalized ladder networks [10].

Theorem 3. *Let the system function $Z(s)$ have the following form,*

$$Z(s) = \frac{K}{s^m + b_{m-1}s^{m-1} + \cdots + b_0} = \frac{K}{D(s)},$$

where $D(s)$ is a Hurwitz polynomial and K is positive. Then $Z(s)$ is a subclass m function.

Proof. The proof depends upon a known property of Hurwitz polynomials, namely, that all the zeros of the even and odd parts of a Hurwitz polynomial of mth degree are simple and purely imaginary [11]. Thus, if m is odd, $R(\omega)$ has $m - 1$ real, simple zeros and, if m is even, $I(\omega)$ has $m - 1$ real, simple zeros. Furthermore, the series expansion of $Z(s)$ is

$$Z(s) = \frac{K}{b_0} - \frac{Kb_1}{b_0}s + \cdots.$$

Since $D(s)$ is a Hurwitz polynomial, all its coefficients are positive [12]. Therefore,

$$R(0) = \frac{K}{b_0} > 0$$

$$\left.\frac{dI}{d\omega}\right]_{\omega=0} = -\frac{Kb_1}{b_0} < 0.$$

Hence, all the conditions of the definition of a subclass m function are satisfied.

Tests for a subclass k function. Two tests have been devised which may be used to determine whether a given system function is a subclass k function. These tests determine the number of real zeros in the real or imaginary parts of the system function whose expression for real frequencies is given by (11).

$$Z(j\omega) = K\frac{N(j\omega)D(-j\omega)}{|D(j\omega)|^2}. \tag{11}$$

Since all the roots of $D(s)$ have negative real parts, $|D(j\omega)|^2$ is positive and finite for all finite ω. Thus, the zeros of the real and imaginary parts of $Z(j\omega)$ are the zeros of the real and imaginary parts of $N(j\omega)\ D(-j\omega)$. Let $P(\omega^2)$ be the even part of $N(j\omega)$ $D(-j\omega)$ and let $\omega Q(\omega^2)$ be its odd part. Replacing the variable ω^2 by x, the following test may be stated which is based on Descartes' rule of signs [13].

Theorem 4. *A system function $Z(s)$ is a subclass $m - n$ function if the following conditions hold. For $m - n$ odd, the number of sign changes in the coefficients of $P(x)$ is $(m - n - 1)/2$ and $P(0)$ is positive. For $m - n$ even, the number of sign changes in the coefficients of $Q(x)$ is $(m - n - 2)/2$ and $Q(0)$ is negative.*

Proof. First consider the case where $m - n$ is odd. The number of real roots of $R(\omega)$ equals twice the number of positive roots of $P(x)$. By Descartes' rule of signs [13] the number of positive roots of $P(x)$ is less than or equal to the number of variations of sign in the coefficients of $P(x)$. So by hypothesis, the number of real roots of $R(\omega)$ is less than or equal to $m - n - 1$. Moreover, by Lemma 3, the number of changes of sign for $R(\omega)$, which is less than or equal to the number of real roots of $R(\omega)$, is at least $m - n - 1$. Thus $W(\omega)$ has exactly $m - n - 1$ real simple roots and the conditions for the definition of a subclass $m - n$ function are fulfilled.

The proof for the case where $m - n$ is even is the same except that now the number of real roots of $I(\omega)$ equals twice the number of positive roots of $Q(x)$ plus one more.

This theorem may be applied to determine another general type of subclass k function having only one zero as given by Eq. (12).

$$Z(s) = K \frac{s + a_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_0} = K \frac{s + a_0}{D(s)}. \tag{12}$$

A condition that the coefficients of Hurwitz polynomials satisfy is

$$b_{m-1} > \frac{b_{m-3}}{b_{m-2}} > \frac{b_{m-5}}{b_{m-4}} > \cdots > 0. \tag{13}$$

The next to the last term in (13) is b_1/b_2 when m is even and b_0/b_1 when m is odd. (In those cases where $m = 3$ or $m = 4$, these inequalities are a consequence of the Hurwitz criterion [14]. Professor C. F. Rehberg has proven that they hold for Hurwitz polynomials of any degree so that the conditions (13) may be omitted from the hypothesis of the following theorem.)

Theorem 5. *If a system function* $Z(s)$, *having one zero, has the form of Eq.* (12), *if its coefficients satisfy the inequalities* (13) *and if the position of the zero satisfies* $b_{m-1} \geq a_0 > b_0/b_1$ *for m odd and* $b_{m-1} \geq a_0 > 0$ *for m even, then* $Z(s)$ *is a subclass* $m - 1$ *function.*

Proof. It will be shown that a system function which satisfies this hypothesis will also satisfy the hypothesis of Theorem 4. First consider the case where m is odd. Since $m - 1$ is even, $Q(x)$ is to be calculated and the number of changes of sign in its coefficients determined.

$$Q(x) = j^{m-1}\left[(b_{m-1} - a_0)x^{(m-1)/2} + (a_0 b_{m-2} - b_{m-3})x^{(m-3)/2} \right.$$
$$\left. + (b_{m-5} - a_0 b_{m-4})x^{(m-5)/2} + \cdots + \frac{b_0 - a_0 b_1}{j^{m-1}}\right].$$

Now it is evident that if the conditions (13) are satisfied and if $b_{m-1} \geq a_0 > b_0/b_1$ then the coefficients of $Q(x)$ have $(m - 3)/2$ changes in sign. But this number equals $(m - n - 2)/2$ for n equal to 1. Furthermore, $Q(0)$ is negative for $a_0 > b_0/b_1$. Thus the hypothesis of Theorem 4 is satisfied.

For the case where m is even, $P(x)$ may be found to be

$$P(x) = j^m\left[(a_0 - b_{m-1})x^{m/2} + (b_{m-3} - a_0 b_{m-2})x^{(m-2)/2} \right.$$
$$\left. + (a_0 b_{m-4} - b_{m-5})x^{(m-4)/2} + \cdots + \frac{a_0 b_0}{j^m}\right].$$

Again, if the conditions (13) are satisfied and if $b_{m-1} \geq a_0 > 0$ then the coefficients of $P(x)$ have $(m - 2)/2$ changes in sign which is $(m - n - 1)/2$ in number for n equal to 1. Furthermore, since $D(s)$ is a Hurwitz polynomial, b_0 is positive. Therefore, $P(0)$ is positive for $a_0 > 0$ and the hypothesis of Theorem 4 is again satisfied.

Descartes' rule of signs only yields an upper bound on the number of positive roots of a polynomial. Sturm's theorem [13] which determines exactly the number of positive roots of a polynomial (a multiple root being counted as a single root) is the basis of the following test.

