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THE FINITE LENGTH PARTIAL FITTED JOURNAL BEARING WITH AXIAL OIL GROOVE*

BY

MELVIN J. JACOBSON

Rensselaer Polytechnic Institute

1. Introduction. Reynolds' lubrication theory gives rise to a differential equation for the pressure distribution in a lubricating film. This film is usually restricted to lie between a stationary surface and a moving surface. In any practical problem, the two surfaces are, of course, of finite dimensions. However, all early solutions of Reynolds' equation involved the assumption of one infinite dimension since this was equivalent to not permitting fluid flow in this direction, thereby reducing the problem to the solution of an ordinary differential equation. The results have severe limitations, however. In the case of journal bearings, the reduction of the axial length from infinity to a finite value can be expected to decrease the circumferential extent of the lubricating film, decrease the pressure developed throughout the film, and cause the point of closest approach of the journal and bearing surfaces to move closer to the plane of the applied load. These changes will in turn affect the performance characteristics of the journal bearing. For example, Shaw and Macks [1]** have compared properties of the full journal bearing predicted by the one-dimensional theory with measured values found by Brad-

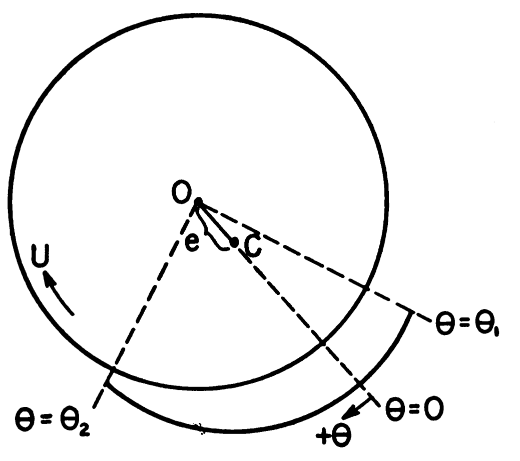

FIG. 1. Cross-section of a partial fitted journal bearing.

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**Numbers in square brackets refer to the bibliography at the end of the paper.

ford [2] and have concluded that the simple theory can lead to optimistic values of load capacity that err by more than 100 per cent.

It is the purpose of this paper to determine an analytical expression for the pressure distribution in the lubricating film of a finite length partial fitted journal bearing when the lubricant is introduced under constant pressure through an axial bearing groove. A partial bearing is one in which the bearing does not completely surround the journal, and the term "fitted" implies that the radii of the journal and bearing are equal. A cross-section of such a journal bearing is shown in Fig. 1 where O and C are the centers of the journal and bearing, respectively, e is the eccentricity, U is the linear velocity of the journal, and θ is an angle measured from the directed line of centers OC. The angle θ_1 is called the inlet edge of the bearing surface and θ_2 is the outlet edge.

2. Reynolds' equation and its solution. For an incompressible and constant viscosity lubricant, Reynolds' equation for a finite length journal bearing may be written as [3]

$$\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(h^3 \frac{\partial p}{\partial \theta}\right) + \frac{\partial}{\partial y}\left(h^3 \frac{\partial p}{\partial y}\right) = \frac{6\mu U}{r}\frac{dh}{d\theta}, \tag{2.1}$$

where p, μ, and h are the pressure, viscosity, and radial thickness of the lubricant, y is the axial variable measured from the central axial section of the journal bearing, and r is the common radius of the journal and bearing. If h is replaced by $e \cos \theta$ and if the dimensionless variable $w = y/r$ is introduced, then (2.1) reduces to

$$\frac{\partial}{\partial \theta}\left(\cos^3 \theta \frac{\partial p}{\partial \theta}\right) + \frac{\partial}{\partial w}\left(\cos^3 \theta \frac{\partial p}{\partial w}\right) = \beta \frac{d \cos \theta}{d\theta}, \tag{2.2}$$

where $\beta = 6\mu Ur/e^2$.

