ASYMPTOTIC SOLUTIONS OF A CLASS OF ELASTIC SHELLS OF REVOLUTION WITH VARIABLE THICKNESS*

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1. Introduction. Since the pioneer work of H. Reissner [1] and Meissner [2, 3] on the small axisymmetric deformation of thin elastic shells of revolution, numerous investigations on this subject have dealt with shells of uniform thickness, but comparatively little attention has been given to shells of non-uniform thickness. Meissner in [3], following an analysis of shells of revolution of variable thickness (to which further reference will be made presently), treated the case of conical shells of linearly varying thickness in detail. Other contributions on shells of revolution of variable thickness have been made by Ekström [4] and Spotts [5] for spherical shells, and by E. Reissner [6] for shallow shells of revolution. In a recent paper by the present authors [7], the differential equations of shells of revolution with small axisymmetric displacements, as given by E. Reissner [6], were combined into a single complex differential equation, a solution of which, valid at a turning point[1] of the differential equation, may be obtained by a more recent method of asymptotic integration due to Langer [8]. These results were subsequently applied to ellipsoidal shells of uniform thickness [9], yielding a solution valid at the apex of the shell where a regular singularity occurs in the differential equation.

In the present paper, with reference to the differential equations given in [6] and [7], a class of shells of revolution of variable thickness is further examined. First, the implications of the thickness variation specified in [7] which led to the complex differential equation in a form amenable to treatment by Langer's method of asymptotic integration are studied in detail and then asymptotic solutions are given which are valid at turning points of the differential equation.

2. The basic equations of shells of revolution. With the use of cylindrical coordinates r, θ, z, the parametric equation of the middle surface of the shell may be written as

$$r = r(\xi), \qquad z = z(\xi). \tag{2.1}$$

Denoting by ϕ the inclination of the tangent to the meridian of the shell, then

$$r' = \alpha \cos \phi, \qquad z' = \alpha \sin \phi, \tag{2.2}$$

$$r_1 = \alpha/\phi', \qquad r_2 = r/\sin \phi, \tag{2.3}$$

where

$$\alpha = [(r')^2 + (z')^2]^{1/2}, \tag{2.4}$$

*Received June 19, 1956. The results presented in this paper were obtained in the course of research sponsored by the Army Office of Ordnance Research under Contract DA-20-018-ORD-12099 with the University of Michigan.

[1]Such points are defined here as ones at which Ψ^2 (see equation (2.6)) vanishes to some degree $\gamma > -2$ and/or Λ has a simple or a double pole. It should be mentioned that this definition of turning point (sometimes called transition point) includes the usual definition given by Langer where γ is restricted to be a positive integer.

r_1 and r_2 are the principal radii of curvature of the middle surface, and prime denotes differentiation with respect to ξ.

The appropriate expressions for the stress resultants N_ξ and N_θ, the stress couples M_ξ and M_θ, and the transverse shear stress resultant Q are given in [6] as well as [7] and will not be repeated here. We recall, however, that as in [6], it is convenient to express N_ξ and Q in terms of "horizontal" and "vertical" stress resultants, H and V, given by

$$\alpha N_\xi = r'H + z'V, \qquad \alpha Q = -z'H + r'V. \tag{2.5}$$

By proper elimination between the stress strain relations and the differential equations of equilibrium and compatibility, E. Reissner in [6] deduced two coupled second-order differential equations governing the small axisymmetric deformation of shells of revolution which, while they differ only slightly from the previous formulation of the theory due to H. Reissner and Meissner, contain certain advantages. It was shown in [7], that the differential equations of shells of revolution, as given in [6], may be combined into the following complex differential equation:

$$W'' + [i^3\mu^2\Psi^2(\xi) + \Lambda(\xi)]W = \left(\frac{h}{h_0}\cdot\frac{r}{\alpha}\right)^{1/2} f(\xi)[F + ikG], \tag{2.6}$$

provided k, given by

$$k = -\frac{i}{\mu^2}\left(\nu\lambda - \frac{\delta}{2}\right) + \left\{1 - \left[\frac{1}{\mu^2}\left(\nu\lambda - \frac{\delta}{2}\right)\right]^2\right\}^{1/2} \tag{2.7}$$

is a constant.

The various quantities appearing in (2.6) and (2.7) are defined by

$$W = \left(\frac{h}{h_0}\right)^{3/2}\left(\frac{r}{\alpha}\right)^{1/2}(\beta + ik\psi), \qquad \psi = \frac{mrH}{Eh^2}$$

$$\Psi^2 = \left(k + i\frac{\nu\lambda}{\mu^2}\right)\left(\frac{h_0}{h}\right)f(\xi);$$

$$\Lambda = -\frac{1}{2}\frac{(r/\alpha)''}{(r/\alpha)} + \frac{1}{4}\left[\frac{(r/\alpha)'}{(r/\alpha)}\right]^2 - \left(\frac{r'}{r}\right)^2 - \frac{3}{2}\frac{r/\alpha)'}{(r/\alpha)}\frac{h'}{h} - \frac{3}{4}\left(\frac{h'}{h}\right)^2 - \frac{3}{2}\frac{h''}{h},$$

$$\mu^2 f(\xi) = \frac{\alpha^2 m}{r_2 h_0}, \qquad m^2 = 12(1 - \nu^2), \tag{2.8}$$

$$\lambda = \left[\frac{h_0}{h}f\right]^{-1}\left\{\frac{(r'/\alpha)'}{(r/\alpha)} + 3\frac{r'}{r}\frac{h'}{h}\right\},$$

$$\delta = 2\left[\frac{h_0}{h}f\right]^{-1}\left(\frac{h''}{h} + 2\nu\frac{r'}{r}\frac{h'}{h} + \frac{(r/\alpha)'}{(r/\alpha)}\frac{h'}{h}\right),$$

where β is the negative change in ϕ due to deformation; h_0 is the value of the thickness h at some reference section; F and G, as in [7], are functions of the load intensity; and E and ν are Young's modulus and Poisson's ratio, respectively.

