HEAT TRANSFER AT RECTANGULAR CORNERS*

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Introduction. In this paper, a solution to the heat equation, $\Delta u = u_t$, appropriate to a small neighborhood of a rectangular corner is developed. It will be assumed throughout that the initial condition is that of zero temperature. The boundary values will be given by a continuous function of position. The equation is solved by means of "Green's function" and the solution expressed as an infinite series. This series is relatively simple to evaluate and the error involved is readily calculated. Moreover, the solution may be differentiated with respect to any one of the variables involved.

I. Let the rectangle be such that a vertex is at the origin of a (ξ, η) coordinate system with its boundary consisting of the positive ξ and η axes, together with two half lines at infinity. The boundary function is defined as

$$\varphi(\xi, \eta) = \begin{cases} k[1 - \xi^2/a^2], & \eta = 0, \quad 0 \le \xi \le a, \\ k[1 - \eta^2/a^2], & \xi = 0, \quad 0 \le \eta \le a, \\ 0, \text{ for all other points } (\xi, \eta) \text{ on the boundary,} \end{cases} \tag{1.1}$$

where a and k are positive, real numbers. From a physical viewpoint, the function φ may represent a good approximation to boundary functions resulting from various modes of heating, at least for ξ and η small. For example, suppose the source is a two-dimensional flame of arbitrary outline, applied to the corner in such a way that its temperature distribution is closely approximated by either $T_1 = k\,[1 - (\xi^3 + \eta^2)/a^2 + f(\xi\eta)]$, where f is a polynomial in $\xi\eta$, or $T_2 = k\,[1 - (\xi^2 + \eta^2)/a^2 +$ (higher powers of ξ and η)], with T_1 and T_2 being positive or zero for all ξ and η. If the flame temperatures are best approximated by T_1, then for $0 \le \xi \le a$, $0 \le \eta \le a$, the boundary temperature must be equally close to φ. If T_2 is best, then, for ξ and η small enough, one has boundary values again arbitrarily close to φ.

II. A solution of $\Delta u = u_t$ will now be obtained which is valid for x, y, and t positive and such that

(a) $$\lim_{t \to 0} u = 0, \quad \text{for} \quad x > 0, \qquad y > 0;$$

(b) $$u = \varphi \text{ for } t > 0 \text{ and } (x, y) \text{ on the boundary.}$$

First a "Green's function" is defined as follows:

$$G(x, y, t; \xi, \eta, \tau) = \tfrac{1}{4}\pi(t - \tau)\{\exp[-(\xi - x)^2/4(t - \tau)] - \exp[-(\xi + x)^2/4(t - \tau)]\} \cdot \{\exp[-(\eta - y)^2/4(t - \tau)] - \exp[-(\eta + y)^2/4(t - \tau)]\}. \tag{2.1}$$

As is well known, by employing Green's formula, one arrives at the solution

$$u(x, y, t) = \int_0^t \left[\int_B \varphi(x, y) \frac{\partial G}{\partial \eta_i} ds \right] d\tau, \tag{2.2}$$

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where the symbol $\int_B$ denotes the integral taken over the boundary of the rectangle and $\partial G/\partial \eta_i$ signifies differentiation in the direction of the inner normal. After substitution, differentiation, and simplification, (2.2) becomes

$$\begin{aligned} u(x, y, t) = y/\pi \int_0^a \varphi(\xi, 0)[&\exp \{(-\tfrac{1}{4}t)[(\xi - x)^2 + y^2]\}/[(\xi - x)^2 + y^2] \\ &- \exp \{(-\tfrac{1}{4}t)[(\xi + x)^2 + y^2]\}/[(\xi + x)^2 + y^2]]\, d\xi + x/\pi \int_0^a \varphi(0, \eta) \\ &\cdot [\exp \{(-\tfrac{1}{4}t)[(\eta - y)^2 + x^2]\}/[(\eta - y)^2 + x^2] \\ &- \exp \{(-\tfrac{1}{4}t)[(\eta + y)^2 + x^2]\}/[(\eta + y)^2 + x^2]]\, d\eta = I_1 - I_2 + I_3 - I_4 , \end{aligned} \tag{2.3}$$

where a is the same as given in (1.1). In order to show that $\lim_{t\to 0} u = 0$, consider the integral I_1. The integrand may be expanded in a series converging absolutely and uniformly for all x, y, and t bounded away from zero. Integration yields the new series,

$$\sum_{n=0}^{\infty} y/\pi \exp(-y^2/4t) \cdot 1/n!\, 1/(4t)^n \int_0^a \varphi(\xi, 0)(\xi - x)^{2n}/[(\xi - x)^2 + y^2]\, d\xi, \tag{2.4}$$

convergent under the same conditions. Now the nth partial sum of (2.4) has limit zero when $t \to 0$. Thus, the limit function approaches zero with t so that $I_1 \to 0$ as $t \to 0$. The same result holds for I_2, I_3, and I_4.