In order to remove the inhomogeneous term in (2.2) we set

$$p(\theta, w) = P(\theta, w) + f(\theta). \tag{2.3}$$

Substituting this expression into (2.2) and separating terms in θ alone from those in θ and w gives rise to two differential equations. The function $f(\theta)$ satisfies

$$\frac{d}{d\theta}\left(\cos^3 \theta \frac{df}{d\theta}\right) = \beta \frac{d \cos \theta}{d\theta} \tag{2.4}$$

and $P(\theta, w)$ satisfies

$$\frac{\partial}{\partial \theta}\left(\cos^3 \theta \frac{\partial P}{\partial \theta}\right) + \frac{\partial}{\partial w}\left(\cos^3 \theta \frac{\partial P}{\partial w}\right) = 0. \tag{2.5}$$

Equation (2.4) is simply the differential equation for the pressure distribution in the corresponding infinite length bearing and its solution is readily found to be

$$\begin{aligned} f(\theta) &= \beta \int_0^\theta \frac{d\theta}{\cos^2 \theta} + K_1 \int_0^\theta \frac{d\theta}{\cos^3 \theta} + K_2 \\ &= \beta g(\theta) + K_1 k(\theta) + K_2, \end{aligned} \tag{2.6}$$

where

$$g(\theta) = \tan \theta, \tag{2.7}$$

$$k(\theta) = \frac{\tan \theta}{2 \cos \theta} + \frac{1}{4} \log \left(\frac{1 + \sin \theta}{1 - \sin \theta}\right) \tag{2.8}$$

and K_1 and K_2 are constants of integration. The solution to (2.5) represents a term in the pressure solution which corrects for finiteness of the bearing or lubricant flow through the end axial sections. It may be determined by seeking a solution having the form

$$P(\theta, w) = Q(\theta)W(w). \tag{2.9}$$

If (2.9) is substituted into (2.5) and the variables are separated, it is found that W and Q satisfy the equations

$$W'' - \lambda^2 W = 0 \tag{2.10}$$

and

$$Q'' - 3 \tan \theta Q' + \lambda^2 Q = 0, \tag{2.11}$$

where $-\lambda^2$ was chosen to be the separation constant. Equation (2.11) together with the boundary conditions to be applied here is a problem of the Sturm-Liouville type and gives rise to a sequence of solutions Q_i corresponding to a sequence of real, non-negative eigenvalues λ_i^2. We therefore write the solution of (2.10) as

$$W_i = A_i \cosh (\lambda_i w) + B_i \sinh (\lambda_i w) \tag{2.12}$$

for $i \neq 0$ ($\lambda_i \neq 0$) and

$$W_0 = A_0 + B_0 w \tag{2.13}$$

for $i = 0$ ($\lambda_i = \lambda_0 = 0$). In order to solve (2.11) we make the substitution $z = \sin^2\theta$ which results in the equation

$$z(1 - z)Q_i'' + \left(\frac{1}{2} - \frac{5}{2}z\right)Q_i' + \frac{\lambda_i^2}{4} Q_i = 0. \tag{2.14}$$

This is the hypergeometric differential equation and has as its solution, for $i \neq 0$,

$$\begin{aligned} Q_i(\theta) &= C_i F(a_i , b_i ; \tfrac{1}{2}; \sin^2 \theta) + D_i \sin \theta F(a_i + \tfrac{1}{2}, b_i + \tfrac{1}{2}; \tfrac{3}{2}; \sin^2 \theta) \\ &= C_i F_i^{(1)}(\theta) + D_i \sin \theta F_i^{(2)}(\theta), \end{aligned} \tag{2.15}$$

