and (16) gives

$$\frac{5}{24}(Y_4^{4,4} + Y_4^{4,-4} + Y_4^{-4,4} + Y_4^{-4,-4}) + \left(\frac{35}{288}\right)^{1/2}(Y_4^{4,0} + Y_4^{-4,0} + Y_4^{0,4} + Y_4^{0,-4}) + \frac{7}{12} Y_4^{0,0} \tag{20}$$

as a properly invariant function.

The author wishes to extend his appreciation to Professor R. Bersohn, without whose assistance this work would have been impossible.

ON THE MOTION OF A SIMPLE PENDULUM*

By BORIS GARFINKEL (*Ballistic Research Laboratories, Aberdeen Proving Ground, Md.*)

Abstract. The vanishing of the tension in a simple pendulum supported by a flexible cord causes the particle to pass from the circular to a parabolic trajectory. The number and the nature of such transitions are related here to the value of the initial energy.

1. When the initial energy of a simple pendulum lies in a certain interval, the tension vanishes at some instant of the motion. Then, if the support is provided by a flexible cord, the particle passes from the circular to a parabolic trajectory. The number and the nature of such transitions are shown here to be precisely related to a dimensionless energy parameter, ξ. Despite the intrinsic interest and the relative simplicity of this motion, it does not appear to have been treated in the literature.

Let l be the length, m the mass of the pendulum, r its distance from the point of support, θ the angular coordinate measured from the downward-drawn vertical line, and g the acceleration of gravity. The constraint

$$l - r \geq 0 \tag{1}$$

can be replaced by the condition

$$\lambda(l - r) = 0, \tag{2}$$

where λ is a multiplier vanishing if $l > r$ and admitting a non-zero value if $l = r$. The Lagrangian of the system,

$$L = \tfrac{1}{2}m(r^{\cdot 2} + r^2\theta^{\cdot 2}) + mgr\cos\theta + \lambda(l - r), \tag{3}$$

leads to the differential equations of motion,

$$m(r^{\cdot\cdot} - r\theta^{\cdot 2} - g\cos\theta) + \lambda = 0, \qquad r\theta^{\cdot\cdot} + 2r^{\cdot}\theta^{\cdot} + g\sin\theta = 0, \tag{4}$$

which together with (2) and (1) determine the functions $r(t)$, $\theta(t)$, $\lambda(t)$ when the initial conditions are prescribed. From (4.1) the multiplier $\lambda(t)$ can be identified with the

*Received July 22, 1957.

tension of the cord. Let the initial conditions now be:

$$r(0) = l, \qquad \theta(0) = 0, \qquad r^{\cdot}(0) = o, \qquad \theta^{\cdot}(0) = (2E/ml^2)^{1/2}, \tag{5}$$

E being the initial energy measured from the lowest point. Then the particle moves in a circular arc $r(t) = l$ until λ vanishes. The integrals of the motion, deduced from (4), are

$$(\theta^{\cdot}/\omega)^2 - 2\cos\theta = 2(2c^2 - 1) = 3\xi, \qquad \sin(\theta/2) = \min(1, c)\,\mathrm{sn}\,[\omega t/\max(1, c)], \tag{6}$$

where the constants ω, c, ξ are defined by

$$\omega^2 = g/l, \qquad c^2 = E/2\,mg\,l, \qquad \xi = \tfrac{2}{3}(2c^2 - 1), \tag{7}$$

and the modulus of the elliptic function sn is

$$\kappa = \min(c, 1/c). \tag{8}$$

The dimensionless parameter ξ is seen from (7) to be proportional to the energy measured from the horizontal configuration $r = l$, $\theta = \pi/2$; its range is $-2/3 \leq \xi < \infty$.

2. At the instant $t = t_1$ of the vanishing of the tension the coordinates and their derivatives, obtained from (4.1) and (6.1) with the substitutions $\lambda = 0$, $r = l$, $r^{\cdot\cdot} = 0$, are given by

$$\cos\theta_1 = -\xi, \qquad r_1 = l, \qquad \theta_1^{\cdot}/\omega = \xi^{1/2}, \qquad r_1^{\cdot} = 0. \tag{9}$$

Hence θ_1 and $\theta_1^{\cdot}$ are real if and only if ξ lies in the range

$$0 \leq \xi \leq 1, \tag{10}$$

and $\pi/2 \leq \theta_1 \leq \pi$, $0 \leq \theta_1^{\cdot} \leq \omega$. At $t = t_1$ there occurs a transition from the circular to a parabolic trajectory. The latter corresponds to the solution of (4) with $\lambda = 0$, $r \leq l$, and the initial conditions (9), and is represented by the equations

$$(r/l)^2 = 1 - (\omega\tau)^3[\xi(1 - \xi^2)]^{1/2} - (\omega\tau)^4/4, \qquad r\sin\theta/l = (1 - \xi^2)^{1/2} - \xi^{3/2}(\omega\tau), \tag{11}$$

where

$$\tau = t - t_1\,. \tag{12}$$

When the particle re-enters the circle, r again assumes the value $r = l$, and (11) yields

$$\omega\tau = 4[\xi(1 - \xi^2)]^{1/2}, \qquad \theta_2 = 2\pi - 3\theta_1\,, \quad r_2 = l, \qquad \theta_2^{\cdot} = \theta_1^{\cdot}(-3 + 12\xi^2 - 8\xi^4), \qquad r_2^{\cdot} = 8l\omega[\xi(1 - \xi^2)]^{3/2}. \tag{13}$$