As pointed out in [7], the condition imposed for the validity of (2.6), namely that k or equivalently $(\nu\lambda - \delta/2)$ is a constant (say $-K/h_0$) results in the following differential equation for the thickness h:

$$\left(\frac{r}{\alpha}\right)h'' + \left(\frac{r}{\alpha}\right)'h' - \nu\left[\left(\frac{r'}{\alpha}\right)h' + \left(\frac{r'}{\alpha}\right)'h\right] = \left(\frac{r}{\alpha}\right)Kf(\xi) \tag{2.9}$$

the solution of which, since

$$\int f(\xi)\,\frac{r}{\alpha}\,d\xi = \frac{m}{\mu^2 h_0}\,z,$$

may be written as

$$h = r^{\nu}\left\{K_1 \int r^{-(1+\nu)}\alpha z\,d\xi + c_1 + c_2 \int r^{-(1+\nu)}\alpha\,d\xi\right\}, \tag{2.10}$$

where c_1 and c_2 are constants of integration, and $K_1 = Km/\mu^2 h_0$. It should be mentioned here that the form of thickness variation specified by (2.10) was first obtained by Meissner [3].

In order to isolate the terms in the coefficient functions of W which involve derivatives of h, we modify (2.6) into a new normal form as follows. From (2.9),

$$-\frac{3}{2}\left[\frac{h''}{h} + \frac{(r/\alpha)'}{(r/\alpha)}\frac{h'}{h}\right] = -\frac{3}{2}K\frac{f}{h} - \frac{3}{2}\nu\frac{(r'/\alpha)'}{(r/\alpha)} - \frac{3}{2}\nu\frac{r'}{r}\frac{h'}{h} \tag{2.11}$$

which expression occurs in Λ. Also, the function Ψ^2 in (2.8) may be written as

$$\Psi^2 = k\,\frac{h_0}{h}\,f + i\,\frac{\nu}{\mu^2}\left\{\frac{(r'/\alpha)'}{(r/\alpha)} + 3\,\frac{r'}{r}\frac{h'}{h}\right\}. \tag{2.12}$$

Substitution of (2.11) and (2.12) into (2.6) after considerable manipulation results in

$$W'' + [i^3\mu^2\Psi_1^2 + (\Lambda_0 + \Lambda_h)]W = R, \tag{2.13}$$

where

$$\Psi_1^2 = \left(k - \frac{3}{2}\,i\,\frac{K_1}{m}\right)\left(\frac{h_0}{h}\right)f(\xi), \tag{2.14}$$

$$\Lambda_0 = -\frac{3}{4}\left(\frac{r'}{r}\right)^2 - \frac{1}{4}\left(\frac{\alpha'}{\alpha}\right)^2 + \frac{1}{2}\left(\frac{\alpha'}{\alpha}\right)' - \frac{(1+\nu)}{2}\frac{(r'/\alpha)'}{(r/\alpha)}, \tag{2.15}$$

$$\Lambda_h = \frac{3}{2}\nu\frac{h'}{h}\left(\frac{r'}{r} - \frac{1}{2\nu}\frac{h'}{h}\right) \tag{2.16}$$

and R denotes the right hand side of (2.6).

We choose $\xi = \phi$ (i.e., $r_1 = \alpha$), and proceed to examine the character of h as given by (2.10) in the neighborhood of $\phi = 0$, and the corresponding behavior of the coefficient functions of W in (2.13), with the aim of obtaining solutions of (2.13) valid at turning points of the differential equation. To this end let the principal radii of curvature be of the form

$$\begin{aligned} r_1 &= \phi^{b_1}\sum_{n=0}^{\infty} p_n\phi^n, \\ r_2 &= \phi^{b_2}\sum_{n=0}^{\infty} q_n\phi^n, \end{aligned} \tag{2.17}$$

where the power series representing non-vanishing analytic functions are uniformly convergent in the real interval I_ϕ: $0 \leq \phi \leq \phi^*$. Substituting (2.17) into the expression for r' involving both r_1 and r_2 (by (2.2) and (2.3)), and equating the coefficients of like powers in ϕ lead to two cases. In one case $b_2 = -1$ (and $b_1 = 0, 1$) and in the other $b_1 = b_2 = b \neq -1$. These cases are considered in detail in Secs. 3 and 4.

3. Case A: Behavior of shell thickness specified by (2.10) and solution of Eq. (2.13), when $b_2 = -1$. In this case two possibilities occur: (1) $b_1 = -b_2 = 1$ and $q_1 = 0$, and (2) $b_2 = -1$, $b_1 = 0$, and $q_1 = p_0$.