To show that u takes on the prescribed boundary values, consider the expression

$$dI_1 = y/\pi \exp \{(-\tfrac{1}{4}t)[(\xi - x)^2 + y^2]\}/[(\xi - x)^2 + y^2].$$

For $x \neq \xi$, $\lim_{y\to 0} dI_1 = 0$ and for $x = \xi$, $dI_1 \to \infty$ as $y \to 0$. Also, $\lim_{y\to 0} \int_{-\infty}^{\infty} dI_1\, d\xi = 1$, so that dI_1 has the "character of a δ" function. Using Schwarz' distribution theory, together with the fact that $\varphi(\xi, 0)$ can be extended to be C^∞ for all ξ, the limit

$$\lim_{y\to 0} \int_{-\infty}^{\infty} \varphi(\xi, 0)\, dI_1\, d\xi = \varphi(x, 0).$$

But for $x > 0$ and bounded away from zero,

$$\lim_{y\to 0} \int_{-\infty}^{\infty} \varphi(\xi, 0)\, dI_1\, d\xi = \lim_{y\to 0} \int_0^a \varphi(\xi, 0)\, dI_1\, d\xi$$

and this limit is uniformly taken in the sense of the Moore-Osgood theorem. Also, for $y > 0$,

$$\lim_{x\to x_0} \int_0^a \varphi(\xi, 0)\, dI_1\, d\xi$$

clearly exists for all $x_0 \geq \delta > 0$. Therefore,

$$\lim_{\substack{y\to 0\\ x\to x_0}} \int_0^a \varphi(\xi, 0)\, dI_1\, d\xi = \varphi(x_0, 0),$$

for all x_0 bounded away from zero. It follows immediately that

$$\lim_{\substack{y\to 0\\ x\to x_0}} [-I_2 + I_3 - I_4] = 0.$$

Using identical arguments,

$$\lim_{\substack{x\to 0\\ y\to y_0}} \int_0^a \varphi(0, \eta)\, dI_3\, d\eta = \varphi(0, y_0) \quad \text{for all} \quad y_0 \geq \delta > 0.$$

Again, the remaining double limits are zero. It remains to show that

$$\lim_{\substack{x\to 0\\ y\to 0}} u = (0, 0).$$

For this purpose, it is sufficient to consider the sum

$$I_1' - I_2' + I_3' - I_4', \tag{2.5}$$

where I'_α is I_α, $\alpha = 1, \cdots, 4$, with the exponential term set equal to 1. After suitable transformations, one can write (2.5) as the sum

$$\begin{aligned} &2/\pi\{\varphi(x + y\lambda, 0) \arctan x/y + \varphi(0, y + x\lambda') \arctan y/x\} \\ &\quad + \frac{1}{\pi}\left\{\int_0^{(a-x)/y} \frac{\varphi(x + y\sigma, 0)}{1 + \sigma^2}\, d\sigma - \int_0^{(a+x)/y} \frac{\varphi(x - y\sigma, 0)}{1 + \sigma^2}\, d\sigma\right\} \\ &\quad + \frac{1}{\pi}\left\{\int_0^{(a-y)/x} \frac{\varphi(0, y + x\sigma)}{1 + \sigma^2}\, d\sigma - \int_0^{(a+y)/x} \frac{\varphi(0, y - x\sigma)}{1 + \sigma^2}\, d\sigma\right\} = A + B + C, \end{aligned}$$

where $-x/y < \lambda < 0$ and $-y/x < \lambda' < 0$. Since φ is a continuous function of x and y in any neighborhood of the origin, it is certainly possible to find a $\delta > 0$ such that for $x < \delta$, $y < \delta$, the following inequalities hold:

$$|A - \varphi(0, 0)| < \epsilon, \ |B| < \epsilon, \quad \text{and} \quad |C| < \epsilon, \quad \text{for} \quad \epsilon > 0.$$

The existence of the limit in question is therefore assured.

