QUARTERLY OF APPLIED MATHEMATICS

Vol. XVI JULY, 1958 No. 2

ON THE APPLICATION OF INFINITE SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS TO PERTURBATIONS OF PLANE POISEUILLE FLOW*

BY

C. L. DOLPH (*The University of Michigan*)

AND

D. C. LEWIS (*The Johns Hopkins University*)

Introduction. While considerable progress has been made in recent years in various stability problems of hydrodynamics (see C. C. Lin [1] and the bibliography contained in the reference) and in the discussion of fully developed turbulence (see G. K. Batchelor [2]), the problems associated with transition from laminar to turbulent motion are in a less satisfactory state. To be sure the phenomenological theory of H. Emmons [3, 4] shows considerable promise, but it replaces explicit dependence on the hydrodynamical equations by general stochastic and probabilistic considerations.

Unfortunately, the usual laminar stability theory of the Orr-Sommerfeld equation seems incapable of reasonable extension to the nonlinear regime. In addition, even the linear stability analysis is very difficult and requires delicate mathematical consideration.

In contrast to this, the techniques used by E. Hopf [5] to establish the existence of a weak solution for all time of the Navier-Stokes equations in a bounded region are in principle simple and constructive, and it therefore seems reasonable to investigate their practicality for, first, the investigation of the stability problems of hydrodynamics and, second, any insight they might be capable of providing for the transistion problem.

In essence, Hopf's method consists of proving the existence of a complete orthonormal set of functions of the space variables which have divergence zero and which vanish outside sets of compact support. A solution to the Navier-Stokes equations is then sought as an orthogonal series expansion with unknown coefficients of time. This, when substituted into the Navier-Stokes equations and sampled by all members of the orthogonal set, leads to an infinite set of ordinary nonlinear differential equations in the infinite number of unknown time dependent coefficients. The existence of a solution of this system is then established by a process of successive approximations involving a square truncation of the system. The idea of the weak solution and Hopf's treatment of the boundary conditions closely parallel those of the direct methods of the calculus of variation.

Certainly, if for a given problem, a sufficiently judicious choice of a complete orthonormal set of functions in the space variables could be made so that the main remaining features of the Navier-Stokes equations were to be contained in a relatively few non-

*Received November 5, 1956. This work was done while the authors were consultants to the Ramo-Wooldridge Corp., Los Angeles, Calif., under contract No. AF 18(600) 1190.

linear ordinary differential equations, the possibility would then exist that some insight would be gained into the mechanism of transition. In fact, it might then be possible to verify the appealing intuitive picture of this phenomenon as conjectured by Hopf [6] in his model. In brief, he pictures the transition from laminar to turbulent motion as a bifurcation phenomenon in phase space in which the number of dimensions of the manifold of central motions changes from zero (the laminar regime where all perturbations die out) by integral values to a large number of dimensions (possibly an infinite number) corresponding to fully developed turbulence.

On the other hand, it could happen that all reasonable and tractable choices for this family could, for a given problem, require the consideration of an enormous number of equations to represent the Navier-Stokes equations adequately, and it might not be practical to replace an existing asymptotic treatment based on the Orr-Sommerfeld equations by one based on the ideas of Hopf. That is, the situation could be intrinsically like Poincaré's famous example of a convergent series requiring a million terms for ten percent accuracy and only two or three for the asymptotic series.

In this paper we will demonstrate that the qualitative and quantitative features of the instability of plane Poiseuille flow can be obtained from low order truncations of a suitably chosen Hopf type of expansion. The qualitative features are already present in the approximate curve of neutral stability obtained from an eight-by-eight truncation of the infinite system of equations satisfied by the coefficients of the expansion even though the minimum critical Reynolds number given by this crude approximate, while of the right order of magnitude, is approximately only half that computed by Lin. In contrast, the incomplete results presently available for the twenty-by-twenty truncation show excellent agreement with the results of Lin with respect to the minimum critical Reynolds number. In any event, our methods are elementary and involve only trigonometric and hyperbolic functions and are limited in accuracy only by machine limitations as to the size of matrices which can be currently treated. Moreover, as will be shown in Sec. II, all solutions of the Orr-Sommerfeld equations are expressible in terms of the solution derived here so that the difference between our approximate neutral curve and the correct one can be attributed to truncation error.

We will limit consideration here to the presentation of the technique for those perturbations whose stream functions are even in y, the running variable perpendicular to the plate. We are unaware of any detailed calculations for those perturbations whose stream functions are odd in y, but certainly they are equally accessible by our methods should any doubt really arise as to whether they are all actually stable, as is generally supposed.

The mathematical justification of our techniques, as well as numerical results of higher order approximations, will be presented elsewhere. In the main the proofs consist of extensions and refinements of several known techniques, some of which have been treated for the first time by one of us [7]. Hopf's original proof is not adequate, since Poiseuille flow is not confined to a bounded region.

In view of the long and often controversial history of this particular problem, we feel fortunate to be able to present additional evidence for the instability of plane Poiseuille flow by elementary means. Moreover, unlike methods based on energy balance, even our eight-by-eight approximations give results of the right order of magnitude. Thus we believe that we have made considerable progress in furnishing a simple alternate approach to problems of linear instability, the need for which has been stressed recently by Lin [1]. Finally, the qualitative success that we have achieved offers considerable promise that consideration of the nonlinear terms would provide insight into the processes involved in transition to turbulence. This particular problem may, however, be unsuited for the study of realistic nonlinear behavior since, like other workers in the field, we have assumed periodicity along the plates.