(1) For the first possibility, it follows that

$$r = \sum_{n=0}^{\infty} q_n^* \phi^n \tag{3.1}$$

and by (2.10)

$$h = \sum_{n=0}^{\infty} h_n \phi^n, \tag{3.2}$$

where

$$q_0^* = q_0, \qquad q_1^* = q_1, \qquad q_2^* = q_2 - \frac{q_0}{3!}, \qquad \text{etc} \tag{3.3a}$$

$$h_0 = h(0) = c_1 q_0^{\nu}, \qquad h_1 = 0, \qquad h_2 = \left(\nu h_0 \frac{q_2^*}{q_0} + \frac{1}{2} c_2 \frac{p_0}{q_0}\right), \qquad \text{etc.} \tag{3.3b}$$

We note that $\lim_{\phi\to 0} h' = \lim_{\phi\to 0} (2h_2 \phi) = 0$ and that h in (3.2) is a non-vanishing analytic function. By (3.1) and (3.2),

$$\begin{aligned} \frac{r'}{r} &= \phi \frac{\sum_{n=0}^{\infty} (n+2) q_{n+2}^* \phi^n}{\sum_{n=0}^{\infty} q_n^* \phi^n}, \\ \frac{h'}{h} &= \phi \frac{\sum_{n=0}^{\infty} (n+2) h_{n+2} \phi^n}{\sum_{n=0}^{\infty} h_n \phi^n}, \end{aligned} \tag{3.4}$$

and from (2.8)

$$\mu^2 = m \left(\frac{p_0}{q_0}\right)\left(\frac{p_0}{h_0}\right), \qquad f(\phi) = (\phi^3) \frac{\left[\sum_{n=0}^{\infty} \frac{p_n}{p_0} \phi^n\right]^2}{\sum_{n=0}^{\infty} \frac{q_n}{q_0} \phi^n}. \tag{3.5}$$

Substitution of (3.4) and (3.5) into (2.14), (2.15) and (2.16) results in

$$\begin{aligned} \Psi_1^2 &= \phi^3 \Psi_1^{*2}, \qquad \Psi_1^{*2}(0) = k - \frac{3}{2} i \frac{K_1}{m} \\ \Lambda_0 &= -\frac{3}{4}\phi^{-2} - \frac{1}{2}\frac{p_1}{p_0}\phi^{-1} + \Lambda_0^*, \end{aligned} \tag{3.6}$$

where Ψ_1^* and Λ_0^* are analytic in I_ϕ. Similarly since

$$\lim_{\phi\to 0} \Lambda_h = \lim_{\phi\to 0} 3\nu \frac{h_2}{h_0}\left(\frac{p_0}{q_0} - \frac{1}{\nu}\frac{h_2}{h_0}\right)\phi^2 \tag{3.7}$$

Λ_h is also analytic in $I\phi$.

Following Langer [8], in view of the character of the coefficient functions Ψ_1^2, Λ_0, and Λ_h (as given by (3.6) and (3.7)), a solution of the homogeneous differential Eq. (2.13) asymptotic with respect to μ^2 ($k - 3i/2\, K_1/m$) to the true solution is[2] given by

$$W = \phi^{-3/4}\Psi_1^{*-1/2}\Phi^{1/2}[AJ_{2/5}(\eta) + BJ_{-2/5}(\eta)] \tag{3.8a}$$

where

$$\eta = i^{3/2}\mu\Phi, \qquad \Phi = \int_0^\phi t^{3/2}\Psi_1^*(t)\, dt \tag{3.8b}$$

and J and Y (to be introduced later) denote Bessel functions of the first and second kind respectively.

Evidently neither h, nor its derivative, influences the order of the Bessel functions in (3.8a). Hence this solution is also valid when h is uniform, provided k is approximated by a constant. It should also be mentioned that the behavior of h as given by (3.2) and (3.3) is similar to that discussed by Meissner [3], where r was assumed to have the form $r = q_0 \cos^n\phi$.

(2) With $b_1 = 0$ and $b_2 = 1$, again r is given by (3.1) and (3.3a), and the function h is of the form (3.2) but the coefficients h_n become

$$h_0 = h(0) = c_1 q_0^r, \qquad h_1 = \nu\frac{p_0}{q_0} + c_2\frac{p_0}{q_0}, \qquad \text{etc.} \tag{3.9}$$

It is clear that h, given by (3.2) and (3.9), as well as h', are non-vanishing analytic functions in $I\phi$.

While μ^2 is again given by the first of (3.5), the function f, by (2.8), becomes

$$f = \phi \frac{\left[\sum_{n=0}^\infty \frac{p_n}{p_0}\phi^n\right]^2}{\sum_{n=0}^\infty \frac{q_n}{q_0}\phi^n} \tag{3.10}$$

and since $q_1 \neq 0$, the expressions which correspond to (3.4) and (3.6), are

$$\frac{r'}{r} = \frac{\sum_{n=0}^\infty (n+1)q_{n+1}^*\phi^n}{\sum_{n=0}^\infty q_n^*\phi^n}, \qquad \frac{h'}{h} = \frac{\sum_{n=0}^\infty (n+1)h_{n+1}\phi^n}{\sum_{n=0}^\infty h_n\phi^n} \tag{3.11}$$

[2]A statement of Langer's theorem may be found in Sec. 5 of Ref. [10].

