A CLASS OF PROBLEMS ON LONGITUDINAL VIBRATIONS*

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1. Introduction. Consider a bar of length l and constant area q formed of n parts with lengths l_k, densities ρ_k and Young's moduli E_k ($k = 1, 2, \cdots, n$). Denoting by $u_k(x, t)$ the displacement in longitudinal vibrations at the point x of the kth part and at any time t, the general form of the Laplace transform

$$U_k(x, p) = p \int_0^\infty e^{-pt} u_k(x, t)\, dt; \qquad k = 1, 2, \cdots, n$$

of $u_k(x, t)$ is

$$U_k(x, p) = A_k \cosh \frac{px}{a_k} + B_k \sinh \frac{px}{a_k} + W_k(x, p), \quad a_k = \sqrt{\frac{E_k}{\rho_k}}; \quad k = 1, 2, \cdots, n. \tag{1}$$

There exists a vast class of cases, important from the technical and physical points of view, where

$$W_k(s_k, p) = W_{k+1}(s_k, p), \qquad W_k'(s_k, p) = e_k W_{k+1}'(s_k, p);$$

$$s_k = \sum_{\kappa=1}^{k} l_\kappa, \qquad e_k = \frac{E_{k+1}}{E_k}, \qquad W_k' = \frac{dW_k}{dx}; \qquad k = 1, 2, \cdots, n-1. \tag{2}$$

All problems of this group may be treated in the following manner.

2. General theory. The subsidiary equations corresponding to the well-known conditions at the sections separating adjacent parts of the rod are (primes denote the x-derivatives):

$$U_k(s_k, p) = U_{k+1}(s_k, p), \qquad U_k'(s_k, p) = e_k U_{k+1}'(s_k, p); \qquad k = 1, 2, \cdots, n-1. \tag{3}$$

Substituting from (1) and using (2) we have

$$A_k \cosh \frac{ps_k}{a_k} + B_k \sinh \frac{ps_k}{a_k} = A_{k+1} \cosh \frac{ps_k}{a_{k+1}} + B_{k+1} \sinh \frac{ps_k}{a_{k+1}}$$

$$A_k \sinh \frac{ps_k}{a_k} + B_k \cosh \frac{ps_k}{a_k} = h_k\left(A_{k+1} \sinh \frac{ps_k}{a_{k+1}} + B_{k+1} \cosh \frac{ps_k}{a_{k+1}}\right), \tag{3.1}$$

$$h_k = \frac{a_{k+1}\rho_{k+1}}{a_k \rho_k}; \qquad k = 1, 2, \cdots, n-1,$$

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or, in the matrix form,

$$\left\| \begin{matrix} A_{k+1} \\ B_{k+1} \end{matrix} \right\| = M_k(p) \left\| \begin{matrix} A_k \\ B_k \end{matrix} \right\|; \qquad k = 1, 2, \cdots, n-1$$

$$M_k(p) = \left\| \begin{matrix} \cosh \frac{ps_k}{a_k} \cosh \frac{ps_k}{a_{k+1}} - \frac{1}{h_k} \sinh \frac{ps_k}{a_k} \sinh \frac{ps_k}{a_{k+1}}, & \sinh \frac{ps_k}{a_k} \cosh \frac{ps_k}{a_{k+1}} - \frac{1}{h_k} \cosh \frac{ps_k}{a_k} \sinh \frac{ps_k}{a_{k+1}} \\ -\cosh \frac{ps_k}{a_k} \sinh \frac{ps_k}{a_{k+1}} + \frac{1}{h_k} \sinh \frac{ps_k}{a_k} \cosh \frac{ps_k}{a_{k+1}}, & -\sinh \frac{ps_k}{a_k} \sinh \frac{ps_k}{a_{k+1}} + \frac{1}{h_k} \cosh \frac{ps_k}{a_k} \cosh \frac{ps_k}{a_{k+1}} \end{matrix} \right\|;$$

$$k = 1, 2, \cdots, n-1. \tag{4}$$

It follows from this that

$$\left\| \begin{matrix} A_{k+1} \\ B_{k+1} \end{matrix} \right\| = Q_k(p) \left\| \begin{matrix} A_1 \\ B_1 \end{matrix} \right\|; \qquad k = 1, 2, \cdots, n-1,$$

$$Q_k(p) = M_k(p) M_{k-1}(p) \cdots M_2(p) M_1(p); \qquad k = 1, 2, \cdots, n-1. \tag{4.1}$$