As a final introductory remark, we would like to express our sincere thanks to Dr. I. Tarnove who carried out the calculations which will be presented here. We would also like to extend our thanks to our colleagues, Professors G. Carrier, F. Clauser, W. Hay, and L. Lees, and Drs. J. Sellars and G. Solomon, who have encouraged us in this work and who have made many constructive suggestions. In particular, the first author is indebted to E. Hopf for his hospitality during that week-end in April 1955 when this approach was still incompletely formulated (see reference [8]).

Section I. Formal transformation of the problem into an infinite system of ordinary differential equations. The Navier-Stokes equations for the two-dimensional motion of an incompressible viscous fluid are written in the familiar form,

$$\begin{aligned} u_t + uu_x + vu_y + p_x &= \nu\Delta u, \\ v_t + uv_x + vv_y + p_y &= \nu\Delta v, \\ u_x + v_y &= 0. \end{aligned} \tag{1}$$

Here $u(x, y, t)$ and $v(x, y, t)$ are, respectively, the x- and y-components of the velocity of the fluid particle which, at time t, is at the point (x, y). The pressure at this point and instant is denoted by $p(x, y, t)$, while ν represents the coefficient of viscosity divided by the density and Δ represents the Laplace operator.

These Navier-Stokes equations are satisfied by the following values for u, v, and p, representing so-called plane Poiseuille motion.

$$u = 1 - y^2, \qquad v = 0, \qquad p = -2\nu x + \text{const}. \tag{2}$$

We are primarily interested in this solution for $-\infty < x < +\infty$ and for $-1 \leq y \leq +1$. It is seen that this solution satisfies the condition requiring that the particles of fluid at the boundary should be at rest, namely

$$u = v = 0 \quad \text{for} \quad y = \pm 1. \tag{3}$$

The problems connected with the stability of this Poiseuille motion require that we consider nearby motions also satisfying the boundary conditions (3). We accordingly introduce the transformation,

$$u = 1 - y^2 + u', \qquad v = v', \qquad p = -2\nu x + \text{const} + p', \tag{4}$$

where for the problem at hand, u', v' and p' will be thought of as small. In these primed variables, the Eqs. (1) appear in the form,

$$u'_t + (1 - y^2)u'_x - 2yv' + p'_x - \nu\Delta u' + u'u'_x + v'u'_y = 0. \tag{5a}$$

$$v'_t + (1 - y^2)v'_x + p'_y - \nu\Delta v' + u'v'_y + v'v'_y = 0. \tag{5b}$$

$$u'_x + v'_y = 0. \tag{5c}$$

It will be useful to eliminate p' from (5a) and (5b) by differentiating the first with respect to y, the second with respect to x, and then forming the difference and simplifying with the aid of (5c). We thus find that

$$(u'_y - v'_x)_t + (1 - y^2)(u'_y - v'_x)_x - 2v' - \nu\Delta(u'_y - v'_x) + u'(u'_y - v'_x)_x + v'(u'_y - v'_x)_y = 0. \tag{6}$$

Equation (5c) permits the introduction of a "stream function," $\psi(x, y, t)$ such that $u' = \psi_y$ and $v' = -\psi_x$, and hence (6) may be written in the form,

$$\Delta\psi_t + (1 - y^2)\Delta\psi_x + 2\psi_x - \nu\Delta^2\psi + \frac{\partial(\Delta\psi, \psi)}{\partial(x, y)} = 0. \tag{7}$$

The linear stability theory is based on the equation obtained by neglecting the second order terms in (7), namely those which are compressed in the expression $\partial(\Delta\psi, \psi)/\partial(x, y)$. We also start out by considering only stability relative to sinusoidal disturbances of a specified period $2\pi/\alpha$ in x. Hence, we set $\psi = e^{i\alpha x}\phi(y, t)$ in the linearized equations (7) and then divide by $e^{i\alpha x}$. The result is

$$(\phi_{yy} - \alpha^2\phi)_t + (1 - y^2)(\phi_{yy} - \alpha^2\phi)i\alpha + 2i\alpha\phi = \nu(\phi_{yyyy} - 2\alpha^2\phi_{yy} + \alpha^4\phi). \tag{8}$$

We attempt the solution of (8) with the appropriate boundary conditions derived from (3) and the subsequent definitions of ψ and ϕ, namely

$$\phi(o, t) = \phi_y(o, t) = 0. \tag{9}$$

We attempt this solution in the form of an infinite series of the form

$$\phi(y, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(y), \tag{10}$$

where $\phi_1(y), \phi_2(y), \cdots$ are associated with certain eigenvalues $\lambda_1, \lambda_2, \cdots$, respectively, satisfy the initial conditions (9), and also satisfy the differential equations

$$\frac{d^4\phi_n}{dy^4} - 2\alpha^2\frac{d^2\phi_n}{dy^2} + \alpha^4\phi_n = \lambda_n\left(\alpha^2\phi_n - \frac{d^2\phi_n}{dy^2}\right). \tag{11}$$