Theorem 6. *Let* $f_0(x)$ *equal* $P(x)$ *for* $m - n$ *odd and equal* $Q(x)$ *for* $m - n$ *even. Let* $f_1(x)$ *equal* df_0/dx. *Let* $f_2(x)$ *be the negative of the remainder obtained by dividing* $f_0(x)$ *by* $f_1(x)$. *Let* $f_3(x)$ *be the negative of the remainder obtained by dividing* $f_1(x)$ *by* $f_2(x)$. *Let this process be continued as indicated below until the last remainder* $f_k(x)$ *is a constant or a polynomial which never changes sign.*

$$f_0(x) = q_1(x)f_1(x) - f_2(x)$$
$$f_1(x) = q_2(x)f_2(x) - f_3(x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_{k-2}(x) = q_{k-1}(x)f_{k-1}(x) - f_k(x).$$

Finally, let V_0 *be the number of sign changes in the sequence* $f_0(0), f_1(0), \cdots, f_k(0)$ *and let* V_∞ *be the number of sign changes in the sequence* $f_0(\infty), f_1(\infty), \cdots, f_k(\infty)$. *If, for* $m - n$ *odd,* $V_0 - V_\infty$ *equals* $(m - n - 1)/2$ *and* $P(0)$ *is positive or, for* $m - n$ *even,* $V_0 - V_\infty$ *equals* $(m - n - 2)/2$ *and* $Q(0)$ *is negative, then* $Z(s)$ *is a subclass* $m - n$ *function.*

Proof. The hypothesis of this theorem is simply a statement of Sturm's test. For instance, when $m - n$ is odd, the excess of V_0 over V_∞ is exactly the number of real positive roots of $P(x)$. When this equals $(m - n - 1)/2$, $R(\omega)$ will have $m - n - 1$ real zeros and the conditions for a subclass $m - n$ function will be satisfied. A similar situation exists when $m - n$ is even.

The phase functions of subclass k functions. The phase angle of a subclass k function for real frequencies has several properties which will be developed in this section. This phase function $\varphi(\omega)$ is defined by

$$Z(j\omega) = | Z(j\omega) | \exp [j\varphi(\omega)]. \tag{14}$$

Of course, any multiple of 2π may be added to $\varphi(\omega)$ without affecting the value of $Z(j\omega)$. In order to deal with a unique phase function, the following convention will be adopted. Consider the factored form of the system function given by (15) where the poles are denoted by the symbols ρ_i and the zeros by the symbols μ_i.

$$Z(s) = K \frac{(s - \mu_1)(s - \mu_2) \cdots (s - \mu_n)}{(s - \rho_1)(s - \rho_2) \cdots (s - \rho_m)}. \tag{15}$$

It will be presumed that at any real frequency the phase angle of any factor in this expression, which is determined by the angle of the vector extending from the pole or zero to the particular frequency in question on the imaginary axis, remains within the bounds $3\pi/2$ and $-\pi/2$. Thus as ω increases from $-\infty$ to $+\infty$, the phase angle for a factor of a pole or zero in the left half s plane will increase from $-\pi/2$ to $\pi/2$ whereas this angle for a pole or zero in the right half s plane will decrease from $3\pi/2$ to $\pi/2$. When the pole or zero occurs on the imaginary axis, the phase angle for its factor will be $-\pi/2$ when ω is smaller than the pole or zero and $\pi/2$ when ω is larger. Such angles are illustrated in Fig. 1 where the symbol ψ_i denotes the phase angle for the factor of a zero and the symbol θ_i denotes this angle for a pole. As was stated previously in Theorem

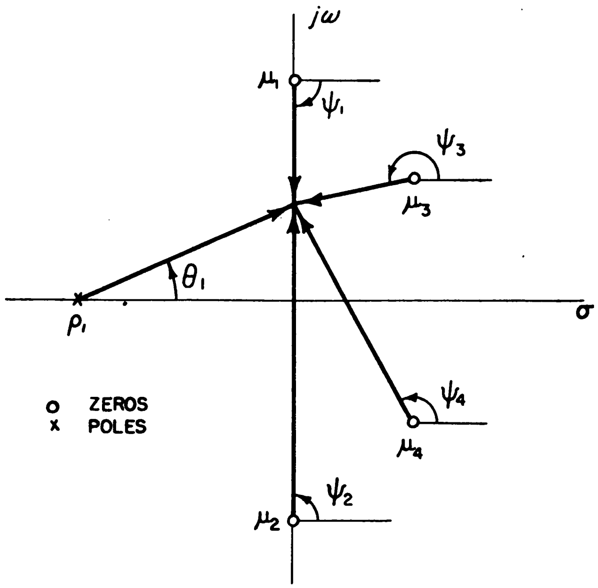

FIG. 1. Illustration of the Convention for Measuring Phase Angles.

1 the constant multiplier K must be a positive number if $Z(s)$ is a class k function. Thus by this convention the phase angle for $Z(j\omega)$ is unique and it is given by

$$\varphi(\omega) = \sum_{i=1}^{n} \psi_i - \sum_{i=1}^{m} \theta_i . \tag{16}$$

Another test for the subclass k functions may be constructed using the phase function $\varphi(\omega)$. It also determines the number of real zeros that $R(\omega)$ and $I(\omega)$ possess.

Theorem 7. If the system function $Z(s)$ is given by Eq. (1), if its phase function at real frequencies $\varphi(\omega)$ is continuous, if $d\varphi/d\omega < 0$, and if $-(m - n)\pi/2 < \varphi(\omega) < (m - n)\pi/2$ for $-\infty < \omega < \infty$, then $Z(s)$ is a subclass $m - n$ function.

Proof. Whenever the phase function $\varphi(\omega)$ equals an odd multiple of $\pi/2$, the real part of the system function at real frequencies $R(\omega)$ equals zero and, whenever $\varphi(\omega)$ equals zero or a multiple of π, the imaginary part $I(\omega)$ equals zero. Thus, under the conditions of the hypothesis, as ω increases continuously from $-\infty$ to $+\infty$, $\varphi(\omega)$ will decrease continuously from $+(m - n)\pi/2$ to $-(m - n)\pi/2$ and the number of times $R(\omega)$ or $I(\omega)$ changes sign may be determined by counting the number of times $\varphi(\omega)$ passes through an odd multiple of $\pi/2$ or a multiple of π. When $m - n$ is odd, $R(\omega)$ has $m - n - 1$ changes of sign and when $m - n$ is even, $I(\omega)$ has $m - n - 1$ changes of sign.