Such an alternation of the trajectory between the circle and a parabola we shall, for the sake of brevity, designate by the term "flip." At the instant $t = t_2$ of re-entry of the circle the ideal cord, assumed to be inextensible and infinitely strong, enforces the constraint (1) by generating an impulse equal and opposite to the radial momentum $m\dot{}$. While the latter is being annihilated, the angular momentum $mr^2\dot{\theta}$ is conserved, and the energy is thus diminished by the quantity $m\dot{r}^2/2$. The corresponding diminution of ξ can be calculated from (13.4) and the definitions (7); the new value ξ' can then be written as a function, $f(\xi)$, in the form

$$\xi' = f(\xi) = \xi - \frac{64}{3}[\xi(1 - \xi^2)]^3, \quad \text{for} \quad 0 \leq \xi \leq 1. \tag{14}$$

Of course, if ξ lies outside the range (10) flips do not occur and the energy is conserved. Therefore

$$f(\xi) = \xi, \quad \text{for} \quad \xi(1 - \xi) < 0. \tag{15}$$

In particular, $-2/3 \leq \xi < 0$ and $\xi > 1$ correspond to the states of oscillation and circulation respectively.

A few curious details of the motion will now be summarized.

1. "Oscillatory" flips, for which the sense of $\dot{\theta}_2$ is opposite to that of $\dot{\theta}_1$, occur if $0 < \xi < \frac{1}{2}[3 - 3^{1/2}]^{1/2} = 0.56$; "circulatory" flips occur if $0.56 \leq \xi < 1$. The separating value corresponds to $\dot{\theta}_2 = 0$.

2. The maximum energy loss per flip corresponds to $\xi = 3^{-1/2} = 0.57$, with the parabola passing through the point of suspension.

3. The "amplitude" of a flip can be defined as $\alpha = |\sin[(\theta_1 - \theta_2)/2]|$. Its maximum $\alpha = 1$, corresponds to $\xi = 2^{-1/2} = 0.71$, $\theta_1 = 3\pi/4$, $\theta_2 = \pi/4$; its minimum, $\alpha = 0$, occurs when $\xi = 0$, $\theta_1 = \theta_2 = \pi/2$ and when $\xi = 1$, $\theta_1 = \theta_2 = \pi$.

4. The entire history of the motion can now be described in terms of the function $f(x)$, which is graphed and tabulated below, together with the corresponding values of θ_1 and θ_2.

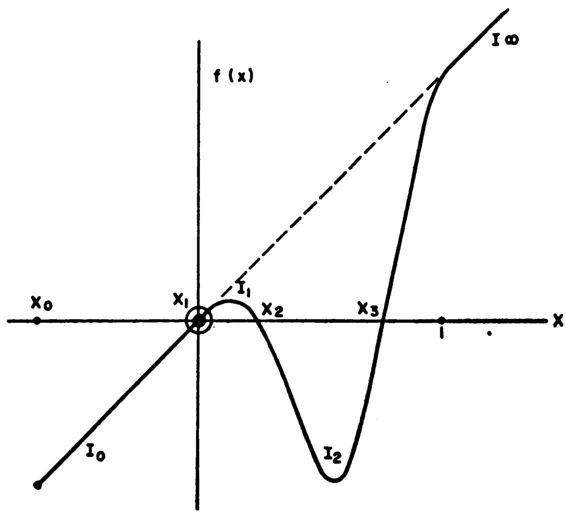

FIG. 1. Graph of the function $f(x)$.

TABLE 1

Function $f(x)$, $0 \leq x \leq 1$

x	$f(x)$	θ_1	θ_2
0.0	0.000	90°.0	90°.0
0.1	0.079	95 .7	72 .8
0.2	0.049	101 .5	55 .4
0.3	−0.134	107 .5	37 .6
0.4	−0.409	113 .6	19 .3
0.5	−0.625	120 .0	0 .0
0.6	−0.608	126 .9	−20 .6
0.7	−0.271	134 .4	−43 .3
0.8	0.290	143 .1	−69 .4
0.9	0.793	154 .2	−102 .5
1.0	1.000	180 .0	−180 .0

Some of the important properties of $f(x)$ are listed below:

1. $f(x)$ is of class C^2; i.e. it has continuous derivatives up to the second order in its entire domain $-2/3 \leq \xi < \infty$.