It follows from (11) and (9) that, if $\lambda_m \neq \lambda_n$, then ϕ_m and ϕ_n are orthogonal to each other in the sense that the integral on the left of the next formula (12) is zero. Even if λ_m could be equal to λ_n (with m not equal to n), which will turn out to be impossible, the Schmidt process of orthogonalization of linearly independent functions permits us to assume that the integral always vanishes when $m \neq n$. Finally, assuming hereby that none of the ϕ_n's is identically zero, we may introduce normalizing factors, so that, without exception, we may write

$$\int_{-1}^{+1} [\alpha^2\phi_m(y)\phi_n(y) + \phi'_m(y)\phi'_n(y)]\, dy = \delta_{mn}. \tag{12}$$

For purposes of abbreviation we introduce the notation

$$\int_{-1}^{+1} [\alpha^2 f(y)g(y) + f'(y)g'(y)]\, dy = (f, g) \tag{13}$$

so that (12), for example, can be written in the form $(\phi_m, \phi_n) = \delta_{mn}$. Assuming that such an expansion as (10) is possible, that this series can be differentiated term by term, and that the differentiated series is uniformly convergent, a simple calculation based on (12) and (13) shows that

$$(\phi, \phi_n) = a_n(t) = a_n . \tag{14}$$

We also have a Bessel inequality

$$\sum_{k=1}^{n} a_k^2 \leq (\phi, \phi), \tag{15}$$

which establishes the convergence of $\sum_{k=1}^{\infty} a_k^2$. Conversely, if $a_1, a_2, \cdots$ were given arbitrarily subject to the condition that this series converges, we can establish by known and straight-forward methods [9] the existence of a function ϕ vanishing at $y = \pm 1$ and satisfying (14). In order that ϕ should also satisfy (8), we seek, in a purely formal manner, a system of differential equations of infinite order to be satisfied by $a_1(t), a_2(t), \cdots$.

Formally differentiating (10) and substituting in (8), it is found that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n'(t)[\phi_n''(y) - \alpha^2\phi_n(y)] + i\alpha(1 - y^2) \sum_{n=1}^{\infty} a_n(t)[\phi_n''(y) - \alpha^2\phi_n(y)] \\ + 2i\alpha \sum_{n=1}^{\infty} a_n(t)\phi_n(y) = \nu \sum_{n=1}^{\infty} a_n(t)\lambda_n[\alpha^2\phi_n(y) - \phi_n''(y)].\end{aligned} \tag{16}$$

In obtaining this result, the terms on the right-hand side are reduced with the help of (11). Integration by parts shows that

$$\int_{-1}^{+1} \phi_m(y)[\alpha^2\phi_n(y) - \phi_n''(y)]\, dy = (\phi_m, \phi_n) = \delta_{mn},$$

inasmuch as $\phi_m(\pm 1) = 0$ in accordance with the boundary conditions. Hence, on multiplying (16) by $\phi_m(y)$ and integrating from -1 to $+1$, we find that

$$\begin{aligned}-a_m'(t) + i\alpha \sum_{n=1}^{\infty} a_n(t) \int_{-1}^{+1} (1 - y^2)\phi_m(y)[\phi_n''(y) - \alpha^2\phi_n(y)]\, dy \\ + 2i\alpha \sum_{n=1}^{\infty} a_n(t) \int_{-1}^{+1} \phi_m(y)\phi_n(y)\, dy = \nu\lambda_m a_m(t), \qquad m = 1, 2, 3, \cdots .\end{aligned}$$

This equation may be written in the form,

$$\frac{da_m}{dt} + \nu\lambda_m a_m = i\alpha\left[-a_m + \sum_{n=1}^{\infty} c_{mn}a_n\right], \qquad m = 1, 2, 3, \cdots, \tag{17}$$

where

$$c_{mn} = \int_{-1}^{+1} y^2\phi_m(y)[\alpha^2\phi_n(y) - \phi_n''(y)]\, dy + 2\int_{-1}^{+1} \phi_m(y)\phi_n(y)\, dy. \tag{18}$$

It is now necessary to examine a bit more explicitly the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$, and the associated eigenfunctions, $\phi_1(y), \phi_2(y), \phi_3(y), \cdots$, which satisfy (11) and vanish together with their first derivatives at $y = \pm 1$. Equation (11) has only constant coefficients. Hence its general solution is readily found by elementary methods and is found to be

$$\phi(y) = A_1 \cosh \alpha y + A_2 \sinh \alpha y + A_3 \cos (\lambda - \alpha^2)^{1/2}y + A_4 \sin (\lambda - \alpha^2)^{1/2}y, \tag{19}$$

where A_1, A_2, A_3, and A_4 are constants of integration. In the sequel we shall systematically use the abbreviation

$$u = (\lambda - \alpha^2)^{1/2}, \qquad u_n = (\lambda_n - \alpha^2)^{1/2}. \tag{20}$$

To determine the A's so that the boundary conditions $\phi(\pm 1) = \phi'(\pm 1) = 0$ are satisfied, we are led to the equations,

$$\begin{aligned} A_1 \cosh \alpha + A_2 \sinh \alpha + A_3 \cos u + A_4 \sin u &= 0, \\ A_1 \cosh \alpha - A_2 \sinh \alpha + A_3 \cos u - A_4 \sin u &= 0, \\ A_1\, \alpha \sinh \alpha + A_2\, \alpha \cosh \alpha - A_3\, u \sin u + A_4\, u \cos u &= 0, \\ -A_1\, \alpha \sinh \alpha + A_2\, \alpha \cosh \alpha + A_3\, u \sin u + A_4\, u \cos u &= 0. \end{aligned}$$