where

$$a_i = \tfrac{1}{4}[3 + (9 + 4\lambda_i^2)^{1/2}] \tag{2.16}$$

and

$$b_i = \tfrac{1}{4}[3 - (9 + 4\lambda_i^2)^{1/2}]. \tag{2.17}$$

The above result is valid for $z = \sin^2\theta < 1$ and this is consistent with the physical requirement that θ must lie in the open interval $-\pi/2, \pi/2$ for there to be a non-zero film thickness at all points between the journal and the bearing. When $i = 0$ ($\lambda_i = \lambda_0 = 0$) the corresponding solution Q_0 is given by

$$\begin{aligned} Q_0 &= C_0 \int_0^\theta \frac{d\theta}{\cos^3 \theta} + D_0 \\ &= C_0\left[\frac{\tan \theta}{2 \cos \theta} + \frac{1}{4} \log \left(\frac{1 + \sin \theta}{1 - \sin \theta}\right)\right] + D_0 . \end{aligned} \tag{2.18}$$

The general solution of (2.5) may therefore be written as

$$P(\theta, w) = Q_0 W_0 + \sum_{i=1}^{\infty} W_i(w) Q_i(\theta). \tag{2.19}$$

3. Application of boundary conditions. The simplest example of a journal bearing having an axial oil inlet is that where the lubricant is introduced at constant pressure p_0 at the inlet edge $\theta = \theta_1$. For this problem, the boundary conditions to be satisfied by the pressure are

$$\begin{aligned} p(\theta_1, w) &= p_0, \\ p(\theta_2, w) &= 0, \\ p(\theta, w_0) &= p(\theta, -w_0) = 0, \end{aligned} \tag{3.1}$$

where $w_0 = L/r$ and $2L$ is the total length of the bearing. In terms of $f(\theta)$ and $P(\theta, w)$ we require that

$$\begin{aligned} f(\theta_1) &= p_0 \\ f(\theta_2) &= 0 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} P(\theta_1, w) &= P(\theta_2, w) = 0 \\ P(\theta, w_0) &= P(\theta, -w_0) = -f(\theta). \end{aligned} \tag{3.3}$$

With the use of these boundary conditions, we can determine the coefficients K_1, K_2, A_i, B_i, C_i, and D_i as well as the eigenvalues λ_i^2. From (2.6) and (3.2) it is easily shown that

$$K_1 = -\frac{\beta[g(\theta_2) - g(\theta_1)] + p_0}{k(\theta_2) - k(\theta_1)} \tag{3.4}$$

and

$$K_2 = -\frac{\beta[g(\theta_1)k(\theta_2) - g(\theta_2)k(\theta_1)] - p_0 k(\theta_2)}{k(\theta_2) - k(\theta_1)}. \tag{3.5}$$

In order to satisfy the first two conditions of (3.3) we require that each $Q_i(\theta)$ satisfy

$$Q_i(\theta_1) = Q_i(\theta_2) = 0. \tag{3.6}$$

From (2.15) and (3.6) we have the two equations

$$C_i F_i^{(1)}(\theta_1) + D_i \sin \theta_1 F_i^{(2)}(\theta_1) = 0 \tag{3.7}$$

and

$$C_i F_i^{(1)}(\theta_2) + D_i \sin \theta_2 F_i^{(2)}(\theta_2) = 0 \tag{3.8}$$

which may be combined to give

$$C_i[\sin \theta_1 F_i^{(1)}(\theta_2) F_i^{(2)}(\theta_1) - \sin \theta_2 F_i^{(1)}(\theta_1) F_i^{(2)}(\theta_2)] = 0 \tag{3.9}$$

and

$$D_i[\sin \theta_1 F_i^{(1)}(\theta_2) F_i^{(2)}(\theta_1) - \sin \theta_2 F_i^{(1)}(\theta_1) F_i^{(2)}(\theta_2)] = 0. \tag{3.10}$$

Since C_i and D_i cannot both be zero for a non-trivial solution we require that

$$\sin \theta_1 F_i^{(1)}(\theta_2) F_i^{(2)}(\theta_1) - \sin \theta_2 F_i^{(1)}(\theta_1) F_i^{(2)}(\theta_2) = 0. \tag{3.11}$$

