Hence, in fully developed incompressible flows, there can be no transverse velocity components. The flow is then found from the second of Eqs. (1), dropping the inertia terms. Of course, for free convective flow (for example, Ref. [2]), an energy equation must also be included.

References

1. D. P. Timo, *Free convection in narrow vertical sodium annuli,* Knolls Atomic Power Laboratory, Report No. KAPL-1082, March 1954
2. S. Ostrach, *Combined natural and forced-convection laminar flow and heat transfer of fluids with and without heat sources in channels with linearly varying wall temperatures,* NACA, TN 3141, 1954

ON AN INEQUALITY OF LIAPOUNOFF*

By AUREL WINTNER (*The Johns Hopkins University*)

1. Let $p(t)$ be a continuous function which is positive on the t-interval under consideration. Instead of $p(t) > 0$, it will be sufficient to assume that $p(t) \geqq 0$, provided that $p(t) > 0$ holds on a dense t-set. The role of this proviso will be that of excluding the existence of a function $x(t)$ which satisfies the differential equation

$$x'' + p(t)x = 0 \tag{1}$$

and is a non-vanishing constant on some t-interval. If such a solution $x(t)$ of (1) is disregarded [and (1) cannot have two, linearly independent, such solutions in any case], then, besides the continuity of $p(t)$, only the assumption $p(t) \geqq 0$ will be needed. Only real-valued solutions $x(t)$ will be considered, and the trivial solution, $x(t) \equiv 0$, will be excluded.

It is clear from (1) that $x''(t) \geqq 0$ or $x''(t) \leqq 0$ at a given t according as $x(t) < 0$ or $x(t) > 0$ at that t. Hence the graph of $x = x(t)$ must turn its concavity toward the t-axis at every t. Since the clustering of zeros of the derivative $x'(t)$ has been excluded, it follows that the zeros of $x(t)$ separate, and are separated by, the zeros of $x'(t)$ [provided that either $x(t)$ or $x'(t)$ has at least two zeros]. Let a closed t-interval $[c, d]$ be called a *primitive* interval of $x(t)$ if neither $x(t)$ nor $x'(t)$ has any zero in the interior of $[c, d]$. Such an interval $[c, d]$ will be called a *complete* primitive interval of $x(t)$ if for no $\epsilon > 0$ is $[c - \epsilon, d + \epsilon]$ a primitive interval of $x(t)$, that is, if $x(t) \neq 0$ and $x'(t) \neq 0$ for $c < t < d$ but either $x(c) = 0$ and $x'(d) = 0$ or $x'(c) = 0$ and $x(d) = 0$. Note that $x'(c) \neq 0$ and $x(d) \neq 0$ in the first case, and that $x(c) \neq 0$ and $x'(d) \neq 0$ in the second case, since the simultaneous vanishing of $x(t)$ and $x'(t)$ leads to the trivial solution.

2. The purpose of this note is to show that, owing to the concept of a primitive interval, a theorem of Liapounoff (see below) can be extended from his "disconjugate" case to the general case, as follows:

If $p(t)$ is continuous and non-negative on a closed t-interval $[a, b]$, and if $[a, b]$ consists of exactly n primitive intervals of some solution $x(t)$ of (1), then

$$\int_a^b p(t)\, dt > n^2/(b - a) \tag{2}$$

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(here n is a preassigned positive integer, and $n = 1$ is allowed; but $n = \infty$ is not allowed, since solutions $x(t)$ which are constant on some t-interval are excluded).

Accordingly, if P is defined by

$$P = \left\{(b - a) \int_a^b p(t)\, dt\right\}^{1/2} \tag{3}$$

and if $[P]$ is the greatest integer not exceeding P, then a solution $x = x(t)$ of (1) on $[a, b]$ cannot consist of more than $[P]$ consecutive stretches on each of which both $x(t)$ and $x'(t)$ are strictly monotone (nowhere constant). The italicized lemma contains however somewhat more, since it does not exclude such linear stretches of the graph $x = x(t)$ as are not parallel to the t-axis. The proof proceeds as follows:

3. First, it is sufficient to prove that (2) must hold when $[a, b]$ consists of exactly n *complete* primitive intervals (rather than of exactly n *arbitrary* primitive intervals) of some solution $x(t)$ of (1). In fact, if the value of n is retained but $[a, b]$ is replaced by $[a^*, b^*]$, where $a^* \leqq a < b \leqq b^*$, then the value of the quotient on the right of (2) is decreased, whereas the value of the integral on the left of (2) is not decreased, since $p(t) \geqq 0$.