By forming half the sum and difference of the first pair of these equations and then doing likewise for the second pair, we see that this system is equivalent to the following:

$$\begin{aligned} A_1 \cosh \alpha \qquad\qquad + A_3 \cos u \qquad\qquad &= 0, \\ A_1\, \alpha \sinh \alpha \qquad\qquad - A_3\, u \sin u \qquad\qquad &= 0, \\ A_2 \sin \alpha \qquad\qquad + A_4 \sin u &= 0, \\ A_2\, \alpha \cosh \alpha \qquad\qquad + A_4\, u \cos u &= 0. \end{aligned}$$

The determinant of this latter system of homogeneous equations in the A's is easily seen to have the value

$$[u \cosh \alpha \sin u + \alpha \sinh \alpha \cos u][u \sinh \alpha \cos u - \alpha \cosh \alpha \sin u].$$

Hence the system is consistent (leading to solutions of (11) that are not identically zero and at the same time satisfy the boundary conditions), if and only if

$$u \cosh \alpha \sin u + \alpha \sinh \cos u = 0 \tag{21}$$

or

$$u \sinh \alpha \cos u - \alpha \cosh \alpha \sin u = 0. \tag{22}$$

It is readily seen that these equations, (21) and (22), cannot be satisfied simultaneously. If (21) is satisfied, $A_2 = A_4 = 0$ and the ϕ given by (19) is an even function. If (22) is satisfied, $A_1 = A_3 = 0$ and ϕ is an odd function. This suggests the possibility of getting more rapid results by considering separately the even disturbances $\phi(y) = \phi(-y)$ and the odd disturbances $\phi(y) = -\phi(-y)$. Any function can be expressed as the sum of an odd function and an even function, and hence there is no loss of generality in this approach, provided that there is no coupling between the odd and even disturbances. Since the coefficients of the linear equation (8) are all even, and since only even-order derivatives occur with respect to y, it is clear that such a coupling is indeed not present in the linearized theory. This fact is also confirmed by formula (18) for the "coupling coefficient" c_{mn}, which is clearly zero if $\phi_m(y)$ and $\phi_n(y)$ have opposite parity.

In view of these considerations we limit ourselves hereafter in this paper to *even* disturbances. Accordingly, we ignore solutions of (22) and denote by $u_1, u_2, u_3, \cdots$, the successive solutions of (21) and by $\lambda_1, \lambda_2, \lambda_3, \cdots$ the corresponding eigenvalues given by (20). We continue to regard α, of course, as fixed. The following may be said about the successive solutions of (21).

In the first place, Eq. (21) can be written in the form

$$u \tan u = \beta, \quad \text{where} \quad \beta = -\alpha \tanh \alpha \tag{23}$$

and we proceed to get an explicit approximate expression for the solutions of (23) valid at least for small values of β, and hence of α. Evidently (23) is satisfied when $\beta = 0$ and $u = k\pi$, $k = 0, \pm 1, \pm 2, \cdots$. By the implicit function theorem, we can solve for u as a function of β, when $k = \pm 1, \pm 2, \cdots$. We have

$$u - k\pi = \tan^{-1} \frac{\beta}{u}, \qquad \frac{du}{d\beta} = \frac{u}{u^2 + \beta + \beta^2}, \qquad \frac{d^2u}{d\beta^2} = -2(\beta + 1) \frac{u^3 + \beta^2 u}{(u^2 + \beta + \beta^2)^3},$$

$$\frac{d^3u}{d\beta^3} = -2(\beta + 1)u \cdot \left[\frac{2\beta(u^2 + \beta + \beta^2)^2 - 2\beta(3u^2 + \beta + 3\beta^2)(u^2 + \beta + \beta^2) - 6u^2(u^2 + \beta^2)}{(u^2 + \beta + \beta^2)^5} \right].$$

Hence the first four terms, in the Taylor series development of u as a function of β, may be written down at once,

$$u = k\pi + \frac{\beta}{k\pi} - \frac{\beta^2}{k^3\pi^3} + 2\frac{\beta^3}{k^5\pi^5} + \cdots,$$

and we may estimate the error in retaining only one, two, or three terms with the help of the remainder theorem for the Taylor series, together with the above formulas for the first three derivatives. In terms of α, then, the kth u is given approximately as follows:

$$u_k = k\pi - \frac{\alpha \tanh \alpha}{k\pi} - \frac{\alpha^2 \tanh^2 \alpha}{k^3\pi^3} - 2\frac{\alpha^3 \tanh^3 \alpha}{k^5\pi^5} + \cdots,$$

which shows that u_k is asymptotically undistinguishable from $k\pi$.