$$\Psi_1^2 = \phi\Psi_1^{*2}, \qquad \Psi_1^{*2}(0) = \left(k - \frac{3}{2}\, i\, \frac{K_1}{m}\right),$$
$$\Lambda_k(0) = \frac{3}{2}\, \nu\, \frac{h_1}{h_0}\left(\frac{q_1}{q_0} - \frac{1}{2\nu}\frac{h_1}{h_0}\right) \tag{3.12}$$

and the function Λ_0 (not recorded) may be shown to be analytic in $I\phi$. Again, an asymptotic solution of the homogeneous equation (2.13) with respect to $\mu^2\,[(k - (3/2)\, i\, (K_1/m)]$ is given by

$$W = \phi^{-1/4}\Psi_1^{*-1/2}\Phi^{1/2}[AJ_{1/3}(\eta) + BJ_{-1/3}(\eta)], \tag{3.13a}$$

where

$$\eta = i^{3/2}\mu\Phi, \qquad \Phi = \int_0^\phi t^{1/2}\Psi_1^*(t)\, dt. \tag{3.13b}$$

It is apparent that the remark which follows (3.8) concerning shells of uniform thickness is equally applicable to the above solution. Before closing this section, we illustrate the foregoing results [Eqs. (3.9) to (3.13)] for a toroidal shell, where the principal radii of curvature of the middle surface may be specified by

$$r_1 = R,$$
$$r_2 = \phi^{-1}a\left[\left(\frac{\phi}{\sin\phi}\right) + \frac{R}{a}\phi\right], \tag{3.14}$$

where R is the radius of the circular cross section and a is the distance from the center of the cross section to the axis of revolution. It follows from (2.17), that

$$p_0 = R, \qquad p_n = 0 \qquad \text{for} \qquad n \geq 1,$$
$$q_0 = a, \qquad q_1 = R, \qquad \text{etc.} \tag{3.15}$$

and by (3.5) and (3.10),

$$\mu^2 = \left(\frac{R}{a}\right)\left(\frac{R}{h_0}\right) m,$$
$$f = \phi\left\{\frac{\phi}{\sin\phi}\left[1 + \frac{R}{a}\sin\phi\right]\right\}^{-1}. \tag{3.16}$$

Hence the function Ψ_1^* and Φ in solution (3.13a) become

$$\Psi_1^{*2} = \left(k - \frac{3}{2}\, i\, \frac{K_1}{m}\right)\left(\frac{h_0}{h}\right)\left\{\frac{\phi}{\sin\phi}\left[1 + \frac{R}{a}\sin\phi\right]\right\}^{-1},$$
$$\Phi = \left(k - \frac{3}{2}\, i\, \frac{K_1}{m}\right)^{1/2}\int_0^\phi t^{1/2}\left\{\left(\frac{h}{h_0}\right)\left(\frac{t}{\sin t}\right)\left[1 + \frac{R}{a}\sin t\right]\right\}^{-1/2} dt, \tag{3.17}$$

where h is given by (3.2) and (3.3).

For toroidal shells of uniform thickness the above solution reduces to that given by Clark [11], in which case $h = h_0 = K_1R/\nu$ conforms to (2.10) with $c_1 = c_2 = 0$, and

$$k = i\frac{\nu}{\mu^2} + \left[1 - \left(\frac{\nu}{\mu^2}\right)^2\right]^{1/2}.$$

4. Case B: Behavior of shell thickness specified by (2.10) and solution of Eq. (2.12) when $b_1 = b_2 \neq -1$. In this case, with $b_1 = b_2 = b \neq -1$, it follows from (2.2), (2.3), and (2.17) that

$$\frac{p_0}{q_0} = 1 + b, \qquad \frac{p_1}{q_1} = 2 + b, \qquad \text{etc.} \tag{4.1}$$

and by (2.10)

$$h = -\frac{c_2}{\nu} F_1(\phi) + c_1 q_0^{\nu} \phi^{(1+b)\nu} F_2(\phi) + K_1 p_0 \frac{b+1}{b+2} \frac{\phi^{b+2}}{b+2-\nu(b+1)} F_3(\phi) \tag{4.2}$$

and

$$h' = -\frac{c_2}{\nu} F_1'(\phi) + c_1 q_0^{\nu}(1+b)\nu\phi^{(1+b)\nu-1} F_4(\phi) + K_1 p_0 \frac{b+1}{b+2-\nu(b+1)} \phi^{b+1} F_5(\phi). \tag{4.3}$$

The functions F_j ($j = 1, 2, 3, 4, 5$) in (4.2) and (4.3) are defined by

$$\begin{aligned} F_1(\phi) &= -\nu r^{\nu} \int r^{-(1+\nu)} \alpha \, d\phi = \sum_{n=0}^{\infty} f_{1n} \phi^n, \\ f_{10} &= F_1(0) = 1, \qquad f_{11} = \frac{\nu(1+b)}{(1+b)\nu - 1} \left[\frac{p_1}{p_0} - (1+\nu) \frac{q_1}{q_0} \right] + \nu \frac{q_1}{q_0}, \qquad \text{etc.} \end{aligned} \tag{4.4}$$

$$F_2(\phi) = [q_0^{\nu} \phi^{(1+b)\nu}]^{-1} r^{\nu}, \qquad F_2(0) = 1, \tag{4.5}$$

$$F_3(\phi) = \left[p_0 \frac{b+1}{b+2} \frac{\phi^{b+2}}{b+2-\nu(b+1)} \right]^{-1} r^{\nu} \int r^{-(1+\nu)} \alpha z \, d\phi, \qquad F_3(0) = 1, \tag{4.6}$$

$$\begin{aligned} F_4(\phi) &= F_2(\phi) + \frac{\phi}{(1+b)\nu} F_2'(\phi) \\ &= 1 + \nu \frac{q_1}{q_0} \frac{(1+b)\nu + 1}{(1+b)\nu} \phi + \phi^2 0(1), \end{aligned} \tag{4.7}$$

$$\begin{aligned} F_5(\phi) &= F_3(\phi) + \frac{\phi}{b+2} F_3'(\phi) \\ &= 1 + \frac{3+b}{2+b} \left\{ \frac{(2+b) - \nu(1+b)}{(3+b) - \nu(1+b)} \left[\frac{p_1}{p_0} \frac{5+2b}{3+b} - (1+\nu) \frac{q_1}{q_0} \right] + \nu \frac{q_1}{q_0} \right\} \phi \\ &\quad + \phi^2 0(1), \end{aligned} \tag{4.8}$$

where $0(1)$ denotes a bounded function.