Furthermore, the denominator of $Z(s)$ is a Hurwitz polynomial so that its magnitude at finite real frequencies is always finite and positive. Moreover $Z(s)$ cannot have any zeros on the imaginary axis for otherwise the phase function $\varphi(\omega)$ would have a discontinuity at each such zero. Thus the magnitude of $Z(j\omega)$ is always finite and non-zero for finite ω. Since $\varphi(0)$ equals zero ($\varphi(\omega)$ being an odd function), $R(0)$ is positive. The fact that $d\varphi/d\omega < 0$ at $\omega = 0$ implies that $dI/d\omega$ is negative at $\omega = 0$.

Therefore, all the conditions for the definition of a subclass $m - n$ function are satisfied.

The hypothesis of this theorem is really much too strong. The condition that $\varphi(\omega)$ is strictly monotonic decreasing may be replaced by the following conditions which encompass a much larger set of functions. *For $m - n$ odd, $\varphi(\omega)$ equals an odd multiple of $\pi/2$ exactly $m - n - 1$ times and, for $m - n$ even, $\varphi(\omega)$ equals a multiple of π or zero exactly $m - n - 1$ times; moreover $d\varphi/d\omega < 0$ at $\omega = 0$.* In this case the proof is practically the same.

It can easily be seen that this theorem may be used in an alternate proof of Theorem 3. Since all the poles of the system function which is the reciprocal of a Hurwitz polynomial are in the left half plane and since there are no zeros, the phase function $\varphi(\omega)$ is strictly monotonic decreasing and satisfies the hypothesis of Theorem 7.

Several properties held by the phase function at real frequencies $\varphi(\omega)$ of a subclass k function are stated by the following theorem.

Theorem 8. If the system function $Z(s)$, given by Eq. (1), is a subclass $m - n$ function, then it satisfies the following conditions.

i. *$Z(s)$ has no zeros in the right half s plane; that is, $Z(s)$ is a minimum phase function.*

ii. *$Z(s)$ has no zeros on the imaginary axis; that is, $\varphi(\omega)$ is a continuous function.*

iii. *$-(m - n)\pi/2 < \varphi(\omega) < (m - n)\pi/2$ for $-\infty < \omega < \infty$.*

Proof. i and ii. It is convenient to prove the first two statements of the conclusion together. Assume that there are p zeros in the right half s plane and q zeros on the imaginary axis so that the number of zeros in the left half s plane is $(n - p - q)$. All the poles are in the left half s plane. As ω increases from $-\infty$ to $+\infty$, the angle θ_i corresponding to the pole ρ_i will increase continuously from $-\pi/2$ to $\pi/2$. The angles ψ_i for any zeros in the left half s plane will behave similarly. However these angles for zeros in the right half s plane will decrease from $3\pi/2$ to $\pi/2$. Finally a zero on the imaginary axis will have an angle for its factor which is $\pm\pi/2$ at all ω except at the zero where there will be a discontinuity of magnitude π.

Now consider the phase functions $\varphi'(\omega)$ determined by all the poles and only those zeros which are off the imaginary axis. This function is continuous.

$$\varphi'(\omega) = \sum_{i=1}^{n-q} \psi_i - \sum_{i=1}^{m} \theta_i .$$

The number of times $R(\omega)$ or $I(\omega)$ changes sign must be at least as great as the number of times the function $\varphi'(\omega)$ varies continuously and monotonically through odd multiples of $\pi/2$ or multiples of π, respectively. For, the contribution to the phase function $\varphi(\omega)$ due to the zeros on the imaginary axis is a step function each of whose discontinuities is a multiple of π and such a contribution will only yield additional zeros to $R(\omega)$ and $I(\omega)$. The function $\varphi'(\omega)$ varies continuously over a range at least as large as $[m - (n - p - q) + p]\pi$. The only way for $R(\omega)$ or $I(\omega)$ to have only $m - n - 1$ changes of sign is for this range to be no greater than $(m - n)\pi$. Thus both p and q must be zero.

iii. By the above argument, all the poles and zeros of $Z(s)$ are in the left half s plane so that $\varphi(0)$ equals zero. Moreover it has been shown that the range of variation for $\varphi(\omega)$ can be no greater than $(m - n)\pi$. Thus $\varphi(\omega)$ is an odd function and it is bounded by $\pm(m - n)\pi/2$ for $-\infty < \omega < +\infty$.

Part II. Bounds on the impulse and step responses. As is well known, the unit impulse response $W(t)$ is related to the system function by the Fourier transform

$$W(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Z(j\omega) \exp(j\omega t)\, d\omega. \tag{17}$$

It will be presumed that all input functions are applied at $t = 0$ so that the response for any physically realizable system must be zero for negative values of time. This result may also be derived from the condition that $Z(s)$ has no poles in the right half s plane. Because of this, $W(t)$ may be represented either by Eqs. (18) or (19) for positive values of time.

$$W(t) = \frac{2}{\pi} \int_0^\infty R(\omega) \cos \omega t\, d\omega, \qquad t \geq 0 \tag{18}$$

$$W(t) = -\frac{2}{\pi} \int_0^\infty I(\omega) \sin \omega t\, d\omega, \qquad t > 0 \tag{19}$$

It has been shown previously that when the frequency responses of networks are restricted in various ways the transient responses are bounded [1–3]. Specifically, when the real part of a system function is of constant sign and the system function has one more pole than the number of its zeros, the magnitude of the corresponding unit impulse response is bounded by the constant multiplier K of the system function. Similarly when the imaginary part of a system function with two more poles than zeros is of constant sign for positive frequencies, then the magnitude of the unit impulse response is bounded by Kt. The restriction on the frequency characteristic of a class k system function leads to a generalization of these two conclusions. This generalization, which is given by the following theorem, states that the magnitude of the unit impulse response is bounded by $Kt^{m-n-1}/(m - n - 1)!$, the symbols being defined in Eq. (1). (By the initial value theorem, the upper bound is seen to be the initial value of the unit impulse response.) It follows that the magnitude of the unit step response will be bounded by $Kt^{m-n}/(m - n)!$ so that the rise time from zero to one will always be greater than $[(m - n)!r/K]^{1/(m-n)}$ where r equals Ka_0/b_0 (presuming that $a_0 > 0$).