With u_k (and hence $\lambda_k = u_k^2 + \alpha^2$) thus characterized, we define a corresponding eigenfunction,

$$\phi_k^*(y) = \cos u_k y - (\cos u_k \cosh \alpha y)/(\cosh \alpha), \tag{24}$$

by taking $A_3 = 1$ arbitrarily in (19) and then determining $A_1 = -(\cos u_k)/(\cosh \alpha)$ so that $\phi_k^*(1) = 0$. Of course, $A_2 = A_4 = 0$ as already stated, in the even case to which we are restricting attention. The other boundary conditions, viz., $\phi_k^{*\prime}(\pm 1) = 0$, are seen to be satisfied in virtue of the fact that u_k, by definition, satisfies (21). These ϕ^*'s are not quite the same as the ϕ's previously introduced because they are not normalized. In order to carry out the normalization, we need to calculate the integral expression for $(\phi_m^*, \phi_n^*) = \mu_m$. This may be done in several obvious ways, using (21), perhaps to simplify the result. We thus find that

$$\mu_m = \lambda_m[1 + (\sin 2u_m)/2u_m]. \tag{25}$$

Then we also have that

$$(\phi_m^*, \phi_n^*) = \delta_{mn}\mu_n. \tag{26}$$

Since ϕ_m is proportional to ϕ_m^* and since $(\phi_m, \phi_n) = \delta_{mn}$, it is evident that

$$\phi_m = \mu_m^{-1/2}\phi_m^*. \tag{27}$$

On account of the awkwardness of the expression for the normalizing factor, it is sometimes desirable to expand directly in terms of the unnormalized ϕ^*'s. Thus, instead of (10) we may sometimes consider an equivalent expression,

$$\phi(y, t) = \sum_{n=1}^{\infty} a_n^*(t)\phi_n^*(y), \tag{28}$$

where

$$a_m^* = \mu_m^{-1}(\phi, \phi_m^*) = \mu_m^{-1/2}(\phi, \phi_m) = \mu_m^{-1/2}a_m . \tag{29}$$

Correspondingly, we sometimes write (17) in the form

$$\frac{da_m^*}{dt} + \nu\lambda_m a_m^* = i\alpha\left[-a_m^* + \sum_{n=1}^{\infty} c_{mn}^* a_n^*\right], \tag{30}$$

where

$$c_{mn}^* = \mu_m^{-1/2} c_{mn} \mu_n^{1/2}. \tag{31}$$

It follows from (27) and (31) that (18) can be written in the form

$$c_{mn}^* = \mu_m^{-1} \int_{-1}^{+1} y^2\phi_m^*(y)[\alpha^2\phi_n^*(y) - \phi_n^{*\prime\prime}(y)]\,dy + 2\mu_m^{-1} \int_{-1}^{1} \phi_m^*(y)\phi_n^*(y)\,dy. \tag{32}$$

Finally, the integrals appearing on the right-hand side of (32) can be evaluated by elementary methods using (21) and (24). We exhibit the results in the following three formulas:

$$\int_{-1}^{1} \phi_m^*(y)\phi_n^*(y)\,dy = \delta_{mn}\mu_m\lambda_m^{-1} + \cos u_m \cos u_n\,(\operatorname{sech}^2 \alpha + \alpha^{-1} \tanh \alpha). \tag{33}$$

$$\int_{-1}^{1} y^2\phi_m^*(y)[\alpha^2\phi_n^*(y) - \phi_n^{*\prime\prime}(y)]\,dy = 2 \cos u_m \cos u_n(\lambda_m - \lambda_n)^{-2}[2\lambda_m\lambda_n + (2\lambda_n - \lambda_m)\lambda_m\lambda_n^{-1}(2\lambda_n - 4\,\alpha^2 \operatorname{sech}^2 \alpha - 4\alpha \tanh \alpha)], \tag{34}$$

if $m \neq n$.

$$\int_{-1}^{1} y^2\phi_n^*(y)[\alpha^2\phi_n^*(y) - \phi_n^{*\prime\prime}(y)]\,dy = \lambda_n(3^{-1} - 2^{-1}u_n^{-2}) + \cos^2 u_n[\lambda_n 2^{-1}u_n^{-4}\alpha \tanh \alpha + \lambda_n u_n^{-2} + 8\lambda_n^{-1}(\alpha^2 \operatorname{sech}^2 \alpha + \alpha \tanh \alpha) - \lambda_n u_n^{-2}\alpha \tanh \alpha - 4], \tag{35}$$

Section II. Application to the question of stability and connections with the Orr-Sommerfeld equation. Truncating the system (17), we obtain.

$$\frac{da_m}{dt} = \sum_{n=1}^{N} k_{mn}a_n , \qquad m = 1, 2, \cdots , N, \tag{36}$$

where N is a "large" finite integer and the constants k_{mn} are given as follows in terms of the previously introduced notation:

$$k_{mn} = i\alpha(c_{mn} - \delta_{mn}) - \delta_{mn}\lambda_m\nu. \tag{37}$$

Assuming that such an approximation is valid for sufficiently large values of N, it is clear that the stability of plane Poiseuille flow relative to perturbations of period $2\pi\alpha^{-1}$ with respect to x can be investigated by a numerical study of the roots $\sigma_1(N)$, $\sigma_2(N)$, $\cdots$, $\sigma_N(N)$, of the determinantal equation,

$$| k_{mn} - \delta_{mn}\sigma | = 0, \qquad m, n = 1, 2, \cdots, N, \tag{38}$$

in σ. In accordance with a familiar theory, if at least one of these roots has a positive real part, the truncated system is certainly unstable. A numerical study of these roots for different values of N also reveals the possibility of ordering them in such a way that the limit

$$\lim_{N\to\infty} \sigma_l(N) = \sigma_l\,, \qquad l = 1, 2, 3, \cdots, \tag{39}$$

exists for each fixed value of l. Evidently, if at least one of these limits appears to have a positive real part, one is justified in concluding that the Poiseuille flow is unstable under the given conditions, that is, the conditions under which α and ν have fixed values. The numerical details will be described in Sect. III.