Writing

$$Q_k(p) = \| q_k^{(\mu\nu)}(p) \|, \qquad \mu, \nu = 1, 2; \qquad k = 1, 2, \cdots, n-1 \tag{5}$$

we obtain from (4.1)

$$A_{k+1} = q_k^{(11)}(p) A_1 + q_k^{(12)}(p) B_1, \qquad B_{k+1} = q_k^{(21)}(p) A_1 + q_k^{(22)}(p) B_1; \qquad k = 1, 2, \cdots, n-1$$

and the expressions (1) become

$$U_1(x, p) = A_1 \cosh \frac{px}{a_1} + B_1 \sinh \frac{px}{a_1} + W_1(x, p) \tag{6.1}$$

$$U_{k+1}(x, p) = \left[q_k^{(11)}(p) \cosh \frac{px}{a_{k+1}} + q_k^{(21)}(p) \sinh \frac{px}{a_{k+1}} \right] A_1 + \left[q_k^{(12)}(p) \cosh \frac{px}{a_{k+1}} + q_k^{(22)}(p) \sinh \frac{px}{a_{k+1}} \right] B_1 + W_{k+1}(x, p);$$

$$k = 1, 2, \cdots, n-1. \tag{6.2}$$

Transforming the conditions at the ends of the bar leads to the equations

$$\begin{aligned} \alpha_0 U_1(0, p) + \beta_0 U_1'(0, p) &= \gamma_0, \\ \alpha_n U_n(l, p) + \beta_n U_n'(l, p) &= \gamma_n, \end{aligned} \tag{7}$$

where $\alpha_0, \beta_0, \cdots, \beta_n, \gamma_n$ depend usually on p.

By means of (6.1), (6.2) relations (7) yield a system of two equations for the unknowns A_1, B_1. Solving it and substituting the resultant values A_1, B_1 into (6.1) and (6.2) leads to the following complicated expressions:

$$U_1(x, p) = \frac{1}{\Delta}\left(\Delta_1 \cosh \frac{px}{a_1} + \Delta_2 \sinh \frac{px}{a_1}\right) + W_1(x, p) \tag{8.1}$$

$$\begin{aligned} U_{k+1}(x, p) = \frac{1}{\Delta}\Bigg\{\Delta_1\left[q_k^{(11)}(p) \cosh \frac{px}{a_{k+1}} + q_k^{(21)}(p) \sinh \frac{px}{a_{k+1}}\right] \\ + \Delta_2\left[q_k^{(12)}(p) \cosh \frac{px}{a_{k+1}} + q_k^{(22)}(p) \sinh \frac{px}{a_{k+1}}\right]\Bigg\} + W_{k+1}(x, p); \\ k = 1, 2, \cdots, n-1 \end{aligned} \tag{8.2}$$

$$\begin{aligned} \Delta_1 = \Bigg[\left(\alpha_n \cosh \frac{pl}{a_n} + \frac{p}{a_n}\beta_n \sinh \frac{pl}{a_n}\right)q_{n-1}^{(12)}(p) + \left(\alpha_n \sinh \frac{pl}{a_n}\right. \\ \left. + \frac{p}{a_n}\beta_n \cosh \frac{pl}{a_n}\right)q_{n-1}^{(22)}(p)\Bigg][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)] \\ - \frac{p}{a_1}\beta_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)] \end{aligned}$$