Theorem 9. *If the system function $Z(s)$, given by Eq.* (1), *is a class $m - n$ function, then the corresponding unit impulse response $W(t)$ is bounded by the following expression for $t \geq 0$.*

$$| W(t) | \leq \frac{Kt^{m-n-1}}{(m - n - 1)!}. \tag{20}$$

Proof. For $m - n = 2\nu + 1$ ($\nu = 0, 1, 2, \cdots$), upon integrating by parts expression (18) 2ν times and using the fact that $R_q(\infty) = 0$ for $q < m - n$, the following may be obtained

$$W(t) = (-1)^\nu t^{2\nu} \frac{2}{\pi} \int_0^\infty R_{2\nu}(\omega) \cos \omega t\, d\omega. \tag{21}$$

But by definition of a class $m - n$ function $(-1)^\nu R_{2\nu}(\omega) \geq 0$ for all ω and so

$$| W(t) | \leq (-1)^\nu t^{2\nu} \frac{2}{\pi} \int_0^\infty R_{2\nu}(\omega)\, d\omega.$$

By integrating $Z_{2\nu}(s)$ around the $\sigma > 0$ half plane, the integral, $\int_0^\infty R_{2\nu}(\omega)\, d\omega$, is found to equal $(-1)^\nu K\pi/2(2\nu)!$. Thus the conclusion of this theorem is obtained for the case where $m - n$ is odd.

For $m - n = 2\nu(\nu = 1, 2, 3, \cdots)$, a similar argument may be applied to (19). Integration by parts $2\nu - 1$ times yields

$$W(t) = (-1)^{\nu+1} t^{2\nu-1} \frac{2}{\pi} \int_0^\infty I_{2\nu-1}(\omega) \cos \omega t \, d\omega. \tag{22}$$

Again by a definition of a class $m - n$ function, $(-1)^{\nu+1} I_{2\nu-1}(\omega) \geq 0$ for all ω so that

$$| W(t) | \leq (-1)^{\nu+1} t^{2\nu-1} \frac{2}{\pi} \int_0^\infty I_{2\nu-1}(\omega) \, d\omega.$$

Integrating $Z_{2\nu-1}(s)$ around the $\sigma > 0$ half plane, the integral on the right hand side of the last expression is found to be equal to $(-1)^{\nu+1} K\pi/2(2\nu - 1)!$. This yields the conclusion once more.

The next two theorems depend upon two inequalities that the sine function satisfies and which were proved in a previous paper [3].

Lemma 4. *For* $0 \leq y < 1$, $x \geq 0$ *and* N *a positive integer,*

$$\sin x \leq Q_0 x + \sum_{p=1}^{N} \frac{y Q_{2p}}{p} \sin \frac{px}{y} \tag{23}$$

and

$$\sin x \geq -Q_0 x + \sum_{p=1}^{N} (-1)^{p+1} \frac{y Q_{2p}}{p} \sin \frac{px}{y}, \tag{24}$$

where

$$Q_0 = 1 + \sum_{k=1}^{N} \frac{(-y^2)(1^2 - y^2) \cdots [(k-1)^2 - y^2]}{(k!)^2}, \tag{25}$$

$$Q_{2p} = (-1)^p 2 \sum_{k=p}^{N} \frac{(-y^2)(1^2 - y^2) \cdots [(k-1)^2 - y^2]}{(k-p)!(k+p)!}, \qquad p = 1, 2, \cdots, N. \tag{26}$$

Any system function which satisfies the inequality (9) or the inequality (10) (as has been noted before, all subclass k functions are of this type) will have the property that, when its corresponding impulse response is bounded beyond a certain time, then other bounds on this response will be determined before this time. The physical significance of this is that the more rapidly an impulse response "settles down" the less violent must this response be. This result is given by the next theorem which is a generalization of Theorems 1 and 2 in [15]. More precisely, the conclusions of [15] are special cases of the following obtained by setting $m - n$ equal to one or two and making some trivial changes in the notation and normalization.

Theorem 10. *Let the system function* $Z(s)$, *given by Eq.* (1), *satisfy the inequality* (9) *if* $m - n$ *is odd or the inequality* (10) *if* $m - n$ *is even. If the magnitude of the corresponding unit impulse response* $W(t)$ *is less than or equal to* M *for* $t \geq \tau$ *where* M *is a positive number, then for* $0 \leq y < 1$,

$$| W(y\tau) | \leq \frac{K \sin \pi y}{(m - n - 1)! \pi y} (y\tau)^{m-n-1} + \frac{2My^{m-n} \sin \pi y}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{m-n-1}(\nu^2 - y^2)}. \tag{27}$$

Proof. First consider the case where $m - n = 2\nu + 1$ ($\nu = 0, 1, 2, \cdots$). Integrating (18) by parts $2\nu - 1$ times, (28) is obtained.

$$W(t) = (-1)^{\nu-1} t^{2\nu-1} \frac{2}{\pi} \int_0^\infty R_{2\nu-1}(\omega) \sin \omega t \, d\omega. \tag{28}$$

By hypothesis, $(-1)^{\nu-1} R_{2\nu-1}(\omega)$ is non-negative. Therefore, setting x equal to ωt, the sine function may be replaced by the right hand side of (23) and the result integrated term by term to yield

$$W(t) \leq (-1)^{\nu-1} t^{2\nu} Q_0 \frac{2}{\pi} \int_0^\infty \omega R_{2\nu-1}(\omega) \, d\omega + \sum_{p=1}^{N} \left(\frac{y}{p}\right)^{2\nu} Q_{2p} W\left(\frac{pt}{y}\right).$$

Moreover,

$$\int_0^\infty \omega R_{2\nu-1}(\omega) \, d\omega = -\int_0^\infty R_{2\nu}(\omega) \, d\omega.$$

The value for this last integral may be found by integrating $Z_{2\nu}(s)$ around the $\sigma > 0$ half plane

$$-\int_0^\infty R_{2\nu}(\omega) \, d\omega = \frac{(-1)^{\nu+1} \pi K}{2(2\nu)!}.$$

Therefore,

$$W(t) \leq \frac{K Q_0 t^{2\nu}}{(2\nu)!} + \sum_{p=1}^{N} \left(\frac{y}{p}\right)^{2\nu} Q_{2p} W\left(\frac{pt}{y}\right). \tag{29}$$

Furthermore, the $W(t)$ are bounded for all t since

$$| W(t) | \leq \frac{2}{\pi} \int_0^\infty | R(\omega) | \, d\omega < \infty .$$

Thus it may be found that the double series obtained by letting N go to infinity in the right hand side of (29) converges absolutely for $0 \leq y < 1$. So summing over the k in Eqs. (25) and (26), the following may be obtained where the Q_{2p} converge to the q_{2p} as shown in [16].

$$W(t) \leq \frac{K q_0 t^{2\nu}}{(2\nu)!} + \sum_{p=1}^{\infty} \left(\frac{y}{p}\right)^{2\nu} q_{2p} W\left(\frac{pt}{y}\right), \tag{30}$$

where

$$q_0 = \frac{\sin \pi y}{\pi y}$$

$$q_{2p} = (-1)^{p+1} \frac{2y \sin \pi y}{\pi(p^2 - y^2)}.$$

Now if t/y is set equal to τ and $W(pt/y)$ is replaced by M for p odd and by $-M$ for p even, the upper bound of (27) is achieved. The lower bound is similarly obtained by use of (24) rather than (23).