It may be noticed that this discussion of stability avoids the Orr-Sommerfeld equation on which previous discussions of stability have been based. That this avoidance is more apparent than actual may be seen in the following way.

The well known Orr-Sommerfeld equation

$$\nu(f'''' - 2\alpha^2 f'' + \alpha^4 f) = i\alpha[(1 - y^2 - c)(f'' - \alpha^2 f) + 2f] \tag{40}$$

is obtained by setting $\phi(y, t) = e^{-i\alpha ct}f(y)$ in the partial differential equation (8) for ϕ and then multiplying by $e^{i\alpha ct}$.

Evidently if, for some value of c, the Eq. (40) has a solution $\neq 0$ which satisfies the boundary conditions $f(\pm 1) = f'(\pm 1) = 0$, then we shall have instability if the imaginary part of c is positive. The direct investigation of the values of c for which this is possible has been the basis of previous studies on stability. This has been carried out by considering the general solution of (40), or approximations thereof, and then attempting to determine the constants of integration so as to satisfy the boundary conditions. This leads to a homogeneous system of linear equations in the constants, the compatibility of which system leads to a very complicated transcendental equation in c.

An alternative way of treating this question is to expand the unknown solution of (40) in terms of our functions $\phi_n(y)$ previously introduced:

$$f(y) = \sum_{n=1}^{\infty} p_n\phi_n(y), \qquad p_n = (f, \phi_n). \tag{41}$$

It is possible to prove that such an expansion is possible if f satisfies the boundary conditions. Conversely, any function $f(y)$ defined by (41) will satisfy the boundary conditions automatically (assuming convergence of the differentiated series), since each of the ϕ_n's satisfies the boundary conditions. Substituting the right member of (41) for f in (40), multiplying by $\phi_m(y)$, and integrating with respect to y from -1 to $+1$, we find with the help of (11), integration by parts, and (18) that

$$(-i\alpha c + \nu\lambda_m)p_m = i\alpha\left(-p_m + \sum_{n=1}^{\infty} c_{mn}p_n\right). \tag{42}$$

With the help of (37), this may be written in the form

$$\sum_{n=1}^{\infty} (k_{mn} + i\alpha c\delta_{mn})p_n = 0, \qquad m = 1, 2, \cdots, \infty.$$

Truncation of this infinite system leads to the finite "approximate" system,

$$\sum_{n=1}^{N} (k_{mn} + i\alpha c\delta_{mn})p_n = 0,$$

the compatibility condition for which is precisely Eq. (38) with σ replaced by $-\ i\alpha c$. The characteristic values c for the Orr-Sommerfeld equation may therefore be presumed to be identical with the σ's of (39) after multiplication by the factor $i\alpha^{-1}$.

Section III. Discussion of numerical results. The majority of the numerical results obtained to date have been based on an eight-by-eight truncation of the system (17). Since there is nothing magical about this choice of order, a few words about the reason for it appear in order. Certainly on an *a priori* basis there is no reason to expect that such a low order system would be capable of representing the complicated behavior of the linearized Navier-Stokes equations quantitatively, for our method is essentially one of sampling, or a method of moments. On the other hand, if our method were to be of any help in understanding the mechanism of transition to turbulence, then it was desirable that a low order system contain the qualitative features of the problem of instability. Now, although we shall not do it here, it is easy to show that all two-by-two truncations of the system (17) are stable. In view of the alternate approach to this problem it is, moreover, reasonable to suppose that at least a fourth order system would be necessary in the method of moments before instability could be detected. The four-by-four truncated system (17) has been explored numerically on an 1103 Computer and no evidence of instability was found. Since we did not know when the phenomenon of instability would make its appearance, we decided to add equations and unknowns two at a time up to the limit of the code then available for complex eigenvalues, ten equations in ten unknowns, and hope for the best. We were fortunate in that this phenomenon did occur in the six-by-six system.*

At this time, then, it was decided that a complete curve of neutral stability for an eight-by-eight system would be computed, and that, in addition, curves of constant amplification would be of some help in indicating the regularities of the approximation. This has been done, and the results of these computations are compared with those of Lin [1] in Fig. 1. These results while most encouraging and of the right order of magnitude did not agree at all well quantitatively with those computed by Lin [1] and checked by Thomas [10]. After some preliminary results with a ten-by-ten truncation which did not indicate any major divergences from the eight-by-eight case, some preliminary evidence was found that the twenty-by-twenty system would give much better agreement with the results of Lin. The twenty-by-twenty truncated system of differential equations was integrated by use of the Runge-Kutta-Gill method and compared with a similar

*The five-by-five system has not been explored.

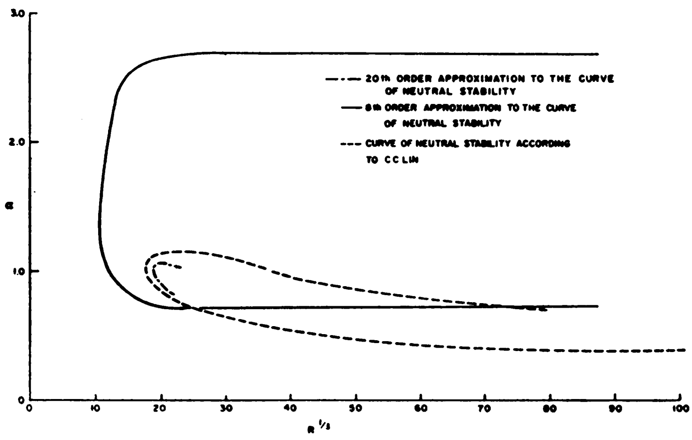

FIG. 1. Curves of neutral stability.