$$\begin{aligned} \Delta_2 = \alpha_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)] - \Bigg[\left(\alpha_n \cosh \frac{pl}{a_n}\right. \\ \left. + \frac{p}{a_n}\beta_n \sinh \frac{pl}{a_n}\right)q_{n-1}^{(11)}(p) + \left(\alpha_n \sinh \frac{pl}{a_n}\right. \\ \left. + \frac{p}{a_n}\beta_n \cosh \frac{pl}{a_n}\right)q_{n-1}^{(21)}(p)\Bigg][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)] \end{aligned}$$

$$\begin{aligned} \Delta = \left[\alpha_0 q_{n-1}^{(12)}(p) - \frac{p}{a_1}\beta_0 q_{n-1}^{(11)}(p)\right]\left(\alpha_n \cosh \frac{pl}{a_n} + \frac{p}{a_n}\beta_n \sinh \frac{pl}{a_n}\right) \\ + \left[\alpha_0 q_{n-1}^{(22)}(p) - \frac{p}{a_1}\beta_0 q_{n-1}^{(21)}(p)\right]\left(\alpha_n \sinh \frac{pl}{a_n} + \frac{p}{a_n}\beta_n \cosh \frac{pl}{a_n}\right). \end{aligned} \tag{8.3}$$

The formulae (8.1) to (8.3) determine the Laplace transforms of the functions $u_k(x, t)$ which are to be found by the methods of operational calculus. The most difficult part of the problem lies in finding an explicit form of the matrix (4.1).

3. An interesting class of problems. We now proceed to treating problems characterised by the equations $h_k = 1$ ($k = 1, 2, \cdots, n-1$), or, by (1) and (3.1),

$$E_k\rho_k = \text{const}; \qquad k = 1, 2, \cdots, n. \tag{9}$$

In this case it is possible to calculate the matrix (4.1) and our former results (8.1) to (8.3) become simpler. Physically the simplification arises from the fact that there is no reflection at the joints of the different parts.

Using the notation

$$\sigma_k = s_k\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right), \qquad \tau_k = \sum_{\kappa=1}^{k} \sigma_\kappa; \qquad k = 1, 2, \cdots, n-1 \tag{10}$$

we easily obtain from (4) and (4.1)

$$
\begin{aligned}
M_k(p) &= \begin{Vmatrix} \cosh p\sigma_k, & \sinh p\sigma_k \\ \sinh p\sigma_k, & \cosh p\sigma_k \end{Vmatrix}; \qquad k = 1, 2, \cdots, n-1 \\
Q_2(p) &= \begin{Vmatrix} \cosh p\tau_2, & \sinh p\tau_2 \\ \sinh p\tau_2, & \cosh p\tau_2 \end{Vmatrix}.
\end{aligned} \tag{11}
$$

However,

$$
M_{\nu+1}(p) \begin{Vmatrix} \cosh p\tau_\nu, & \sinh p\tau_\nu \\ \sinh p\tau_\nu, & \cosh p\tau_\nu \end{Vmatrix} = \begin{Vmatrix} \cosh p\tau_{\nu+1}, & \sinh p\tau_{\nu+1} \\ \sinh p\tau_{\nu+1}, & \cosh p\tau_{\nu+1} \end{Vmatrix},
$$

so that (4.1) yields the closed forms

$$
Q_k(p) = \begin{Vmatrix} \cosh p\tau_k, & \sinh p\tau_k \\ \sinh p\tau_k, & \cosh p\tau_k \end{Vmatrix}; \qquad k = 1, 2, \cdots, n-1 \tag{12}
$$

of the fundamental matrices $Q_k(p)$.

Equations (8.3) then give

$$
\Delta_1 = \left[\alpha_n \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{p}{a_n}\beta_n \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right)\right][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)] - \frac{p}{a_1}\beta_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)]
$$

$$
\Delta_2 = \alpha_0[\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)] - \left[\alpha_n \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{p}{a_n}\beta_n \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right)\right][\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]
$$

$$
\Delta = \left(\alpha_0\alpha_n - \frac{p^2}{a_1 a_n}\beta_0\beta_n\right)\sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + p\left(\frac{\alpha_0\beta_n}{a_n} - \frac{\alpha_n\beta_0}{a_1}\right)\cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right) \tag{13}
$$