The conclusion of this theorem may also be achieved in the case where $m - n = 2\nu$ ($\nu = 1, 2, 3, \cdots$) by integrating (19) by parts $2\nu - 2$ times and then proceeding in the same way.

Finally bounds which are similar to those of Theorem 10 exist on the step responses of those system functions which satisfy either inequality (9) or (10) and whose constant a_0 is greater than zero. Once more, it may be noted that all subclass k functions are of this type. The unit step response $A(t)$ is related to the unit impulse response $W(t)$ by the expression,

$$A(t) = \int_0^t W(\xi)\, d\xi. \tag{31}$$

The following theorem is once again a generalization of Theorem 3 of [15].

Theorem 11. *Let the system function* $Z(s)$, *given by Eq.* (1), *satisfy the inequality* (9) *if* $m - n$ *is odd or the inequality* (10) *if* $m - n$ *is even and let* a_0 *be positive. If the corresponding unit step response* $A(t)$ *is bounded by* $(1 \pm \gamma)r$ *for* $t \geq \tau$ *where* $r = Ka_0/b_0 > 0$ *and* γ *is a positive number, then, for* $0 \leq y < 1$,

$$A(y\tau) \leq \frac{y^{m-n-1} \sin \pi y}{\pi} \left\{ \frac{K\tau^{m-n}}{(m-n)!} + 2y^2 r \left[\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{\nu^{m-n}(\nu^2 - y^2)} + \gamma \sum_{\nu=1}^{\infty} \frac{1}{\nu^{m-n}(\nu^2 - y^2)} \right] \right\} \tag{32}$$

and

$$A(y\tau) \geq -\frac{y^{m-n-1} \sin \pi y}{\pi} \left\{ \frac{K\tau^{m-n}}{(m-n)!} - 2y^2 r(1-\gamma) \sum_{\nu=1}^{\infty} \frac{1}{\nu^{m-n}(\nu^2 - y^2)} \right\}. \tag{33}$$

Proof. To obtain the upper bound (32) for both cases where $m - n$ is odd and $m - n$ is even, the proof proceeds in the same way as in the preceding theorem until the inequality given by (29) is obtained where 2ν is replaced by $m - n - 1$. Integrating according to (31),

$$A(t) \leq \frac{KQ_0 t^{m-n}}{(m-n)!} + \sum_{p=1}^{N} \left(\frac{y}{p}\right)^{m-n} Q_{2p} A\left(\frac{pt}{y}\right). \tag{34}$$

However,

$$|\, A(t) \,| \leq \frac{2t}{\pi} \int_0^{\infty} |\, R(\omega) \,|\, d\omega \quad \text{for} \quad t \geq 0.$$

Thus $A(t)$ is bounded for $0 \leq t < \infty$ and so it may be seen that the double series obtained by letting N go to infinity in the right hand side of (34) converges absolutely. Upon letting N go to infinity and summing over the k in Eqs. (25) and (26), the following may be obtained where the q_{2p} are given below expression (30)

$$A(t) \leq \frac{Kq_0 t^{m-n}}{(m-n)!} + \sum_{p=1}^{N} \left(\frac{y}{p}\right)^{m-n} q_{2p} A\left(\frac{pt}{y}\right). \tag{35}$$

Now setting t/y equal to τ and replacing $A(pt/y)$ by $(1 + \gamma)r$ for p odd and by $(1 - \gamma)r$ for p even, expression (32) is achieved.

Expression (33) may be derived in a similar fashion using (24). In this case, the expression corresponding to (34) is the following

$$A(t) \geq -\frac{KQ_0 t^{m-n}}{(m-n)!} + \sum_{p=1}^{N} (-1)^{p+1} \left(\frac{y}{p}\right)^{m-n} Q_{2p} A\left(\frac{pt}{y}\right).$$

As N goes to infinity the double series converges absolutely again. Also the $A(pt/y)$ must be replaced by $(1 - \gamma)r$ for all p to maintain the inequality. This completes the proof.

For $m - n$ equal to one or two, some of the infinite series given in (27), (32) and (33) may be expressed in closed form as follows.

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2 - y^2} = \frac{1}{2y^2} - \frac{\pi}{2y} \cot \pi y, \qquad 0 \leq y < 1$$

$$2y^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu^2 - y^2)} = -[\Psi(y) + \Psi(-y) + 2g], \qquad 0 < y < 1.$$

Here $\Psi(y)$ is the digamma function [17] defined by

$$\Psi(y) = \frac{d}{dy} \log \Gamma(y)$$

and g is Euler's constant defined by

$$g = \lim_{n \to 0} \left(\sum_{\nu=1}^{n} \frac{1}{\nu} - \log n \right) = .5772 \cdots.$$

The results of Theorems 10 and 11 and the following corollaries 11a and 11b are not the best possible and can be improved. For it was presumed in the proofs that $W(p\tau)$ and $A(p\tau)$ equal their bounds $\pm M$ and $(1 \pm \gamma)r$ for all positive integers p. This behavior is impossible since $W(\infty)$ equals zero and $A(\infty)$ equals r.

Restrictions on the rise time and settling time. Theorem 11 yields restrictions on the shape factors of the unit step response which are again generalizations of results that have appeared previously [3]. Defining the rise time T_γ as the time it takes for the unit step response to first cross the final value line after the input has been impressed (see Fig. 2), it is found that a lower bound exists on this rise time which becomes larger

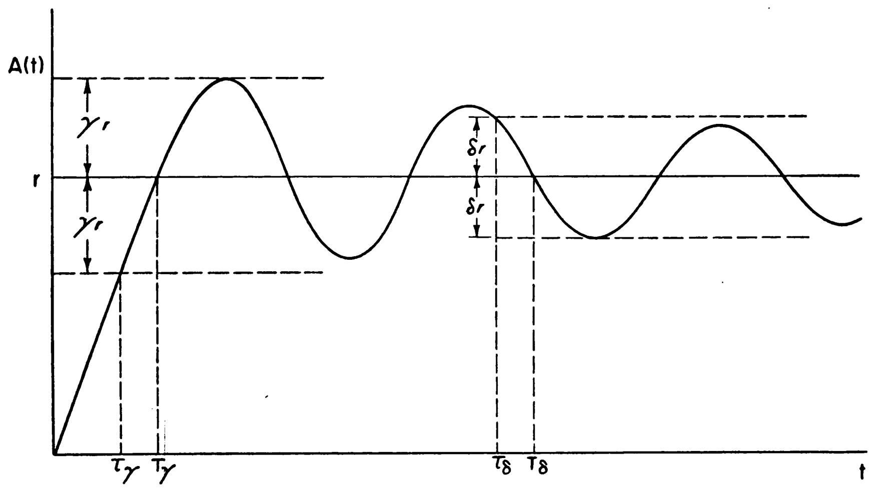

FIG. 2. Illustration of the Shape Factors for the Unit Step Response.