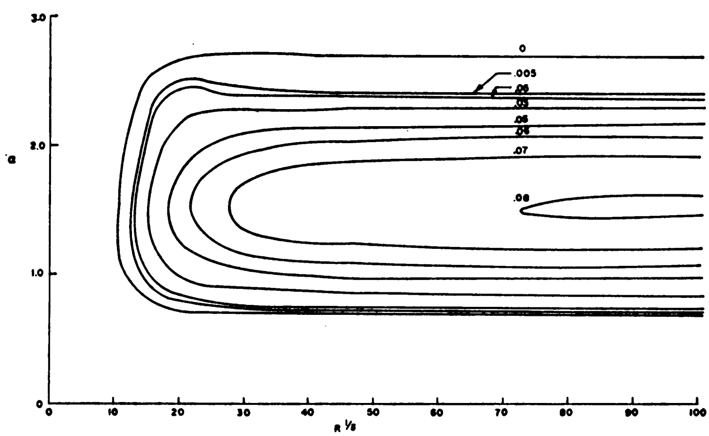

FIG. 2. Curves of constant amplification.

integration of the eight-by-eight system. In the case of the eight-by-eight system, the integration was in agreement with the result of the eigenvalue calculations in that the system appeared either stable or unstable depending upon whether the corresponding eigenvalue has a real part which was negative or positive. While no finite run can be considered conclusive, a corresponding integration of the twenty-by-twenty system for $\alpha = 1.5$ and $\nu = 10^{-4}$ indicated stability in agreement with the curve of Lin. This comparison is indicated in Fig. 3.

It turned out to be a major problem to develop machine codes and programs which would permit the calculation of the complex eigenvalues of the twenty-by-twenty unsymmetric complex matrices with sufficient accuracy. Fortunately this problem has recently been solved by the Computer Division of the Ramo-Wooldridge Corp. and it is possible to report a few numerical results at this time. The details and techniques necessary for these calculations will be reported elsewhere by members of the Computing Division. For the eight-by-eight case the computations were based on a routine programmed at Convair, San Diego, Calif., and kindly furnished the Ramo-Wooldridge Corp. The routine was derived from a paper by Leppert [11]. The computations were

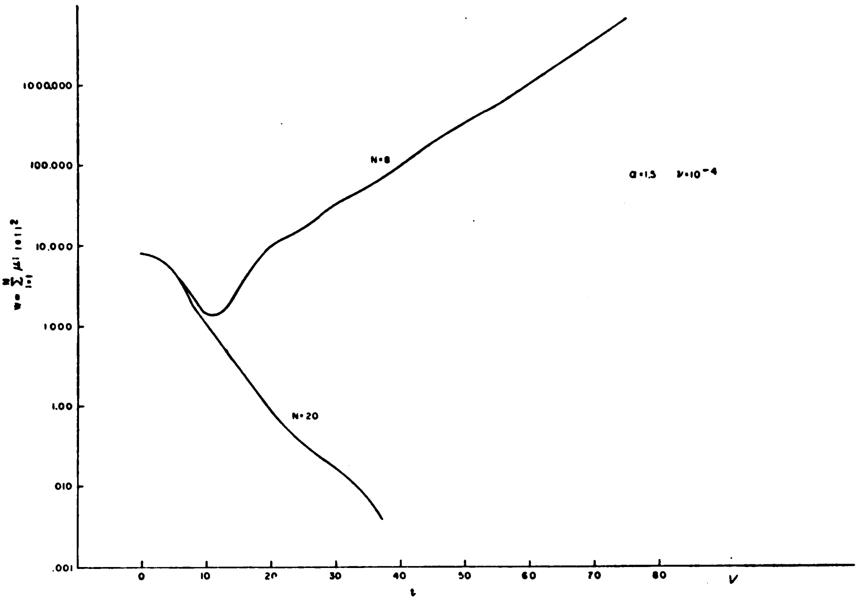

FIG. 3. Numerical integration of system of differential equations.

TABLE 1

Maximum of the real parts of the eigenvalue
$N = 8$

$\alpha \diagdown \nu$	6×10^{-4}	4×10^{-4}	2×10^{-4}	10^{-4}	10^{-5}	10^{-6}
0.7				−0.014652	−0.001594	−0.000159
0.8			−0.003858	+0.007952	+0.022386	+0.024191
0.9			+0.009241	+0.022902	+0.038748	+0.040589
1.0		−0.002491	+0.020052	+0.034649	+0.051136	+0.053007
1.1		+0.004910	+0.028679	+0.043931	+0.060904	+0.062810
1.2	−0.008579	+0.010234	+0.035119	+0.050919	+0.068348	+0.070295
1.3	−0.006618	+0.013419	+0.039433	+0.055734	+0.073613	+0.075610
1.4	−0.006915	+0.014613	+0.041783	+0.058538	+0.076864	+0.078909
1.5	−0.009132	+0.014120	+0.042410	+0.059539	+0.078274	+0.080370
1.6	−0.012737	+0.012323	+0.041588	+0.058961	+0.078027	+0.080175
1.7	−0.017162	+0.009597	+0.039580	+0.057007	+0.076278	+0.078473
1.8	−0.021947	+0.006241	+0.036604	+0.053832	+0.073121	+0.075358
1.9	−0.026806	+0.002468	+0.032828	+0.049534	+0.068558	+0.070828
2.0		−0.001592	+0.028380	+0.044152	+0.062457	+0.064752
2.1			+0.023364	+0.037698	+0.054473	+0.056768
2.2			+0.017894	+0.030233	+0.043764	+0.046007
2.3			+0.012123	+0.022052	+0.027878	+0.029702
2.4			+0.006261	+0.013936	+0.005404	+0.000594
2.5			+0.000555	+0.006936	+0.001535	+0.000155
2.6			−0.004764	+0.001538	+0.000449	+0.000045
2.7				−0.002473	−0.000118	−0.000012