Next, it is sufficient to prove (2) for the particular case $n = 1$ For suppose that (2) is true for $n = 1$. Then, if $[a, b]$ consists of n primitive intervals, and if the latter are denoted by $[c_{k-1}, c_k]$, where $c_0 = a$, $c_n = b$ and $k = 1, \cdots, n$, then an application of the case $n = 1$ of (2) to each of the intervals $[c_{k-1}, c_k]$ leads to

$$\int_a^b p(t)\, dt = \sum_{k=1}^n \int_{c_{k-1}}^{c_k} p(t)\, dt > \sum_{k=1}^n 1^2/d_k\,,$$

where $d_k = c_k - c_{k-1} > 0$. Since $\sum_{k=1}^n d_k = b - a$, this implies (2) for an arbitrary n if it is ascertained that

$$n^2 \Big/ \sum_{k=1}^n d_k \geqq \sum_{k=1}^n 1/d_k\,. \tag{4}$$

But (4) is true, since it merely expresses the fact that the harmonic mean of n positive numbers d_k cannot exceed their arithmetical mean.[1]

Accordingly, it is sufficient to prove (2) under the following two (simultaneous) assumptions: $n = 1$, and $[a, b]$ is a *complete* primitive interval of some solution $x(t)$ of (1). Then, if $a = 0$ without loss of generality, the assertion (2) reduces to

$$b \int_0^b p(t)\, dt > 1, \tag{5}$$

and the assumption of (5) is the existence of a solution $x(t)$ for which both $x(t) \neq 0$ and $x'(t) \neq 0$ hold on the open interval $0 < t < b$ and *either* $x(0) = 0$ and $x'(b) = 0$ *or* $x'(0) = 0$ and $x(b) = 0$. But it will be clear (by interchanging "past" and "future") that it will be sufficient to prove (5) for the first of these two alternative cases.

In addition, since $x(t) \neq 0$ for $0 < t < b$, and since the solution $x(t)$ can be replaced by $-x(t)$, it can be assumed that $x(t) > 0$ for $0 < t \leqq b$, while $x(0) = 0$. Since, as pointed out above, the graph of $x = x(t)$ always turns its concavity toward the t-axis, and since $x'(t) \neq 0$ for $0 \leqq t < b$ [while $x'(b) = 0$], it follows that, if $0 < t < b$, the

[1]See, for instance, G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, 1924, p. 50.

ordinate of the graph of $x = x(t)$ is positive and increasing, while its slope is positive and non-increasing. Since the slope at $t = 0$ is $x'(0) > 0$, and since $x'(b) = 0$ prevents that $x(t)$ be linear on the whole of $[0, b]$, it now follows from $x(0) = 0$ that $x(b) < bx'(0)$.

On the other hand, since $x'(b) = 0$, integration of (1) between $t = 0$ and $t = b$ shows that

$$x'(0) = \int_0^b p(t)x(t)\,dt. \tag{6}$$

But since $x(t)$ increases from $x(0) = 0$ to $x(b) > 0$ on the t-range of (6), and since $p(t) \geqq 0$, the representation (6) of $x'(0)$ implies that, unless $p(t)$ vanishes identically,

$$0 < x'(0) < x(b) \int_0^b p(t)\,dt.$$

Hence (5) follows from the inequality $x(b) < bx'(0)$, found at the end of the preceding paragraph.

4. This completes the proof of the italicized assertion. The following fact is a corollary: *If* (1), *where* $p(t) \geqq 0$ *and* $a \leqq t \leqq b$, *possesses some solution for which* $x(t)$ *and* $x'(t)$ *together do not have more than one zero on the open interval* $a < t < b$, *then*

$$\int_a^b p(t)\,dt > 4/(b - a). \tag{7}$$

In fact, since (7) is the case $n = 2$ of (2), it is sufficient to ascertain that the assumptions of (2) are satisfied by $n = 1$ or $n = 2$ according as $x(t)$ and $x'(t)$ together have no or a single zero on the interval $a < t < b$. But this is clear for reasons of concavity (see above), since the whole of $[a, b]$ is a primitive interval of $x(t)$ in the first case, while $[a, b]$ consists of two primitive intervals of $x(t)$ in the second case.