With $K_1 \neq 0$, examination of (4.2) reveals that h is bounded at $\phi = 0$ if $b > -2$ and either $c_1 = 0$ or $(1 + b)\nu > 0$. Likewise with $K_1 \neq 0$, $h'(0)$ is bounded provided $b > -1$ and either $c_1 = 0$ or $(1 + b)\nu > 1$. Thus, the requirement that both h and h' be bounded at $\phi = 0$ when $K_1 \neq 0$ gives rise to the following restrictions[3].

$$b > -1, \qquad \nu > 0, \qquad K_1 \neq 0 \tag{4.9a}$$

$$c_1 = 0 \qquad \text{or} \qquad (1 + b)\nu > 1. \tag{4.9b}$$

[3]Although in the linear theory of elasticity, Poisson's ratio ν may have the range $-1 < \nu \leq 1/2$, as is usual on physical grounds, the negative values of ν are ruled out here.

It may be noted that, in addition to conditions (4.9), if in (4.4) we set $f_{11} = 0$, then $h'(0) = 0$.

If only condition (4.9a) is imposed on (4.2) and $c_1 \neq 0$, then h is continuous in I_ϕ and h' at $\phi = 0$ reduces to

$$\lim_{\phi \to 0} h' = \lim_{\phi \to 0} \left\{ -\frac{c_2}{\nu} f_{11} + \nu c_1 q_0^\nu (1 + b) \phi^{(1+b)\nu - 1} \right\}. \tag{4.10}$$

It is clear by (4.9b) that, if $(1 + b)\nu < 1$, then $h'(0)$ will not be bounded and, consequently, a drastic change of h will take place in the neighborhood of $\phi = 0$. In the case of ellipsoidal shells, for example, this situation leads to the occurrence of a cusp-like variation in h about $\phi = 0$.

With the restriction imposed by (4.9a), i.e., $b > -1$, $\nu > 0$, it follows from (2.8), (2.17), and (4.1), that

$$\begin{aligned} \mu^2 &= m \frac{p_0}{h_0} (1 + b), \\ f &= (\phi^b) \frac{\left[\sum_{n=0}^{\infty} \frac{p_n}{p_0} \phi^n \right]^2}{\sum_{n=0}^{\infty} \frac{q_n}{q_0} \phi^n}. \end{aligned} \tag{4.11}$$

We now proceed to examine the character of Λ_0 and Λ_h given by (2.15) and (2.16) in the neighborhood of $\phi = 0$. For this purpose, we record the following expression:

$$\begin{aligned} \frac{3}{2} \nu \frac{h'}{h_0} \frac{r'}{r} - \frac{3}{4} \left(\frac{h'}{h_0} \right)^2 &= \frac{3}{2} \nu^2 \frac{(1 + b)^2}{\phi^2} \frac{c_1 q_0^\nu}{h_0} \phi^{(1+b)\nu} \\ &\cdot \left\{ 1 + \phi^2 O(1) - \frac{c_1 q_0^\nu}{h_0} \phi^{(1+b)\nu} [1 + \phi^2 O(1)] \right\} \\ &+ \frac{3}{2} \frac{\nu(1 + b)}{\phi} f_{11} \left\{ 1 + \phi O(1) \right. \\ &+ \frac{K_1}{h_0} \frac{1 + b}{f_{11}} \frac{p_0}{2 + b - \nu(1 + b)} \phi^{1+b} (1 + \phi O(1)) \\ &- \frac{c_1 q_0^\nu}{h_0} \phi^{(1+b)\nu} \left[1 + \phi O(1) + \frac{K_1}{h_0} \frac{p_0}{f_{11}} \frac{1 + b}{2 + b - \nu(1 + b)} \phi^{1+b} \right. \\ &\cdot \left. (1 + \phi O(1)) \right] + \nu \frac{1 + b}{f_{11}} \frac{c_1 q_0^\nu}{h_0} \phi^{(1+b)\nu} \left[\left(\frac{p_1}{p_0} - \frac{q_1}{q_0} \right. \right. \\ &+ \left. \left. \left. \nu \frac{q_1}{q_0} \frac{1 + (1 + b)\nu}{(1 + b)\nu} \right) - 2\nu \frac{c_1 q_0^\nu}{h_0} \frac{q_1}{q_0} \frac{1 + (1 + b)\nu}{(1 + b)\nu} \phi^{(1+b)\nu} \right] \right\} + G_1(\phi) \end{aligned} \tag{4.12}$$

and write Λ_0 as

$$\Lambda_0 = \frac{A_1}{\phi^2} + \frac{B_1}{\phi} + \Lambda_0^*, \tag{4.13}$$

where

$$A_1 = -\frac{1}{4}(4b^2 + 8b + 3),$$
$$B_1 = -\frac{3}{2}(1 + b)^2\left(\frac{p_1}{q_0} - \frac{q_1}{q_0}\right) + \frac{b}{2}\frac{p_1}{p_0} \tag{4.14}$$

and G_1 and Λ_0^* are analytic in $I\phi$.