as the maximum overshoot or undershoot γ becomes smaller. That is, the rise time and maximum overshoot or undershoot of the unit step response define a point on the plane of Fig. 3 and this point must lie above the curve for the appropriate $m - n$. Furthermore, let T_δ be the least time at which the unit step response crosses the final value line and beyond which the overshoots and undershoots are less than or equal to δ. (Some of the overshoots and undershoots may be greater than δ before T_δ). In this case the curves of Fig. 3 still apply giving lower bounds on this shape factor T_δ . These results may be stated as follows.

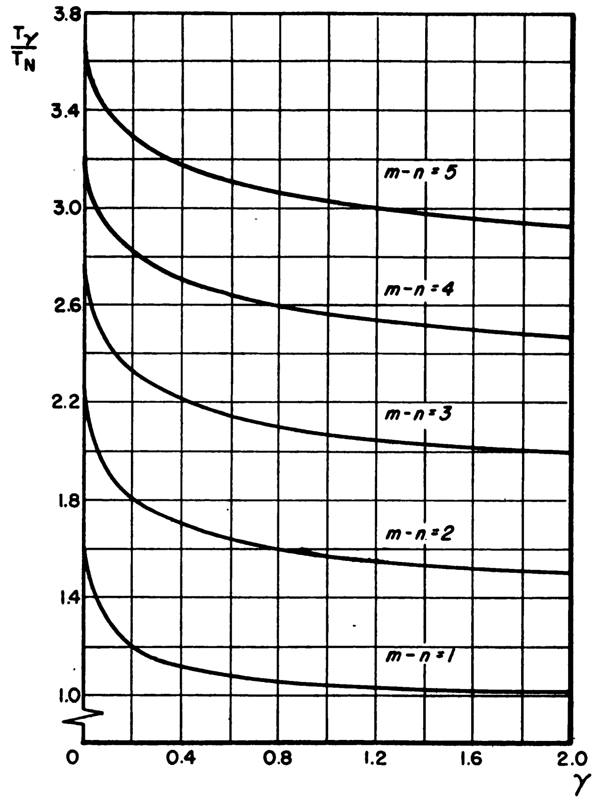

FIG. 3. The Envelopes of the Lower Bounds on the Rise Time of tho Unit Step Response. Given by Corollary 11a.

Corollary 11a. *Let the system function* $Z(s)$, *given by Eq.* (1), *satisfy the inequality* (9) *if* $m - n$ *is odd or the inequality* (10) *if* $m - n$ *is even and let* a_0 *be positive. If the corresponding unit step response* $A(t)$ *is bounded by* $(1 \pm \gamma)r$ *for* $t \geq T_\gamma$ *where* $r = Ka_0/b_0$, γ *is a positive number, and* $A(T_\gamma) = r$, *then*

$$T_\gamma > \left\{\frac{(m-n)!r}{K}\left[\frac{\pi y}{\sin \pi y} - 2y^{m-n+2}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu^{m-n}(\nu^2 - y^2)} - 2\gamma y^{m-n+2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^{m-n}(\nu^2 - y^2)}\right]\right\}^{1/(m-n)}, \tag{36}$$

where $0 \leq y < 1$.

This corollary follows immediately from expression (32) if yT is set equal to T_γ so that $A(yT)$ equals r. Since $A(\infty) = r$, equality between the two sides of (36) is impossible.

For a given y and $m - n$, the right hand side of expression (36) is a function of γ. As the parameter y is varied between zero and one, a family of curves is generated on the plane of Fig. 3 and T_γ must be greater than the envelope of this family of curves.

These envelopes for several values of $m - n$ are shown in Fig. 3, where T_N is a normalization factor given by

$$T_N = \left(\frac{r}{K}\right)^{1/(m-n)} \tag{37}$$

As y approaches one, the envelopes approach a point on the ordinate axis given by

$$(m - n)!\left[\frac{2(m - n) + 1}{2} + 2 \sum_{\nu=2}^{\infty} \frac{(-1)^{\nu}}{\nu^{m-n}(\nu^2 - 1)}\right]. \tag{38}$$

As y approaches zero, the envelopes progress infinitely to the right approaching the horizontal line whose ordinate is $(m - n)!$.

A settling time τ_δ for the unit step response may be defined as the least time beyond which the unit step response remains within the bounds $(1 \pm \delta)r$. This is illustrated in Fig. 2. Again Theorem 11 implies a lower bound on τ_δ as shown in Fig. 4. For a given δ,

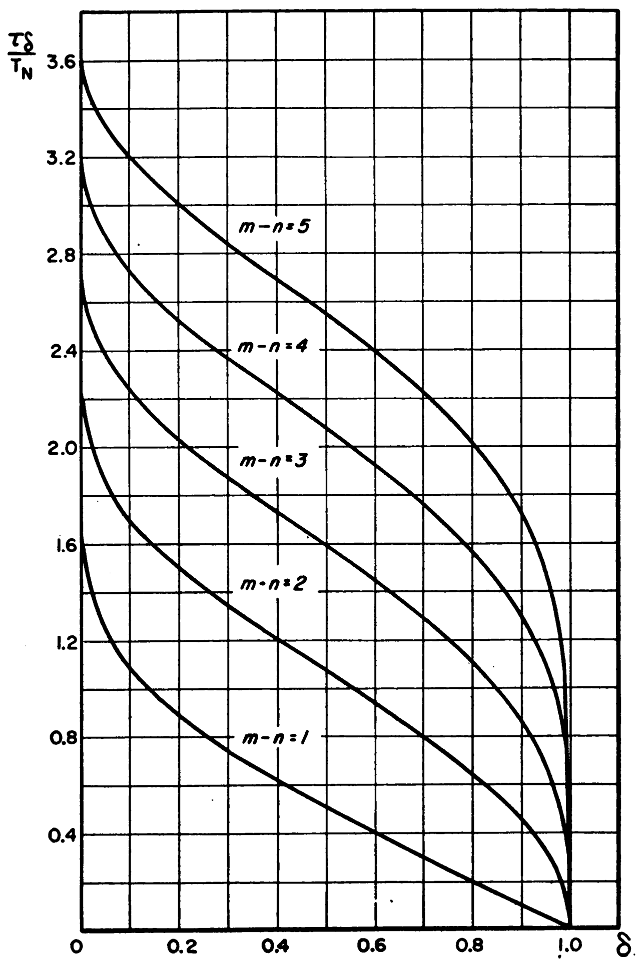

Fig. 4. The Envelopes of the Lower Bounds on the Settling Time of the Unit Step Response Given by Corollary 11b.

the settling time τ_δ must be greater than the curve corresponding to the appropriate $m - n$. These curves are an immediate consequence of corollary 11b.