and the formulas (8.1), (8.2) become

$$
\begin{aligned}
U_1(x, p) = \frac{1}{\Delta}\Big\{&[\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]\left[\alpha_n \sinh p\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right)\right. \\
&\left. + \frac{p}{a_n}\beta_n \cosh p\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{a_1}\right)\right] + [\gamma_n - \alpha_n W_n(l, p) - \beta_n W_n'(l, p)] \\
&\cdot\left(\alpha_0 \sinh \frac{px}{a_1} - \frac{p}{a_1}\beta_0 \cosh \frac{px}{a_1}\right)\Big\} + W_1(x, p),
\end{aligned} \tag{14.1}
$$

$$
\begin{aligned}
U_{k+1}(x, p) = \frac{1}{\Delta}\Big\{&[\gamma_0 - \alpha_0 W_1(0, p) - \beta_0 W_1'(0, p)]\left[\alpha_n \sinh p\left(\frac{l}{a_n} - \frac{x}{a_{k+1}}\right.\right. \\
&\left.\left. + \tau_{n-1} - \tau_k\right) + \frac{p}{a_n}\beta_n \cosh p\left(\frac{l}{a_n} - \frac{x}{a_{k+1}} + \tau_{n-1} - \tau_k\right)\right] + [\gamma_n - \alpha_n W_n(l, p) \\
&- \beta_n W_n'(l, p)]\left[\alpha_0 \sinh p\left(\tau_k + \frac{x}{a_{k+1}}\right) - \frac{p}{a_1}\beta_0 \cosh p\left(\tau_k + \frac{x}{a_{k+1}}\right)\right]\Big\} \\
&+ W_{k+1}(x, p); \qquad k = 1, 2, \cdots, n-1.
\end{aligned} \tag{14.2}
$$

It is easy to verify that the expressions (13) to (14.2) satisfy in fact the boundary conditions (7).

Formulae (14.1), (14.2) represent the Laplace transforms of $u_k(x, t)$ and from this we obtain in each concrete case the completed solution by the methods of operational calculus. Several examples are given in the following section.

4. Special problems. We illustrate the preceding theoretical considerations by typical examples generalising the results given occasionally in the literature [1]. In order to have reasonable expressions we assume (9) to be valid.

All the calculations needed in each case proceed by five stages. First, one applies the Laplace transformation to the fundamental equations governing the problem and to the initial conditions. This yields the general form of $U_k(x, p)$, as indicated in (1), and especially also the expressions $W_k(x, p)$.

When the relations (2) hold, we write down the subsidiary equations corresponding to the boundary conditions at $x = 0$, $x = l$ and obtain the values α_0, β_0, γ_0, α_n, β_n, γ_n. The third and fourth steps consist in calculating the expression Δ by (13) and the Laplace transforms $U_k(x, p)$ according to (14.1), (14.2). Finally, we deduce the wanted solution $u_k(x, t)$ by the methods of operational calculus.

4.1. Vibrations of a bar under its own weight. The upper end $x = 0$ is fixed, the lower $x = l$ supported so that the displacement is zero at all points. At $t = 0$ the end $x = l$ is released.

Putting $s_0 = 0$ fundamental equations of the problem with belonging initial conditions are

$$\frac{\partial^2 u_k}{\partial t^2} = a_k^2 \frac{\partial^2 u_k}{\partial x^2} + g, \qquad s_{k-1} < x < s_k, \qquad t > 0; \qquad k = 1, 2, \cdots, n \tag{15.1}$$

$$u_k(x, 0) = 0, \qquad \frac{\partial u_k(x, 0)}{\partial t} = 0, \qquad s_{k-1} < x < s_k; \qquad k = 1, 2, \cdots, n. \tag{15.2}$$

Solving the corresponding subsidiary equations with respect to $U_k(x, p)$ and comparing with (1) gives

$$W_k(x, p) = \frac{g}{p^2}; \qquad k = 1, 2, \cdots, n. \tag{15.3}$$

Thus, the relations (2) hold.

