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SOME ASPECTS OF THREE-DIMENSIONAL BOUNDARY LAYER FLOWS*

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Abstract. The equations for laminar boundary layer flow over a general smooth surface in three dimensions are analyzed in a normal coordinate system. The invariance properties of these equations are found using the concept of subtensors. The boundary layer equations are not tensor equations but subtensor equations. Conditions for the Cartesian form of the equations are given and a criterion for no secondary flow is found in terms of the geodesics of the body surface. The displacement effect of the boundary layer is also discussed.

Introduction. In recent times interest in three-dimensional boundary layer flows has grown considerably. A review of the subject has been given by Sears [1]. Most of the work cited by Sears is concerned with the solutions to particular problems. To the author's knowledge the only general discussions of boundary layers in three-dimensional flows are contained in the work of Howarth [2], Moore [3], and Hayes [4]. However, there is also some work of C. C. Lin contained in Chap. 18 of [5], in which the appropriate equations are derived without much discussion. This latter work had been overlooked by the previously mentioned authors. The method presented by Lin proves to be useful in discussing the invariance properties of the boundary layer equations. (C. C. Lin has informed me that the material on general boundary layer equations [5] was supplied to the late Professor A. D. Michal not as new results, but as a presentation of some previous work by Levi-Civita.)

In this paper it is first shown that Lin's approach carries over to a more general class of coordinate systems. Certain invariance properties follow from examining the resulting equations. Now it is known that the two-dimensional boundary layer equations are not tensor equations and one would expect the same result in three-dimensions (In [4], Hayes seems to imply the contrary). However, in the language of [6], the boundary layer equations are subtensor equations which means that they are invariant with respect to certain types of coordinate transformations. This does not contradict the work of Lagerstrom and Kaplun [7] and [8], where the non-tensor character of the boundary layer equations is used to define an "optimum coordinate system". A similar procedure could be developed for three-dimensional flows, however a more general coordinate system than that used here would have to be considered.

From the subtensor form of the equations, it is easily shown that they reduce to the "Cartesian form" (as is true for any two-dimensional flow) for any surface whose

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Gaussian curvature is zero if appropriate coordinates are used. Howarth [2], concludes that this happens only for planes and cylinders. The curvature effects that Howarth describes include the effects of a curvilinear coordinate system. (Moore [3] conjectured that the Cartesian form applies to any curved surface.)

Using the streamline coordinates as in [4], it follows that a boundary layer flow is essentially two-dimensional if there is no secondary flow. A simple criterion for no secondary flow is given in terms of the geodesics of the surface. Finally the displacement effect of the boundary layer is discussed. This is done in a different way than by Moore [9], and some differences between the two- and three-dimensional cases are discussed.

Only laminar flows of an incompressible, non-conducting fluid are discussed. Some of the conclusions should hold in more general cases, however. Also body forces have been neglected and difficulties due to separation effects are not discussed.

The boundary layer equations. In applying the boundary layer concept to the flow of a viscous fluid we have some given surface to consider, for example a solid body or an interface between two fluids. This basic surface is used to define a coordinate system: it is one of the coordinate surfaces. A convenient class of coordinate systems is the normal coordinate systems, [6], which is defined as follows. A given one-parameter family of surfaces in a Riemannian N-space has a family of orthogonal trajectories such that, under very general conditions, only one trajectory passes through each point of space. The parameter of the family of surfaces is denoted by x^N and on one of the surfaces a coordinate system $x^1, \cdots, x^{N-1}$ is set up. We shall be concerned with a Euclidean 3-space although the derivation of the boundary layer equations would proceed in the same way for more general spaces.

The convention is adopted that Greek suffixes have the range 1, 2 and small Latin suffixes the range 1, 2, 3. The given surface over which we consider the boundary layer flow is the one on which the coordinates x^α are defined and for this surface $x^3 = 0$. Normal coordinate systems have the property that the distance expression is given by

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + g_{33}(dx^3)^2$$

that is,

$$g_{3\alpha} = 0.$$

There is no unique way of choosing our normal coordinate system since all that is required is that the given surface be one of the family. A simple special case is the geodesic normal coordinate system in which the family of surfaces is obtained by measuring off constant distances from a given surface along the geodesics which cut the surface orthogonally. In this special case

$$g_{33} = 1.$$

This is the coordinate system used by Lin [5].