This is an equation for the eigenvalues λ_i^2. The use of this result is reserved for Sec. 4. With the λ_i^2 determined, it is now possible to reduce the two unknown constants C_i and D_i in (2.15) to only one unknown by making use of (3.7) or (3.8). In order to determine C_0 and D_0 we apply (3.6) to (2.18) to get the two equations

$$C_0 \int_0^{\theta_1} \frac{d\theta}{\cos^3 \theta} + D_0 = 0 \tag{3.12}$$

and

$$C_0 \int_0^{\theta_1} \frac{d\theta}{\cos^3 \theta} + C_0 \int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos^3 \theta} + D_0 = 0. \tag{3.13}$$

Subtracting these expressions gives $C_0 = 0$ after which $D_0 = 0$ from (3.12). Consequently, there is no solution corresponding to $i = 0$.

The remaining coefficients may now be evaluated by making use of the last two conditions of (3.3). From (2.12), (2.19), and (3.3) we have the two equations

$$\sum_{i=1}^{\infty} [A_i \cosh (\lambda_i w_0) + B_i \sinh (\lambda_i w_0)] Q_i(\theta) = -f(\theta) \tag{3.14}$$

and

$$\sum_{i=1}^{\infty} [A_i \cosh (\lambda_i w_0) - B_i \sinh (\lambda_i w_0)] Q_i(\theta) = -f(\theta), \tag{3.15}$$

where it is to be understood that the single unknown coefficient in $Q_i(\theta)$ has been incorporated into the A_i and B_i. Before employing these expressions, we first note from (2.11) and (3.6) that the $Q_i(\theta)$ are orthogonal with respect to the weighing function $\cos^3\theta$ over the interval θ_1, θ_2. That is,

$$\int_{\theta_1}^{\theta_2} \cos^3 \theta Q_i(\theta) Q_j(\theta)\, d\theta = 0 \tag{3.16}$$

for $i \neq j$. If (3.14) and (3.15) are multiplied by $\cos^3\theta\, Q_j(\theta)$ and the results integrated over θ_1, θ_2 two equations in A_i and B_i result which, when solved simultaneously, give

$$A_i = -\frac{\int_{\theta_1}^{\theta_2} f(\theta) \cos^3 \theta Q_i(\theta)\, d\theta}{\cosh (\lambda_i w_0) \int_{\theta_1}^{\theta_2} \cos^3 \theta Q_i^2(\theta)\, d\theta} \tag{3.17}$$

and

$$B_i = 0. \tag{3.18}$$

The latter result makes the pressure solution an even function in w which is to be expected from the symmetry of the problem. This completes the evaluation of the coefficients, and the pressure solution

$$p(\theta, w) = f(\theta) + \sum_{i=1}^{\infty} W_i(w) Q_i(\theta) \tag{3.19}$$

is now completely determined for the case where the lubricant enters at constant pressure through the inlet edge of the journal bearing.

For the case where the lubricant enters through an axial bearing groove at θ_G where $\theta_1 < \theta_G < \theta_2$, the analysis is analogous to that preceding except that two pressure solutions of the type (3.19) must be developed, one for the range θ_1, θ_G and another for the range θ_G, θ_2. Once the pressure distribution has been determined, it is possible to compute such quantities as load carrying capacity, coefficient of friction, axial thrust, and rate of flow of lubricant. These will not be discussed here, however, since their definitions are readily available in any one of numerous lubrication texts.

4. Evaluation of eigenvalues. In the preceding development, (3.11) gives the eigenvalues λ_i^2. In this section one technique for determining them will be described which employs Whittaker's method for finding the root of smallest absolute value of a power series [4].