III. To obtain the series development of the solution, first the exponentials in each integrand of (2.3) are expanded. The resulting expressions are series converging absolutely and uniformly for x, y, and t in any closed, bounded domain, minus the origin. After substituting for φ as given by (1.1) and collecting similar terms, one may integrate, obtaining the expression

$$\begin{aligned} u = {} & k/\pi a^2\{1 - (a^2 + y^2 - x^2) \arctan 2xy/(a^2 + y^2 - x^2) \\ & - (a^2 + x^2 - y^2) \arctan 2xy/(a^2 + x^2 - y^2) \\ & + xy \ln (x^2 + y^2)^4/[a^8 - 2a^4(x^4 - 6x^2y^2 + y^4) + (x^2 + y^2)^4]\} \\ & - ka^2xy/16\pi t^2 + ka^2xy/96\pi t^3[a^2/3 + x^2 + y^2] \\ & - ka^2xy/3072\pi t^4[a^4/2 + 8a^2/3(x^2 + y^2) + 3(x^2 + y^2)^2] \\ & + ka^2xy/30{,}720\pi t^5[a^6/5 + 5a^4/3(x^2 + y^2) + 10a^2/3(x^2 + y^2)^2 + 2(x^2 + y^2)^3] - \cdots, \end{aligned} \tag{3.1}$$

where the principal value is used in computing the arctan.

The series is alternating and converges absolutely for all x, y, and t positive. For applied purposes, one may have to restrict the values of t to some definite range in order that the solution give results in agreement with actual conditions. Appropriate ranges for t would, of course, be governed by the nature of the heat source and the material heated. In any case, it is clear that the most rapid convergence is obtained for small x and y and comparatively large t. Although the first term appears formidable, it turns out that in many instances the arctan and log terms are negligible. For example, if $k = 10^3$, $a = 2$, and $xy \leq 4/10^6$, the arctan and log may be neglected if an error within $\pm 1/10$ is satisfactory. Since the series is alternating, the error involved is less, in absolute value, than the first term not employed in the calculation. Thus, once k and a are fixed, one might check terms, beginning with the third, i.e., $ka^2xy/96\pi t^3\ [a^2/3 + x^2 + y^2]$, in order to obtain a range for x, y, and t so that a given error tolerance is not exceeded. From the form of each term, an inequality involving the product, xy, is simplest to handle and furnishes a good check for x and y small. As a usual occurrence the calculation of three or four terms gives sufficient accuracy for applied purposes provided a reasonable balance is maintained among the variables.

VISCO-ELASTIC STRESS ANALYSIS*

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1. Introduction. In his paper on stress analysis in visco-elastic bodies [1]** E. H. Lee bases his reasoning on the concept of an associated elastic problem to which a visco-elastic problem reduces after removal of its time dependence by application of the Laplace transform. Thus Lee's method requires the application of the Laplace transform to the boundary conditions as well as to the basic equations and it might be expected that it is restricted to problems whose boundary conditions admit such an operation. As a result, for example, the problem of indentation of a half-space by a curved punch could not be solved by this method, since at any one point of the boundary at different times stresses or displacements are specified.

It is the purpose of this paper to extend the applicability of Lee's method to problems of the above type and to show that the apparent restriction is due to the process by which Lee deduced his method, in particular, due to the concept of the associated elastic problem.

At the same time, the Laplace transform method will be restated in a different form which may be called the method of functional equations. This method is completely equivalent to Lee's method, since both these methods coincide, if the functional equations are solved by operational methods. However, the extension of the applicability of the Laplace transform method to the wider range of problems requires the functional equation approach for its justification.

2. The method of functional equations. The basic, quasi-static equations governing the linear theories of isotropic, elastic or visco-elastic media, referred to orthogonal, rectilinear coordinates x_k, may be written in the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = 0, \tag{2.1}$$

$$P^p s_{ij} = Q^q e_{ij}, \qquad R^r \sigma_{ii} = S^s \epsilon_{ii}, \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.2}$$

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**Numbers in square brackets refer to the bibliography at the end of the paper.