The boundary conditions at $x = 0$ and $x = l$ are

$$u_1(0, t) = 0, \qquad \frac{\partial u_n(l, t)}{\partial x} = 0; \qquad t > 0 \tag{16.1}$$

and this leads to

$$U_1(0, p) = 0, \qquad U_n'(l, p) = 0.$$

We therefore have

$$\alpha_0 = \beta_n = 1, \qquad \beta_0 = \gamma_0 = \alpha_n = \gamma_n = 0 \tag{16.2}$$

and (13) gives

$$\Delta = \frac{p}{a_n} \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right). \tag{16.3}$$

It then follows from (14.1), (14.2) that

$$U_1(x, p) = \frac{g}{p^2}\left[1 - \frac{\cosh p\left(\dfrac{l}{a_n} + \tau_{n-1} - \dfrac{x}{a_1}\right)}{\cosh p\left(\dfrac{l}{a_n} + \tau_{n-1}\right)}\right],$$

$$U_{k+1}(x, p) = \frac{g}{p^2}\left[1 - \frac{\cosh p\left(\dfrac{l}{a_n} - \dfrac{x}{a_{k+1}} + \tau_{n-1} - \tau_k\right)}{\cosh p\left(\dfrac{l}{a_n} + \tau_{n-1}\right)}\right]; \quad k = 1, 2, \cdots, n-1 \quad (17.1)$$

and from this we find (with the aid of a table of Laplace transforms or by contour integration) the complete solution in the form

$$u_1(x, t) = \frac{gx}{a_1}\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{2a_1}\right) - \frac{16g}{\pi^3}\left(\frac{l}{a_n} + \tau_{n-1}\right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos \frac{(2n+1)\pi\left(\dfrac{l}{a_n} + \tau_{n-1} - \dfrac{x}{a_1}\right)}{2\left(\dfrac{l}{a_n} + \tau_{n-1}\right)} \cdot \cos \frac{(2n+1)\pi t}{2\left(\dfrac{l}{a_n} + \tau_{n-1}\right)}$$

$$u_{k+1}(x, t) = g\left(\frac{x}{a_{k+1}} + \tau_k\right)\left(\frac{l}{a_n} + \tau_{n-1} - \frac{x}{2a_{k+1}} - \frac{\tau_k}{2}\right) - \frac{16g}{\pi^3}\left(\frac{l}{a_n} + \tau_{n-1}\right)^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos \frac{(2n+1)\pi\left(\dfrac{l}{a_n} + \tau_{n-1} - \dfrac{x}{a_{k+1}} - \tau_k\right)}{2\left(\dfrac{l}{a_n} + \tau_{n-1}\right)} \cos \frac{(2n+1)\pi t}{2\left(\dfrac{l}{a_n} + \tau_{n-1}\right)};$$

$$k = 1, 2, \cdots, n-1. \quad (17.2)$$

4.2. Bar with a mass at the end. The end $x = 0$ is fixed, a mass m is attached to the other end $x = l$. The bar is initially stretched by the axial force qS and at $t = 0$ its end $x = l$ is released.

The fundamental equations

$$\frac{\partial^2 u_k}{\partial t^2} = a_k^2 \frac{\partial^2 u_k}{\partial x^2}, \quad s_{k-1} < x < s_k, \quad t > 0; \quad k = 1, 2, \cdots, n \quad (18.1)$$

with

$$u_k(x, 0) = S\left(\sum_{\kappa=1}^{k-1} \frac{l_\kappa}{E_\kappa} + \frac{x - s_{k-1}}{E_k}\right), \quad \frac{\partial u_k(x, 0)}{\partial t} = 0, \quad s_{k-1} < x < s_k;$$

$$k = 1, 2, \cdots, n \quad (18.2)$$

and with the boundary conditions

$$u_1(0, t) = 0$$

$$m \frac{\partial^2 u_n(l, t)}{\partial t^2} = -qE_n \frac{\partial u_n(l, t)}{\partial x}, \quad u_n(l, 0) = S \sum_{\kappa=1}^{n} \frac{l_\kappa}{E_\kappa}, \quad \frac{\partial u_n(l, 0)}{\partial t} = 0; \quad t > 0 \tag{18.3}$$