2. The stationary values of $f(x)$ are given by min $f(x) = -0.655 = f(0.544)$, max $f(x) = +0.086 = f(0.130)$.

3. The real zeros of $f(x)$ are:

$$x_1 = 0, \qquad x_2 = 0.236, \qquad x_3 = 0.751.$$

4. These zeros, together with the boundary point $x_0 = -2/3$ and the recursive relation

$$f(x_{k+4}) = x_{k+2}\,, \qquad k = 0, 1, \cdots, \tag{16}$$

define a monotone increasing sequence $\{x_k\}$; $k = 0, 1, \cdots$, converging to $x_\infty = 1$. This result follows from the facts that x_2 and x_3 lie in the interval max $f < x < 1$, that in this interval the inverse function $f^{-1}(x)$ is defined and is bounded by the inequality $x < f^{-1}(x) < 1$. The first nine terms of the sequence are:

$$-0.667, \quad 0.000, \quad 0.236, \quad 0.751, \quad 0.791, \quad 0.889, \quad 0.899, \quad 0.930, \quad 0.935, \cdots, \tag{17}$$

The value ξ lies in some interval I_k, such that

$$\begin{aligned} x_k \leq \xi < x_{k+1} \quad &\text{if} \quad -2/3 \leq \xi < 1, \\ k = \infty \qquad &\text{if} \quad 1 \leq \xi < \infty. \end{aligned} \tag{18}$$

The three intervals $k = 0$, $k = 1$, $k = \infty$ are the only stable states of ξ, in the sense that the system once in such a state will remain in that state forever. In particular, the two extreme states, $k = 0$, $-2/3 \leq \xi < 0$, and $k = \infty$, $1 \leq \xi < \infty$ are the classical oscillation and circulation respectively. The state $k = 1$, $0 \leq \xi < 0.236$ contains an infinite succession of oscillatory flips whose amplitude converges to zero; the motion converges to an oscillation with $\xi = 0$, max $\theta = \pi/2$. In all other states a flip results in the jump $\Delta k = -2$ in the step-function $k(t)$. Consequently the particle executes

$[k(0)/2]$ flips before descending to the state $k = 0$ if $k(0)$ is even or to the state $k = 1$ if $k(0)$ is odd. The bracket above denotes the integral part of a number. The total number N of flips and the ultimate state $k(\infty)$ of the system are hence given by the following table.

TABLE 2

History of the motion

$k(0)$	N	$k(\infty)$	Ultimate state
even	$[k(0)/2]$	0	oscillation
odd	∞	1	flipping
∞	0	∞	circulation

As an example, consider $\xi(0) = 0.777$. From inspection of the sequence (17) it is seen that $k(0) = 3$. After one flip the system reaches the state $k = 1$, characterized by an infinite succession of flips, as indicated in Table 2. It is to be observed that if the support were rigid the pendulum would circulate, since ξ exceeds the critical value $\xi = 2/3$, $(c = 1)$, which separates the oscillatory and the circulatory states in the classical case.

4. Summary. In the case of a simple pendulum supported by a flexible cord, whenever the tension vanishes the particle passes from the circular to a parabolic trajectory. The energy loss occurring upon re-entry into the circle is given by the expression

$$f(x) - x = -\frac{64}{3}[x(1 - x^2)]^3, \qquad (0 < x < 1)$$
$$= 0, \qquad (x(1 - x) < 0),$$

where x and $f(x)$ are the old and the new values of the energy, measured from the horizontal configuration $\theta = \pi/2$, and normalized by a divisor 3 mgl/2. The state of the pendulum is characterized by an integer k, such that

$$x_k \le x < x_{k+1}, \qquad (x < 1),$$
$$k = \infty, \qquad (x \ge 1),$$

where the sequence $\{x_k\}$; $k = 0, 1, \cdots$, is monotone increasing, converging to $x_\infty = 1$, and defined by

$$f(x_{k+4}) = x_{k+2}, \qquad k = 0, 1, \cdots$$
$$x_0 = -\tfrac{2}{3}, \qquad x_1 = 0, \qquad x_2 = 0.236, \qquad x_3 = 0.751.$$

Here x_1, x_2, and x_3 are the three real zeros of $f(x)$, and x_0 is the lowest possible energy. The two extreme states $k = 0$ and $k = \infty$ correspond respectively to pure oscillation and pure circulation. The state $k = 1$, which does not arise in the usual case of a rigid support, contains an infinite succession of parabola-circle transitions. The three states $k = 0$, $k = 1$ and $k = \infty$ are the only stable states, in the sense that the system once in such a state will remain in that state forever. From an unstable state the system will ultimately descend to the state $k = 0$ if $k(0)$ is even, or to the state $k = 1$ if $k(0)$ is odd. In this process there occurs a finite number of parabola-circle transitions, this number being equal to $[k(0)/2]$, the largest integer not exceeding $k(0)/2$.