Denoting by Λ the sum of Λ_0 and Λ_k, then with the aid of (2.16), (4.14), and (4.13), we have

$$\Lambda = \frac{A_1^*}{\phi^2} + \frac{B_1^*}{\phi} + \Lambda_1 , \tag{4.15}$$

where Λ_1 is continuous in $I\phi$, and

$$A_1^* = A_1 + \frac{3}{2}\nu^2(1 + b)^2 \frac{c_1 q_0^\nu}{h_0}\phi^{(1+b)\nu}\left[1 - \frac{c_1 q_0^\nu}{h_0}\phi^{(1+b)\nu}\right],$$

$$\begin{aligned} B_1^* = B_1 + \frac{3}{2}\nu(1 + b)f_{11}\Bigg\{&1 + \frac{K_1}{h_0}\frac{1 + b}{f_{11}}\frac{p_0}{2 + b - (1 + b)\nu}\phi^{1+b} \\ &- \frac{c_1 q_0^\nu}{h_0}\phi^{(1+b)\nu}\left[1 + \frac{K_1}{h_0}\frac{p_0}{f_{11}}\frac{1 + b}{2 + b - (1 + b)\nu}\phi^{1+b}\right] \\ &+ \nu\frac{1 + b}{f_{11}}\frac{c_1 q_0^\nu}{h_0}\phi^{(1+b)\nu}\left[\left(\frac{p_1}{p_0} - \frac{q_1}{q_0} + \nu\frac{q_1}{q_0}\frac{1 + (1 + b)\nu}{(1 + b)\nu}\right)\right. \\ &\left.- 2\nu\frac{c_1 q_0^\nu}{h_0}\frac{q_1}{q_0}\frac{1 + (1 + b)\nu}{(1 + b)\nu}\phi^{(1+b)\nu}\right]\Bigg\}. \end{aligned} \tag{4.16}$$

If now condition (4.9b) is also imposed, two possibilities will result which will be considered separately: (1) $c_1 = 0$ and (2) $(1 + b)\nu > 1$.

(1) With $c_1 = 0$, the functions A_1^* and B_1^* in (4.16) simplify as follows:

$$A_1^* = A_1 ,$$
$$B_1^* = B_1 + \frac{3}{2}\nu(1 + b)f_{11}\left[1 + \frac{1 + b}{f_{11}}\frac{K_1}{h_0}\frac{p_0}{2 + b - (1 + b)\nu}\phi^{1+b}\right]. \tag{4.17}$$

Since B_1^* , as given by (4.17), makes no contribution[4] to the asymptotic solution of (2.13), it follows that, in the neighborhood of $\phi = 0$, the behavior of Λ is dominated by A_1/ϕ^2. Also the function Ψ_1^2 by (2.14) and (4.11) may be written as

$$\Psi_1^2 = \phi^b\Psi_1^{*2}(\phi), \qquad \Psi_1^{*2}(0) = k - \frac{3}{2}i\frac{K_1}{m}. \tag{4.18}$$

Hence, with (4.15), (4.17), and (4.18), an asymptltic solution of (2.13) with respect to $u^2[k - 3iK_1/(2m)]$ is given by

$$W = \phi^{-b/4}\Psi_1^{*-1/2}\Phi^{1/2}[AJ_{2(b+1)/b+2}(\eta) + BY_{2(b+1)/b+2}(\eta)], \tag{4.19a}$$

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[4]This may readily be seen if one introduces the transformation $\phi = s^2/4$, $W = s^{1/2}U$ into (2.13), in which case the transformed Λ reads as $(-3/4 + 4A_1^*)/s^2 + B_1^* + s^2/4\ \Lambda_1$ where the last two terms are continuous in I_s .

where

$$\eta = i^{3/2}\mu\Phi, \qquad \Phi = \int_0^\phi t^{b/2}\Psi_1^*(t)\, dt. \tag{4.19b}$$

(2) Here Eqs. (4.11) and (4.16), which have been deduced under condition (4.9a), i.e., $b > -1$ and $\nu > 0$, are to be further restricted so that $(1 + b)\nu > 1$.

Since for $(1 + b)\nu \geq +2$, Λ_k is continuous in $I\phi$, solution (4.19) is again valid in the neighborhood of $\phi = 0$. Also, when $0 < (1 + b)\nu < 2$, $\lim_{\phi \to 0} A_1^* = A_1$ and hence, (4.19) is a valid asymptotic form of W in the neighborhood of $\phi = 0$.

As an illustration of the above results consider an ellipsoidal shell, the middle surface of which is specified by $r^2/a^2 + z^2/c^2 = 1$. The principal radii of curvature may then be written as

$$\begin{aligned} r_1 &= a(c/a)^{-1}[1 - \epsilon^2 \sin^2 \phi]^{-3/2}, \\ r_2 &= a(c/a)^{-1}[1 - \epsilon^2 \sin^2 \phi]^{-1/2}, \end{aligned} \tag{4.20a}$$

where

$$\epsilon^2 = 1 - (a/c)^2. \tag{4.20b}$$

Comparison of (4.20) and (2.17) yields

$$\begin{aligned} &b = 0, \\ &p_0 = a(c/a)^{-1}, \qquad p_1 = 0, \qquad p_2 = (3a/2)(c/a)^{-1}\epsilon^2 \qquad \text{etc.,} \\ &q_0 = a(c/a)^{-1}, \qquad q_1 = 0, \qquad q_2 = (a/2)(c/a)^{-1}\epsilon^2 \qquad \text{etc.} \end{aligned} \tag{4.21}$$