give, as in the preceding case

$$\begin{gathered} W_k(x, p) = S\left(\sum_{\kappa=1}^{k-1} \frac{l_\kappa}{E_\kappa} + \frac{x - s_{k-1}}{E_k}\right), \qquad \alpha_0 = 1, \qquad \beta_0 = \gamma_0 = 0, \\ \alpha_n = 1, \qquad \beta_n = \frac{qE_n}{mp^2}, \qquad \gamma_n = S \sum_{\kappa=1}^{n} \frac{l_\kappa}{E_\kappa}. \end{gathered} \tag{19}$$

Thus, we find without difficulty

$$\Delta = \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{qE_n}{mpa_n} \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right) \tag{20}$$

and formulae (14.1), (14.2) yield

$$\begin{aligned} U_1(x, p) &= \frac{Sx}{E_1} - \frac{qS}{m} \cdot \frac{\sinh \frac{px}{a_1}}{p\left[p \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{qE_n}{ma_n} \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right)\right]} \\ U_{k+1}(x, p) &= S\left(\sum_{\kappa=1}^{k} \frac{l_\kappa}{E_\kappa} + \frac{x - s_k}{E_{k+1}}\right) \\ &\quad - \frac{qS}{m} \cdot \frac{\sinh p\left(\tau_k + \frac{x}{a_{k+1}}\right)}{p\left[p \sinh p\left(\frac{l}{a_n} + \tau_{n-1}\right) + \frac{qE_n}{ma_n} \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right)\right]}; \\ &\qquad k = 1, 2, \cdots, n-1. \end{aligned} \tag{21.1}$$

The solution is

$$\begin{aligned} u_1(x, t) &= \frac{Sx}{E_1}\left(1 - \frac{a_n}{a_1}\frac{E_1}{E_n}\right) + \frac{2qS}{m}\left(\frac{l}{a_n} + \tau_{n-1}\right)^2 \\ &\cdot \sum_{\rho=1}^{\infty} \frac{\sec \delta_\rho}{\delta_\rho\left[\left(\frac{l}{a_n} + \tau_{n-1}\right)\frac{qE_n}{ma_n} + \left(\frac{l}{a_n} + \tau_{n-1}\right)^2\left(\frac{qE_n}{ma_n}\right)^2 + \delta_\rho^2\right]} \sin \frac{\frac{\delta_\rho x}{a_1}}{\frac{l}{a_n} + \tau_{n-1}} \cos \frac{\delta_\rho t}{\frac{l}{a_n} + \tau_{n-1}} \\ u_{k+1}(x, t) &= S\left[\sum_{\kappa=1}^{k} \frac{l_\kappa}{E_\kappa} - \frac{s_k}{E_{k+1}} - \frac{a_n \tau_k}{E_n} + \frac{x}{E_{k+1}}\left(1 - \frac{a_n}{a_{k+1}}\frac{E_{k+1}}{E_n}\right)\right] \\ &+ \frac{2qS}{m}\left(\frac{l}{a_n} + \tau_{n-1}\right)^2 \sum_{\rho=1}^{\infty} \frac{\sec \delta_\rho}{\delta_\rho\left[\left(\frac{l}{a_n} + \tau_{n-1}\right)\frac{qE_n}{ma_n} + \left(\frac{l}{a_n} + \tau_{n-1}\right)^2\left(\frac{qE_n}{ma_n}\right)^2 + \delta_\rho^2\right]} \\ &\cdot \sin \frac{\delta_\rho\left(\tau_k + \frac{x}{a_{k+1}}\right)}{\frac{l}{a_n} + \tau_{n-1}} \cos \frac{\delta_\rho t}{\frac{l}{a_n} + \tau_{n-1}}; \qquad k = 1, 2, \cdots, n-1. \end{aligned} \tag{21.2}$$