We first recall that the hypergeometric function may be expanded in the form

$$F(a, b; c; x) = 1 + \frac{a}{1}\frac{b}{c}x + \frac{a}{1}\frac{b}{c}\frac{a+1}{2}\frac{b+1}{c+1}x^2 + \cdots \tag{4.1}$$

The functions $F_i^{(1)}(\theta)$ and $F_i^{(2)}(\theta)$ are easily expressed in this form if we note that the coefficients in (4.1) contain terms of the type

$$(a + n)(b + n) = ab + n(a + b) + n^2, \tag{4.2}$$

where n is zero or any positive integer. From (2.16) and (2.17) we have that $ab = -\lambda_i^2/4$ and $a + b = 3/2$ so that, for $F_i^{(1)}(\theta)$,

$$(a + n)(b + n) = -\frac{\lambda_i^2}{4} + \frac{n}{2}(2n + 3). \tag{4.3}$$

Similarly, for $F_i^{(2)}(\theta)$,

$$(a + n)(b + n) = -\frac{\lambda_i^2}{4} + \frac{1}{2}(n + 2)(2n + 1). \tag{4.4}$$

The coefficients in the expansions of $F_i^{(1)}(\theta)$ and $F_i^{(2)}(\theta)$ are therefore polynomials in λ_i^2. These expansions may now be substituted into (3.11) and the series rearranged to give a power series in λ_i^2 where each coefficient is, in general, a rapidly converging infinite series. If we represent the result as

$$e_0 + e_1\lambda_i^2 + e_2(\lambda_i^2)^2 + e_3(\lambda_i^2)^3 + \cdots = 0, \tag{4.5}$$

then, by Whittaker's method, the smallest root λ_1^2 is given by

$$-\frac{e_0}{e_1} - \frac{e_0^2 e_2}{e_1}\begin{vmatrix} e_1 & e_2 \\ e_0 & e_1 \end{vmatrix}^{-1} - e_0^3\begin{vmatrix} e_2 & e_3 \\ e_1 & e_2 \end{vmatrix}\begin{vmatrix} e_1 & e_2 \\ e_0 & e_1 \end{vmatrix}^{-1}\begin{vmatrix} e_1 & e_2 & e_3 \\ e_0 & e_1 & e_2 \\ 0 & e_0 & e_1 \end{vmatrix}^{-1} - \cdots \tag{4.6}$$

The second root may now be found by factoring out the first root and repeating the procedure, and similarly for all other roots.

5. Asymptotic forms. Because of the complex nature of the eigenfunctions and Eq. (3.11) for the eigenvalues, it is convenient to develop approximate forms which are valid for large values of λ_i. Watson has derived asymptotic expansions for $F(a, b; c; x)$ when various combinations of the parameters are large [5]. When applied to $F_i^{(1)}(\theta)$, the appropriate expansion gives

$$F_i^{(1)}(\theta) \sim \frac{2\cos[\frac{1}{2}\theta(9+4\lambda_i^2)^{1/2}]}{(9+4\lambda_i^2)^{1/2}\cos^{3/2}\theta} \frac{\Gamma(\frac{1}{4}[1+\{9+4\lambda_i^2\}^{1/2}])}{\Gamma(\frac{1}{4}[-1+\{9+4\lambda_i^2\}^{1/2}])}, \tag{5.1}$$

where Γ represents the gamma function. If all functions in λ_i are expanded and only terms of order 1 are kept, (5.1) reduces to

$$F_i^{(1)}(\theta) \sim \frac{\cos\lambda_i\theta}{\cos^{3/2}\theta}. \tag{5.2}$$

Similarly, it can be shown that

$$F_i^{(2)}(\theta) \sim \frac{\sin\lambda_i\theta}{\lambda_i\sin\theta\cos^{3/2}\theta}. \tag{5.3}$$

If these results are substituted into (3.11), it is found that

$$\lambda_i \sim \frac{i\pi}{\theta_2-\theta_1}. \tag{5.4}$$

The same result can, of course, be determined from the general theory of the asymptotic behavior of the eigenvalues in the Sturm-Liouville problem [6].