and by (4.11) and (2.14)

$$\begin{aligned} \mu^2 &= m(a/h_0)(c/a)^{-1}, \\ f &= (1 - \epsilon^2 \sin^2 \phi)^{-5/2}, \end{aligned} \tag{4.22a}$$

$$\Psi_1^{*2} = \Psi_1^{*2} = \left(k - \frac{3}{2}\, i\, \frac{K_1}{m}\right)\left(\frac{h}{h_0}\right)^{-1}(1 - \epsilon^2 \sin^2 \phi)^{-5/2}. \tag{4.22b}$$

Thus by (4.19) we finally have

$$W = \Psi_1^{*-1/2}\Phi^{1/2}[AJ_1(\eta) + BY_1(\eta)], \tag{4.23}$$

where η and Φ are evaluated by (4.19b) and (4.22b).

It may be noted that for ellipsoidal shells of uniform thickness, solution (4.23) reduces to that given previously in [9], where k [defined by (2.7)] is approximated to a constant. In the case of spherical shells of uniform thickness, $(c/a = 1)$, however, no such approximation is necessary, since $k = i\,(\nu/\mu^2) + [1 - (\nu/\mu^2)^2]^{1/2}$ and $h = h_0 = (K_1/\nu)a$.

5. Reduction to the theory of shallow shells. We conclude the present paper with the special forms of the several solutions given in Secs. 3 and 4 (cases A and B) appropriate in the theory of shallow shells of revolution, where attention is confined to small values of the meridional coordinate ϕ. Thus in what follows, in the series representation of the functions involved only terms up to and including ϕ will be retained.

In addition we also note that since $(\nu\lambda - \delta/2) = -K/h_0 = -\mu^2(K_1/m)$ then with the restriction $K/h_0 \ll \mu^2$, it follows from (2.7) that

$$k = 1$$

and

$$k - \frac{3}{2} i \frac{K_1}{m} = 1.$$

By imposing the above stipulations, we now proceed to reduce the solutions given in Secs. 3 and 4 to those valid for shallow shells of revolution.

Case A:

(1) $b_1 = -b_2 = 1,\ q_1 = 0.$

From (3.2), $h/h_0 = 1$

and hence by the first of (2.8), (2.17) and (3.1)

$$W = \phi^{-1/2}\left(1 - \frac{1}{2}\frac{p_1}{p_0}\phi\right)(\beta + i\psi). \tag{5.1}$$

Also by (2.14), (3.5) and (3.6)

$$\Psi_1^* = 1 + \frac{p_1}{p_0}\phi$$

and hence (3.8) becomes

$$W = \phi^{1/2}\left[1 - \frac{1}{7}\frac{p_1}{p_0}\phi\right][AJ_{2/5}(\eta) + BJ_{-2/5}(\eta)], \tag{5.2}$$

$$\eta = i^{3/2}\frac{2}{5}\left[m\frac{p_0}{q_0}\frac{p_0}{h_0}\right]^{1/2}\left[1 + \frac{5}{7}\frac{p_1}{p_0}\phi\right]\phi^{5/2}. \tag{5.3}$$

(2) $b_1 = 0,\ b_2 = -1,\ q_1 = p_0$.

From (3.9) and (3.2)

$$\frac{h}{h_0} = 1 + \frac{h_1}{h_0}\phi, \tag{5.4}$$

where

$$\frac{h_1}{h_0} = \frac{p_0}{q_0}\left(\nu + \frac{c_2}{h_0}\right). \tag{5.5}$$

Again by the first of (2.8), (2.17), (3.1) and (5.4),

$$W = \left[1 + \frac{1}{2}\left(\frac{p_0}{q_0} + 3\frac{h_1}{h_0} - \frac{p_1}{p_0}\right)\phi\right](\beta + i\psi) \tag{5.6}$$

and by (2.14), (3.10) and (3.12)

$$\Psi_1^* = 1 - \frac{1}{2}\left(\frac{p_0}{q_0} + \frac{h_1}{h_0} - 2\frac{p_1}{p_0}\right)\phi$$

with the aid of which (3.13) yields

$$W = \phi^{1/2}\left[1 + \frac{1}{10}\left(\frac{p_0}{q_0} + \frac{h_1}{h_0} - 2\frac{p_1}{p_0}\right)\phi\right][AJ_{1/3}(\eta) + BJ_{-1/3}(\eta)], \tag{5.7}$$

$$\eta = i^{3/2}\frac{2}{3}\left[m\frac{p_0}{q_0}\frac{p_0}{h_0}\right]^{1/2}\left[1 - \frac{3}{10}\left(\frac{p_0}{q_0} + \frac{h_1}{h_0} - 2\frac{p_1}{p_0}\right)\phi\right]\phi^{3/2}. \tag{5.8}$$

Case B: $b_1 = b_2 = b \neq -1$.

Because of the several possibilities for the form of the thickness variation discussed at length in the previous section, the determination of the specific coefficients for the series expansion of any one of these possible forms of h (which can be deduced in a straightforward manner) will not be considered here.