The surface $x^3 = 0$ is, in general, a Riemannian 2-space. In this subspace we can carry out tensor operations and these will be intimately related to the tensor operations of the parent 3-space. We use a comma to denote covariant differentiation in the 3-space and a semi-colon for the same operation in the subspace. A set of quantities T_α which transform according to the law

$$T'_\alpha = T_\beta(\partial x^\beta/\partial x'^\alpha)$$

under a transformation of the form

$$x'^{\alpha} = f^{(\alpha)}(x^{\beta}) \qquad x'^{3} = x^{3} \tag{1}$$

is called a subtensor. If a tensor T_i is split up into two groups T_α, T_3 then for (1) T_α is a subtensor and T_3 is a subinvariant, and similarly for higher order tensors. Also the Christoffel symbols, in which one or more of the indices have the value 3, are subtensors. A more detailed discussion of substensors is given in [6].

The derivation of the boundary layer equations is straightforward but tedious. The procedure here is essentially the same as Lin's for the special case of a geodesic normal coordinate system and therefore will not be discussed in detail. (Some errors were found in the details of [5] but these only affected terms which drop out in the boundary layer approximation.) Starting from the Navier-Stokes equations in tensor form

$$(\partial u_i/\partial t) + u^j u_{i,j} = \nu g^{jk} u_{i,jk} - \pi_{,i}, \qquad u^i_{,i} = 0, \tag{2}$$

where π is the ratio of pressure to the constant density, the following steps are necessary. The momentum equations are split up into two groups (as illustrated above with T_i); the covariant derivatives are expressed in terms of the "sub-covariant derivatives"; a transformation

$$\zeta = x^3/\nu^{1/2}, \qquad U_3 = u_3/\nu^{1/2}, \tag{3}$$

is applied; all quantities, including the metric tensor and Christoffel symbols, are expanded in a power series in $\nu^{1/2}$; the equations for the lowest order terms then yield the boundary layer equations

$$\begin{aligned} (\partial u_\alpha/\partial t) + u^\beta u_{\alpha;\beta} + (U_3/g_{33}^{(0)})(\partial u_\alpha/\partial \zeta) &= -\pi_{;\alpha} + (\partial^2 u_\alpha/\partial \zeta^2)/g_{33}^{(0)} \\ \partial \pi/\partial \zeta &= 0 \\ u^\beta_{;\beta} + C_\beta u^\beta + (\partial U_3/\partial \zeta)/g_{33}^{(0)} &= 0, \end{aligned} \tag{4}$$

where

$$C_\alpha = (\partial g_{33}^{(0)}/\partial x^\alpha)/2g_{33}^{(0)}$$

$$g_{33}^{(0)} = g_{33}(x^1, x^2, 0)$$

and the metric tensor for (4) is $a_{\alpha,\beta}$ where

$$a_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2, 0).$$

The boundary layer equations (4) reduce to those given by Lin for $g_{33} = 1$. (Note that, in the procedure outlined above, nothing is implied about the higher approximations obtained from the series expansion. Consideration of these involves significant difficulties, [13].)

Under transformations of the form (1) it is easily seen that equations (4) are invariant since U_3 and g_{33} are invariant. Thus the boundary layer equations have subtensor character. This is not too surprising since what destroys the tensor character of this approximation is the transformation (3) and this is not affected by (1). Also in the subtensor form it becomes obvious that the boundary layer equations reduce to "Cartesian form" for any surface of zero curvature (flat 2-space) in which the coordinates x^α are Cartesian coordinates provided that a geodesic normal coordinate system is used, i.e. $g_{33} = 1$. This conclusion differs from Howarth's [2], because he restricts his analysis (which uses a geodesic normal system) to coordinate systems x^α which can be Cartesian only for planes and cylinders. Because of the subtensor character of (4) there is complete freedom in the choice of x^α. However, the choice of Cartesian coordinates x^α on a developable surface may not always be the most advantageous. For example, the form of the terms $\pi_{;\alpha}$ which are determined by the external flow, may be complicated by such a choice. It must be remembered that the conclusion concerning when the boundary layer equations reduce to Cartesian form applies only to within the approximations of the standard theory. For example, for the flow over on open-ended cylinder the equations are in Cartesian form (for appropriate x^α) but the usual boundary layer approximations become invalid as the distance from the leading edge increases. Finally, it may be noted, even for flow over a plane the equations will have curvature terms appearing if x^α are not Cartesian coordinates.