The $\delta_\rho (\rho = 1, 2, \cdots)$ are the positive roots of

$$\delta \tan \delta = \left(\frac{l}{a_n} + \tau_{n-1}\right) \frac{qE_n}{ma_n}. \tag{21.3}$$

4.3. Collision of two rods. Two equal rods of the kind treated above move longitudinally in opposite directions with equal speeds v and collide at $t = 0$, at the origin $x = 0$. By symmetry we consider only the rod $0 < x < l$ and so long as the rods are in contact we have the equations

$$\frac{\partial^2 u_k}{\partial t^2} = a_k^2 \frac{\partial^2 u_k}{\partial x^2}, \quad s_{k-1} < x < s_k, \quad t > 0; \quad k = 1, 2, \cdots, n \tag{22.1}$$

with the initial conditions

$$u_k(x, 0) = 0, \quad \frac{\partial u_k(x, 0)}{\partial t} = -v, \quad s_{k-1} < x < s_k; \quad k = 1, 2, \cdots, n \tag{22.2}$$

and with

$$u_1(0, t) = 0, \quad \frac{\partial u_n(l, t)}{\partial x} = 0, \quad t > 0. \tag{22.3}$$

From this we obtain, as before

$$W_k(x, p) = -\frac{v}{p} (k = 1, 2, \cdots, n); \quad \alpha_0 = \beta_n = 1, \quad \beta_0 = \gamma_0 = \alpha_n = \gamma_n = 0 \tag{23.1}$$

$$\Delta = \frac{p}{a_n} \cosh p\left(\frac{l}{a_n} + \tau_{n-1}\right), \tag{23.2}$$

and formulae (14.1), (14.2) become

$$U_1(x, p) = -\frac{v}{p}\left[1 - \frac{\cosh p\left(\dfrac{l}{a_n} + \tau_{n-1} - \dfrac{x}{a_1}\right)}{\cosh p\left(\dfrac{l}{a_n} + \tau_{n-1}\right)}\right]$$

$$U_{k+1}(x, p) = -\frac{v}{p}\left[1 - \frac{\cosh p\left(\dfrac{l}{a_n} + \tau_{n-1} - \tau_k - \dfrac{x}{a_{k+1}}\right)}{\cosh p\left(\dfrac{l}{a_n} + \tau_{n-1}\right)}\right];$$

$$k = 1, 2, \cdots, n-1. \tag{24.1}$$

Hence, the solution is

$$u_1(x, t) = \frac{8v}{\pi^2}\left(\frac{l}{a_n} + \tau_{n-1}\right) \cdot \sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{(2r+1)^2} \cos \frac{(2r+1)\pi\left(\dfrac{l}{a_n} + \tau_{n-1} - \dfrac{x}{a_1}\right)}{2\left(\dfrac{l}{a_n} + \tau_{n-1}\right)} \sin \frac{(2r+1)\pi t}{2\left(\dfrac{l}{a_n} + \tau_{n-1}\right)}$$

$$u_{k+1}(x, t) = \frac{8v}{\pi^2}\left(\frac{l}{a_n} + \tau_{n-1}\right)$$

$$\cdot \sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{(2r+1)^2} \cos \frac{(2r+1)\pi\left(\dfrac{l}{a_n} + \tau_{n-1} - \tau_k - \dfrac{x}{a_{k+1}}\right)}{2\left(\dfrac{l}{a_n} + \tau_{n-1}\right)} \sin \frac{(2r+1)\pi t}{2\left(\dfrac{l}{a_n} + \tau_{n-1}\right)};$$

$$k = 1, 2, \cdots, n-1. \tag{24.2}$$

5. Final remarks. For $n = 1$ the formulae for $u_1(x, t)$ in (17.2), (21.2) and (24.2) change into the well-known elementary results given in the literature [1].

Our preceding deductions may serve as a kind of guide in treating other problems on vibrations of composite continua. A special example has been discussed in detail in the author's paper given in the literature [2]. The method of matrix analysis can be used also in treating other important questions of engineering such as torsional oscillations [3], conduction of heat [4, 5], etc.

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