Corollary 11b. *Let the system function* $Z(s)$, *given by Eq.* (1), *satisfy the inequality* (9) *if* $m - n$ *is odd or the inequality* (10) *if* $m - n$ *is even and let* a_0 *be positive. If the corresponding unit step response* $A(t)$ *is bounded by* $(1 \pm \delta)r$ *for* $t \geq \tau_\delta$ *where* $r = Ka_0/b_0$ *and* δ *is a positive number between zero and one, then*

$$\tau_\delta > \left\{\frac{(m-n)!r}{K}\left[\frac{\pi y}{\sin \pi y}(1-\delta) - 2y^{m-n+2}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu^{m-n}(\nu^2 - y^2)} - 2\delta y^{m-n+2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^{m-n}(\nu^2-y^2)}\right]\right\}^{1/(m-n)}, \tag{39}$$

where $0 \leq y < 1$.

Setting $y\tau$ equal to τ_δ, $A(y\tau)$ may be replaced by $(1 - \delta)r$, τ by τ_δ/y and γ by δ in expression (32). This will yield expression (39) upon rearrangement. Again $A(\infty) = r$ so that equality between the two sides of expression (39) cannot be achieved.

Once again, a family of curves is generated on the plane of Fig. 4 by the right hand side of expression (39) when the parameter y is varied between zero and one. The settling time τ_δ must be greater than the envelopes of these families of curves for various values of $m - n$. These envelopes are shown in Fig. 4. As y approaches one, the envelopes approach a point on the ordinate axis which is given by expression (38) and, as y approaches zero, they approach the value, one, on the abscissa axis.

APPENDIX I

Examples and applications. 1. As an illustration of the bounds holding on the impulse response of a system function which is the reciprocal of a Hurwitz polynomial, the following system function is considered.

$$Z(s) = \frac{1.241}{(s + .591)^2 + (.806)^2}.$$

The corresponding unit impulse response is

$$W(t) = 1.537e^{-.591t}\sin .806t$$

and this is plotted in Fig. 5. Since $Z(s)$ is the reciprocal of a Hurwitz polynomial of second degree, it is automatically a Subclass 2 function by Theorem 3. Thus all the bounds developed in Part II apply to its corresponding unit impulse and unit step responses. The bound, $1.241t$, on the magnitude of the unit impulse response given by Theorem 9 is readily seen to hold. Furthermore, as seen in Fig. 5, $|W(t)|$ is less than .063 for all t greater than 3.486. Thus by Theorem 10, $|W(y\tau)|$ is bounded in the interval $0 \leq t < 3.486$ and these bounds are also shown in Fig. 5. Of course other bounds on $|W(t)|$ could be obtained by choosing other possible values of M and τ.

2. An example of a Subclass 4 function is the following system function

$$Z(s) = \frac{(s + 1.25)^2 + .5625}{(s+1)(s+2)[(s+1)^2+1][(s+3)^2+9]}.$$

To show that this is indeed a Subclass 4 function, the function $Q(x)$, which is defined

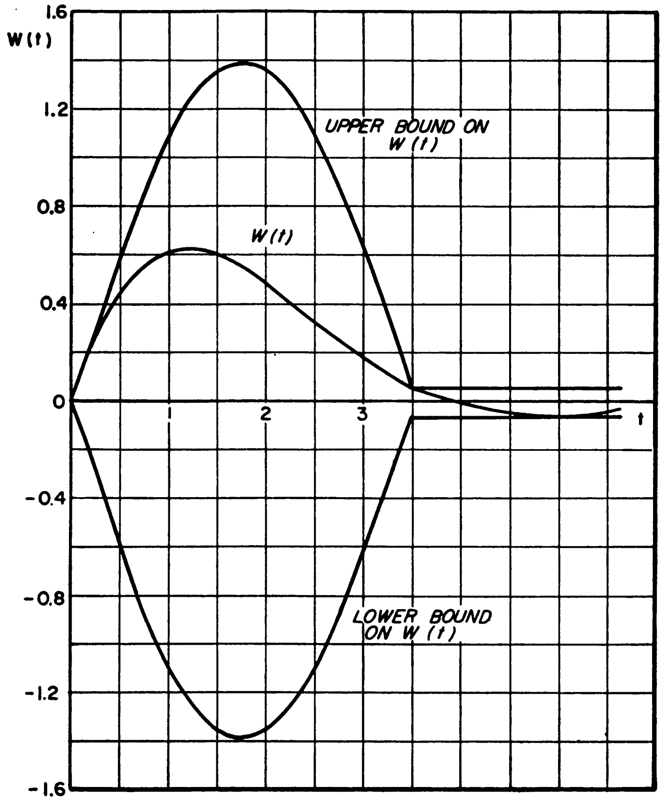

FIG. 5. Illustration of the Bounds of Theorem 10 in the Case Where $K = 1.241$, $M = 0.063$, $m - n = 2$, and $\tau = 3.486$.

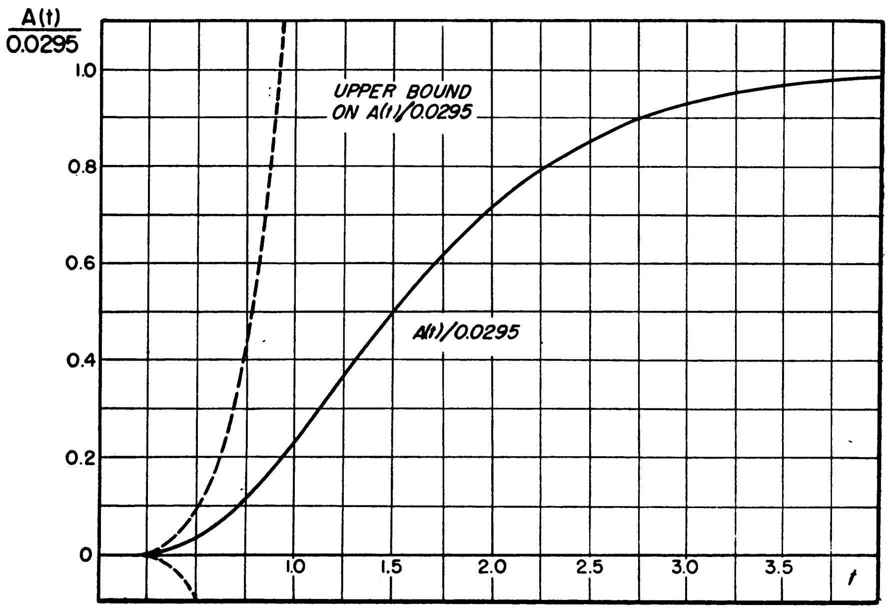

FIG. 6. Bound on the Unit Step Response Discussed in the Second Example.