carried out by assuming a given value of Reynolds number based on a channel half-width of unity and a maximum laminar velocity of unity, and exploring the system for different values of α. In this normalization the Reynolds number is numerically equal to the reciprocal of the viscosity. In none of the cases so far computed have we found more than one unstable eigenvalue for any given value of α, and our computations have furnished N distinct eigenvalues.

According to Lin [1] the minimum critical Reynolds number as defined above is 5300. The eight-by-eight system yielded an approximate minimum critical Reynolds number of around 2000, a result of the right order but quantitatively unsatisfactory. The currently available results shown in Table II for the twenty-by-twenty case imply

TABLE 2

Maximum of the real parts of the eigenvalue
$N = 20$

$\alpha \diagdown \nu$	2×10^{-4}	1.5×10^{-4}	10^{-4}
0.80			−0.000046
0.85			+0.002010
0.90		−0.000774	+0.003327
0.95	−0.003076	+0.000540	+0.003772
1.00	−0.002007	+0.001012	+0.003222
1.05		+0.000540	+0.001562
1.10		−0.000962	−0.001311

that the minimum critical Reynolds number of this order of approximation is not less than 5000 nor greater than 6600. Linear interpolation of them indicates a value of approximately 5800. This value may well show even closer agreement with the results of Lin after further exploration of the nose of the neutral curve. It should also be observed that the minimum critical Reynolds number occurs in the neighborhood of $\alpha = 1$, in general agreement with the results of Lin although the meager numerical results so far available make it impossible to decide the extent of the agreement. In view of the vast differences which exist between our techniques and those of the asymptotic theory the above agreement is remarkably good.

Since the methods used in this paper are not asymptotic it is not reasonable to expect that the agreement given by the twenty-by-twenty approximation near the nose would hold uniformly for larger Reynolds numbers. In fact it does not. While nothing like a complete neutral curve has been computed in this case, isolated calculations for a Reynolds number of 10,000 show that the lower branch lies between $\alpha = 9.5$ and $\alpha = 9$ while a similar calculation for a Reynolds number of 100,000 shows that the lower branch lies between $\alpha = 9$ and $\alpha = 8.5$. Thus while this is evidence that the lower branch is monotonically decreasing it is also evidence that the lower branch lies above that of the curve computed by Lin. No corresponding calculations are yet available for the upper branch at large Reynolds numbers. This merely reflects the fact that our method involves series expansions which, while convergent for all Reynolds number, require the use of more terms the higher the Reynolds number.

In spite of this inherent limitation we believe that we have furnished additional evidence of the instability of plane Poiseuille flow and have indicated an alternative general method of attack on the problems of stability which is not only capable of complete mathematical justification but which involves only techniques familiar to almost all workers in the field. Last but not least is the fact that potentially these same techniques appear capable of furnishing a rigorous mathematical way of discussing the mechanism of transition to turbulence since in principle their scope is not limited to linear problems. Their success in the linear problem of this paper lends credence to the hope that it may eventually be possible to verify that transition to turbulence takes place as envisaged by E. Hopf [6].

References

1. C. C. Lin, *The theory of hydrodynamical stability*, Cambridge University Press, 1955
2. G. K. Batchelor, *The theory of homogeneous turbulence*, Cambridge University Press, 1953
3. H. Emmons, *The laminar-turbulent transition in a boundary layer*, J. Aero. Sci. 18, 490-498 (1951)
4. H. Emmons, and A. E. Bryson, *The laminar-turbulent transition in a boundary layer*, Proc. 1st U.S. Natl. Congr. Appl. Mech., Edwards Bros., Ann Arbor, 1952
5. E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachrichten 4, 213-231 (1950)
6. E. Hopf, *A mathematical example displaying features of turbulence*, Communs. Pure and Appl. Math. 1, 87-123 (1952)
7. D. C. Lewis, *Infinite systems of ordinary differential equations with applications to certain second order partial differential equations*, Trans. Am. Math. Soc. 35, 792-823 (1933)
8. C. L. Dolph, *Quadratic functionals, quadratic forms, and transition to turbulence*, The Ramo-Wooldridge Corp. Rep., 15 June 1955
9. D. C. Lewis, *Remarks on the expansion of arbitrary functions in series appropriate to the problem of transition for plane Poiseuille flow*, The Ramo-Wooldridge Corp. Rep. No. GM-TM-97 1956
10. L. H. Thomas, *The stability of plane Poiseuille flow*, Phys. Rev. 91, 780-783 (1953)
11. L. Leppert, *A fractional series solution for characteristic values useful in some problems in airplane dynamics*, J. Aero. Sci. 22, 326-328 (1955)