5. It is well-known that a result of Liapounoff[2] is substantially equivalent to the following assertion: If $p(t) \geqq 0$, then (7) must hold whenever (1) has on $[a, b]$ a solution $x(t)$ satisfying

$$x(a) = 0, \qquad x(b) = 0, \qquad x(t) > 0 \quad \text{for} \quad a < t < b. \tag{8}$$

It is also known[3] that, under the assumption (8), the 4 is the best absolute constant in (7). A simple proof of the fact that (7) is necessary for (8) was given by Shukovski (a contemporary of Liapounoff)[4] and recently by Borg.[5]

Clearly, this result is a particular case of the italicized corollary ($n = 2$) of the general inequality (2). In fact, the assumption (8) means that the graph of $x = x(t)$ is a convex arch over $[a, b]$, and so $x'(t)$ has just one zero on $[a, b]$. But (7) turns out to hold also for boundary conditions more general than those of (8). For instance, (7) must hold also if (1) possesses a solution $x(t)$ which satisfies the boundary condition $x'(a) = 0$, $x'(b) = 0$ but has only one zero on $[a, b]$.

The general result, expressed by (2) for an arbitrary n, suggested itself by Shukovski's

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[2]Actually, Liapounoff considered only the case of an equation (1) having a periodic coefficient $p(t)$, of period $b - a$; see A. Liapounoff, Comptes Rendus **123**, 1248-1252 (1896).

[3]E. R. van Kampen and A. Wintner, Amer. J. Math. **59**, 270-274 (1937).

[4]See J. L. Geronimus, *Alexander Michailowitsch Ljapunow*, Berlin, 1954, pp. 52-53.

[5]G. Borg, Amer. J. Math. **71**, 67-70 (1949).

and Borg's proof of (7) for the case (8). In fact, even the use of (4) is[6] between the lines of that step in Borg's proof of (7) in Liapounoff's case which leads from $[a, c]$ and $[c, b]$ to $[a, b]$, where c is defined by $x(c) = \max x(t)$.

MAXIMUM SPEED IN STEADY SUBSONIC FLOWS*

By CHIA-SHUN YIH (*University of Michigan*)

The purpose of this note is to show that the maximum speed in a singularity-free region of any steady subsonic flow must occur on its boundary, provided the flow is isentropic and irrotational. The corresponding result for irrotational flows of an incompressible fluid is well known.

Since the flow under consideration is irrotational, the square of the speed q is given by

$$q^2 = \varphi_{,i}\varphi_{,i} \tag{1}$$

in which φ is the velocity potential, the summation convention is used, and, with Cartesian coordinates,

$$\varphi_{,i} = \frac{\partial \varphi}{\partial x_i}$$

is the jth component of the velocity vector—with the usual minus sign suppressed for convenience. The equation of continuity is, for steady flow,

$$(\rho\varphi_{,i})_{,i} = 0 \tag{2}$$

in which ρ is the density. The equations of motion are

$$(\ln \rho)_{,i} = -\frac{1}{2c^2}(q^2 + 2gx_3)_{,i} \tag{3}$$

if x_3 is measured in a direction opposite to that of the gravitational acceleration g, and c is the local speed of sound. Finally, the Bernoulli equation can be written in the form (with p indicating pressure)

$$c^2 = k\frac{p_0}{\rho_0} - \frac{k-1}{2}(q^2 + 2gx_3) \tag{4}$$

in which k is the ratio of the specific heat at constant pressure to that at constant volume, and the subscripts zero indicate that the quantities involved are taken at some point of reference, where the gas is at rest and from which x_3 is measured.

For a proof, it is sufficient to show that the Laplacian of q^2 is non-negative. Whereas this is easily done for irrotational flows of an incompressible fluid, it cannot be readily established for the flows under discussion. Professors D. Gilbarg (through his publications) and C. Truesdell (by oral communication) have indicated to the writer that

[6] G. Borg, Amer. J. Math. 71, 68 (1949); see also P. Hartman and A. Wintner, *ibid.*, p. 209.

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