immediately before the statement of Theorem 4, may be found to be

$$Q(x) = 8.500x^3 - 38.38x^2 - 66.00x - 253.5.$$

Since the coefficients of $Q(x)$ have only one change in sign and since $Q(0)$ is negative, $Z(s)$ is truly a Subclass 4 function by Theorem 4. Therefore all the bounds given in Part II hold on the transient responses corresponding to this $Z(s)$.

In particular, the unit step response is

$$A(t) = .0295 - .0481e^{-t} + .0281e^{-2t} + .0247e^{-t} \cos (t + 1.89) + .0064e^{-3t} \cos (3t + 4.43)$$

and this is plotted in Fig. 6. The bound, $t^4/1.413$, on the magnitude of $A(t)/r$, which is a consequence of Theorem 9, is shown by the dotted curves. Moreover pairs of values for the variables τ_δ and δ are seen to be well above the curve for $m - n = 4$ in Fig. 4. For instance, for t greater than 2.60, $A(t)/r$ is bounded by $(1 \pm .129)$. Moreover the normalization factor T_N has a value of .429 for this example. The point determined by $\tau_\delta/T_N = 6.06$ and $\delta = .129$ is above the appropriate curve in Fig. 4.

3. A network [18], which has satisfactory responses for appropriate choices of the network parameters is shown in Fig. 7. Both the series-shunt peaking network and the

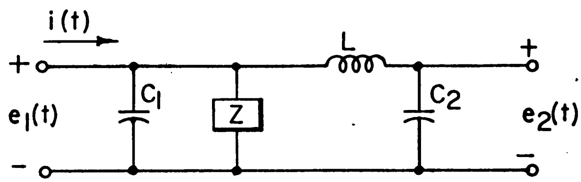

FIG. 7. A network whose transfer function of output voltage $E_2(s)$ divided by input current $I(s)$ is always a Subclass 3 function for all positive values of C_1, C_2, and L and all driving point impedances Z having a finite, non-zero value at DC.

Dietzold network have this form and their transient responses of the output voltage $e_2(t)$ as a function of the input current $i(t)$ are plotted in [18].

It is easily seen that if the driving point impedance $Z(s)$ has a finite non-zero value at DC, then the transfer function $Z_T(s)$ of output voltage $E_2(s)$ divided by input current $I(s)$ will always be a Subclass 3 function for all positive values of C_1, C_2, and L and for all permissible driving point impedances $Z(s)$. For, the natio of the output voltage to input voltage $E_1(s)$ is

$$\frac{E_2}{E_1} = \frac{1}{LC_2s^2 + 1}.$$

But since $E_1(s) = I(s)Z_D(s)$ where $Z_D(s)$ is the input impedance of this network,

$$Z_T(s) = \frac{E_2}{E_1} \cdot \frac{E_1}{I} = \frac{1}{LCs^2 + 1} Z_D(s).$$

Thus,

$$R_T(\omega) = \text{Re}\,[Z_T(j\omega)] = \frac{R_D(\omega)}{1 - LC\omega^2}.$$

Since $R_D(\omega)$ is the real part of the passive driving point impedance $Z_D(j\omega)$, it is non-negative for all ω. And so, $R_T(\omega)$ will have exactly two changes of sign. This means that $Z_T(s)$ is a Subclass 3 function provided that $R_D(0)$ is non-zero and finite.

An inspection of the unit step responses corresponding to $Z_T(s)$ for the special cases of the series-shunt peaking network and the Dietzold network [18] shows that these transient responses satisfy all the restrictions developed in Part II of this paper. It should be noted that these networks are suitable for video amplifiers where a small rise time from 10 to 90 per cent coupled with small overshoot is of importance. If a small initial time delay is also desired, then these networks would not be particularly useful since they have Subclass 3 transfer impedances and their step responses must have appreciable initial time delays.

APPENDIX II

List of symbols. Those symbols which have physical significance are defined as follows:

$A(t)$, the response to a unit step function applied at time, $t = 0$;
a_k, the coefficients in the numerator of a system function;
b_k, the coefficients in the denominator of a system function;
$D(s)$, the denominator of a system function;
$f_k(x)$, the Sturm functions defined in Theorem 6;
$I(\omega)$, the imaginary part of a system function for real frequencies;
$I_a(\omega)$, a function defined by Eq. (4);
K, the constant multiplier of a system function;
m, the degree of the denominator of a system function;
$N(s)$, the numerator of a system function;
n, the degree of the numerator of a system function;
$P(\omega^2)$, the even part of $N(j\omega)\, D(-j\omega)$;
$Q(\omega^2)$, the even part of $N(j\omega)\, D(-j\omega)/\omega$;
$R(\omega)$, the real part of a system function for real frequencies;
$R_a(\omega)$, a function defined by Eq. (3);
r, the resistance of a system function under DC conditions;
s, the complex frequency variable;
T_N, a time normalization factor defined by expression (37);
T_γ, the rise time from zero to one of the step response;
t, the time variable;
$W(t)$, the response to a unit impulse function applied at time, $t = 0$;
x, the square of angular velocity $= \omega^2$;
$Z(s)$, a system function;
$Z_a(s)$, a function defined by Eq. (2);
γ, the least upper bound on all the fractional overshoots and undershoots of the unit step response;
δ, the least upper bound on $|A(t) - r|/r$ for $t \geq \tau_\delta$;
θ_i, the phase angle for a pole factor defined in Fig. 1;
μ_i, a zero of a system function;
ρ_i, a pole of a system function;
σ, the real part of the complex frequency variable;
τ, any time beyond which the unit step response remains within the bounds $(1 \pm \gamma)r$;

τ_δ, the least time beyond which the unit step response remains within the bounds $(1 \pm \delta)r$ where $0 \leq \delta \leq 1$;
$\varphi(\omega)$, the phase angle of a system function defined by Eq. (14);
Ψ_i, the phase angle for a zero factor defined in Fig. 1;
ω, the imaginary part of the complex frequency variable, s.

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