6. Example. In order to give some indication of the nature of the solution determined above, we shall consider the simple case where the lubricant enters under constant pressure at $\theta = \theta_1$ and we shall suppose that $\theta_1 = -\theta_2$. That is, the clearance between the journal and the bearing surface will be taken to be symmetric about the line of centers $\theta = 0$. In addition, we shall take θ_2 to be small. This corresponds to a small bearing arc and makes λ_i large from (5.4), thereby justifying the use of the asymptotic forms developed above.

From (3.4) and (3.5) we have that

$$K_1 = -\frac{2\beta g(\theta_2) + p_0}{2k(\theta_2)} \tag{6.1}$$

and

$$K_2 = \frac{p_0}{2}, \tag{6.2}$$

so that, from (2.6),

$$f(\theta) = \beta g(\theta) - \frac{2\beta g(\theta_2) + p_0}{2k(\theta_2)} k(\theta) + \frac{p_0}{2}. \tag{6.3}$$

It will be recalled that this is not only a portion of the finite length bearing solution, but is also the pressure distribution in the corresponding infinite length bearing.

From (5.4) the square root of the first eigenfunction is given by

$$\lambda_1 \sim \frac{\pi}{2\theta_2} \tag{6.4}$$

and (2.15), (5.2), (5.3), and (6.4) give

$$Q_1(\theta) \sim C_1 \frac{\cos [\pi\theta/(2\theta_2)]}{\cos^{3/2} \theta} + D_2 \frac{2\theta_2 \sin [\pi\theta/(2\theta_2)]}{\pi \cos^{3/2} \theta}. \tag{6.5}$$

From (3.6) it is found that $D_1 = 0$ so that

$$Q_1(\theta) \sim C_1 \frac{\cos [\pi\theta/(2\theta_2)]}{\cos^{3/2} \theta}. \tag{6.6}$$

Similarly,

$$Q_2(\theta) \sim D_1 \frac{\theta_2 \sin (\pi\theta/\theta_2)}{\pi \cos^{3/2} \theta} \tag{6.7}$$

and the pressure solution from (3.19) is

$$\begin{aligned} p(\theta, w) \sim \beta g(\theta) - \frac{2\beta g(\theta_2) + p_0}{2k(\theta_2)} k(\theta) + \frac{p_0}{2} \\ + A_1 \cosh \left(\frac{\pi}{2\theta_2} w\right) \frac{\cos [\pi\theta/2\theta_2]}{\cos^{3/2} \theta} + A_2 \frac{\theta_2}{\pi} \cosh \left(\frac{\pi}{\theta_2} w\right) \frac{\sin (\pi\theta/\theta_2)}{\cos^{3/2} \theta} + \cdots , \end{aligned} \tag{6.8}$$

where $g(\theta)$ and $h(\theta)$ are given by (2.7) and (2.8), and, from (3.17),

$$A_1 = \frac{-p_0}{\theta_2 \cosh \left(\dfrac{\pi}{2\theta_2} w_0\right)} \int_0^{\theta_2} \cos^{3/2} \theta \cos \frac{\pi}{2} \frac{\theta}{\theta_2} \, d\theta \tag{6.9}$$

and

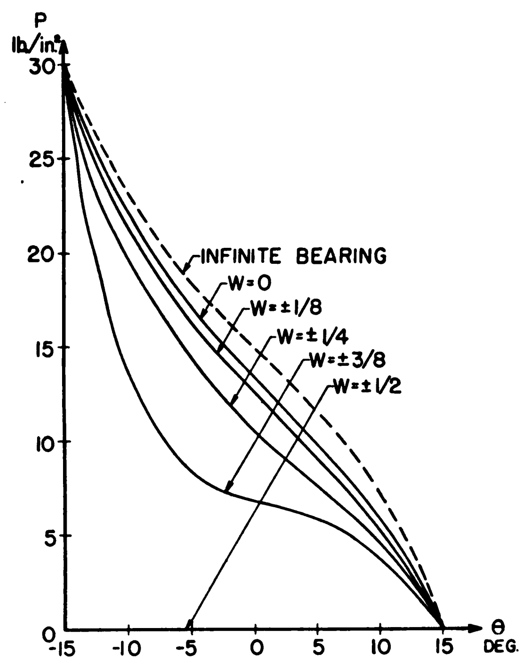

FIG. 2. Circumferential variation of pressure.