It follows from the first of (2.8) that

$$W = \left(\frac{h}{h_0}\right)^{3/2}\phi^{1/2}\left[1 + \frac{1}{2}\left(\frac{q_1}{q_0} - \frac{p_1}{p_0}\right)\phi\right](\beta + i\psi) \tag{5.9}$$

and by (2.14), (3.6) and (4.11),

$$\Psi_1^* = \left(\frac{h}{h_0}\right)^{-1/2}\left[1 + \frac{1}{2}\left(2\frac{p_1}{p_0} - \frac{q_1}{q_0}\right)\phi\right]. \tag{5.10}$$

Hence (4.19a) becomes

$$W = \phi^{-b/4}\left(\frac{h}{h_0}\right)^{1/4}\left[1 - \frac{1}{4}\left(2\frac{p_1}{p_0} - \frac{q_1}{q_0}\right)\phi\right]\Phi^{1/2}[AJ_{2(b+1)/b+2}(\eta) + BY_{2(b+1)/b+2}(\eta)], \tag{5.11}$$

where η is defined by (4.19b) and (5.10).

When h is uniform (in which case, as discussed earlier, k is approximated by a constant), with the aid of (4.19b) and (5.10), (5.11) reduces to

$$W = \phi^{1/2}\left[1 - \frac{1/4(2p_1/p_0 - q_1/q_0)}{(b/2 + 2)}\phi\right][AJ_{2(b+1)/(b+2)}(\eta) + BY_{2(b+1)/(b+2)}(\eta)], \tag{5.12}$$

where

$$\eta = i^{3/2}\left[4m\frac{p_0}{h_0}\frac{(1 + b)}{(b + 2)^2}\right]^{1/2}\phi^{(b+2)/2}\left[1 + \frac{1}{2}\left(2\frac{p_1}{p_0} - \frac{q_1}{q_0}\right)\frac{b/2 + 1}{b/2 + 2}\phi\right]. \tag{5.13}$$

As an example of the foregoing solution (5.12), we now consider the case of shallow paraboloidal shells of revolution, the middle surface of which is specified by

$$z = \left(\frac{r}{a_0}\right)^n, \tag{5.14}$$

where $n (n > 1)$ is a rational number. Also, the principal radii of curvature of (5.14) are

$$r_1 = \frac{a}{n - 1}\sec^3\phi(\tan\phi)^{(n-1)^{-1}-1}, \tag{5.15}$$

$$r_2 = \frac{a}{n - 2}\frac{(\tan\phi)^{1/(n-1)}}{\sin\phi},$$

where a, as a representative length, is given by

$$a = a_0\left(\frac{a_0}{n}\right)^{1/(n-1)}.$$

Expansion of (5.15) in the series form of (2.17) leads to

$$\begin{aligned} b &= \frac{1}{n-1} - 1, \\ p_0 &= \frac{1}{n-1}a, \qquad p_1 = 0, \qquad \text{etc.}, \\ q_0 &= a, \qquad q_1 = 0 \qquad \text{etc.}, \end{aligned} \tag{5.16}$$

the substitution of which in (5.12) and (5.13) results in the following solution for uniform shallow paraboloidal shells

$$W = \phi^{1/2}[AJ_{2/n}(\eta) + BY_{2/n}(\eta)], \tag{5.17}$$

where

$$\eta = i^{3/2}\left[4m\frac{a}{h_0}\frac{1}{n^2}\right]^{1/2}\phi^{(n/2)/(n-1)}. \tag{5.18}$$

The above example of shallow paraboloidal shells of uniform thickness has been treated earlier by E. Reissner [6] and it would be of interest to show the correspondence of the two solutions. Using his own notation (except for a subscript R which, when necessary, will be added in order to distinguish the symbols from those of the present paper), Reissner in [6] defines the middle surface of paraboloidal shells by

$$r = a\xi, \qquad z = a\mu_R f_R(\xi), \qquad f_R(\xi) = \xi^n, \tag{5.19}$$

"where a is a reference length, f_R' is of order unity and μ_R a number small compared with unity. ⋯ A shallow shell is defined by the stipulation that for it, terms of order μ_R^2 may be neglected compared with terms of order unity." With this stipulation, $\alpha = a$, $\Lambda_0 = -3/(4\xi^2)$, $\mu^2 = m/h_0\ a\ n\mu_R$, $\Psi_1^2 = \xi^{n-2}$, and (2.13) becomes (which, except for a constant, is the solution given in [6])

$$W_R = \xi^{1/2}\begin{Bmatrix} J_{2/n}(\eta^*) \\ Y_{2/n}(\eta^*) \end{Bmatrix}, \tag{5.20}$$

where

$$\eta^* = i^{3/2}\left[4m\frac{a}{h_0}\frac{\mu_R}{n}\right]^{1/2}\xi^{n/2}. \tag{5.21}$$

Prior to establishing the relationship between the two solutions (5.17) and (5.20), we note that if terms of $0(\phi^2)$ are neglected in the series expressions of r and z of this paper, these quantities read:

$$r = a\phi^{1/(n-1)}, \qquad z = \frac{a}{n}\phi^{n/(n-1)}. \tag{5.22}$$

Thus, the form (5.19) is equivalent to (5.22) of the present paper if

$$\mu_R = \frac{1}{n}, \qquad (5.23)$$

$$\xi = \phi^{1/(n-1)}.$$

It follows from (5.23) that $\eta^* = \eta$, and hence

$$W = \xi^{(n-2)/2} W_R . \qquad (5.24)$$

It may be noted that though the two solutions W and W_R differ, due to the particular choice of the independent variable, they yield identical expressions for $\beta + i\psi$ defined in (5.9).

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