$$A_2 = \frac{-2\pi}{\theta_2^2 \cosh (\pi w_0/\theta_2)} \int_0^{\theta_2} [\beta g(\theta) + K_1 k(\theta)] \cos^{3/2} \theta \sin \pi \frac{\theta}{\theta_2} \, d\theta. \tag{6.10}$$

The integrals in these expressions are easily evaluated by successive integrations by parts if we recall that $\pi/2\theta_2$ was assumed large.

We shall now apply (6.8) to the specific problem where

radius of bearing $= r = 1$ in.
length of bearing $= 2L = 1$ in.
speed $= 2000$ r.p.m.
eccentricity $= e = .001$ in.
viscosity $= \mu = 1.2 \times 10^{-6}$ reyn
oil inlet pressure $= p_0 = 30$ lb./in.^{2}
outlet edge of bearing $= \theta_2 = 15°$

With these conditions, the pressure solution to two eigenfunctions reduces to

$$\begin{aligned} p(\theta, w) \sim 1510 \tan \theta - 387\left[2 \frac{\tan \theta}{\cos \theta} + \log \left(\frac{1 + \sin \theta}{1 - \sin \theta}\right)\right] + 15 \\ - 1.80 \cosh (6w) \frac{\cos 6\theta}{\cos^{3/2} \theta} + .0452 \cosh (12w) \frac{\sin 12\theta}{\cos^{3/2} \theta}. \end{aligned} \tag{6.11}$$

This result can be expected to represent a good approximation to the complete pressure solution for all w not close to $w_0 = \pm \frac{1}{2}$ since, in these neighborhoods, the hyperbolic functions in (6.11) are relatively large. However, in these cases, the behavior of the pressure solution is known from the boundary conditions.

Equation (6.11) appears graphically in Figs. 2 and 3. Figure 2 illustrates how the

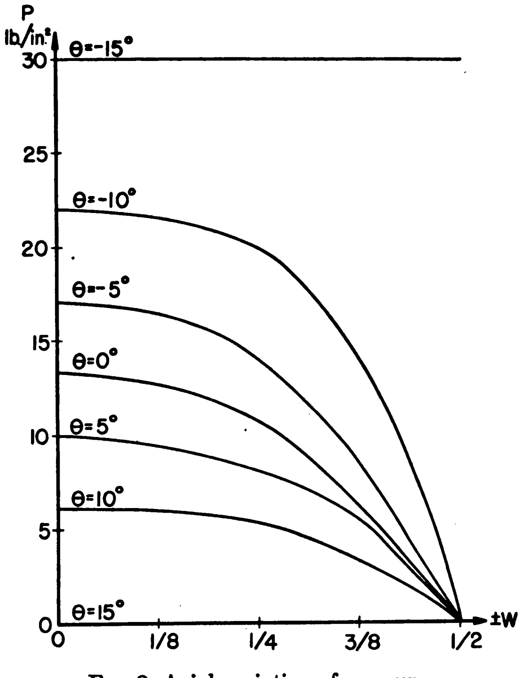

FIG. 3. Axial variation of pressure.

pressure distribution at various axial sections differs from that for the corresponding infinite bearing. While the distribution at the central axial section $w = 0$ does not differ greatly from the infinite distribution in this case, the difference would be considerable had the bearing length been taken shorter or the bearing arc larger. Figure 3 shows the variation in pressure as a function of axial distance for various fixed values of θ. The pressure decreases from a peak value at the central axial section to zero at both end sections. The corresponding pressure for the infinite bearing would be, of course, constant for all w and each fixed θ.

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