The boundary layer equations (4) can be written in the more conventional form in terms of the physical components of the velocity. Before doing this let us specialize to orthogonal coordinates x^α and change the notation for the metric tensor

$$\begin{aligned} g_{ii} &= H_i^2 , \\ a_{\alpha\alpha} &= h_\alpha^2 , \qquad \text{(no summation)} \\ g_{33}(x^1, x^2, 0) &= h_3^2 = g_{33}^{(0)} , \end{aligned} \tag{5}$$

where h_α is not a function of x^3. Also denote the physical components of the velocity by u, v, and w. Thus

$$\begin{aligned} u &= u_1/h_1 , \\ v &= u_2/h_2 , \\ w &= u_3/h_3 , \end{aligned}$$

since for (4) the metric tensor components are the lower case h's. In terms of the physical components, using x, y, and z as coordinates, Eqs. (4) become

$$\begin{aligned} u_t + uu_x/h_1 + vu_y/h_2 + uvh_{1y}/h_1h_2 - v^2h_{2x}/h_1h_2 + wu_z/h_3 &= -\pi_x/h_1 + \nu u_{zz}/h_3^2 , \\ v_t + uv_x/h_1 + vv_y/h_2 + uvh_{2x}/h_1h_2 - u^2h_{1y}/h_1h_2 + wv_z/h_3 &= -\pi_y/h_2 + \nu v_{zz}/h_3^2 , \\ \pi_z &= 0 \\ [(h_2h_3u)_x + (h_1h_3v)_y]/h_1h_2 + w_z/h_3 &= 0, \end{aligned} \tag{6}$$

where the subscripts t, x, y, and z denote partial differentiation. These reduce to Howarth's equations [2] for $h_3 = 1$, i.e. a geodesic normal coordinate system. Hayes [4], gives these equations (if compressibility is neglected in his equations) except that he, in some way, allows h_3 to depend on z.

Finally, it can be remarked that, to examine the flow in the neighborhood of a stagnation point, as Howarth has done [10], it is only necessary to introduce a Riemannian coordinate system for x^α with origin at the stagnation point.

Secondary flow. Since the boundary layer equations are invariant under transformations of the surface coordinates x^α, it is natural to look for coordinates which, under certain conditions, simplify the general equations. An ingenious choice of coordinates was made by Hayes [4]. For a steady external flow the coordinate x^1 is chosen along the streamlines of the external flow evaluated on the surface and x^2 is along the orthogonal trajectories of these streamlines. If U and V are the physical components of the external flow, evaluated on the surface, in the x and y (or x^1 and x^2) directions, then for this choice of coordinates $V = 0$ and

$$\begin{aligned} -\pi_x/h_1 &= UU_x/h_1 , \\ -\pi_y/h_2 &= -U^2k, \end{aligned} \tag{7}$$

where

$$k = h_{1y}/h_1h_2$$

i.e. k is the geodesic curvature of the streamlines. Consider steady flow in the boundary layer. The boundary conditions for v are

$$v = 0, \qquad x \to \infty$$

and, if the surface over which the flow is considered is a solid, non-spinning body,

$$v = 0, \qquad z = 0.$$

Now if $k = 0$, it is seen from (7) and the second equation of (6) that $v = 0$ is a solution. The term "secondary flow" (also cross flow) is used to indicate that the streamlines in the boundary layer flow do not coincide with the external flow streamlines evaluated on the body. Also $k = 0$ has a simple geometric interpretation: the streamlines are geodesics of the surface. Thus, for steady flow over a non-spinning body, there is no secondary flow if the external flow streamlines are geodesics of the body surface.

A simple example of the above conclusion is provided by the flow over a flat plate with an arbitrary leading edge placed in a uniform stream at no angle of attack; see Fig. 1. The external flow streamlines are just straight lines on the plate so there is no

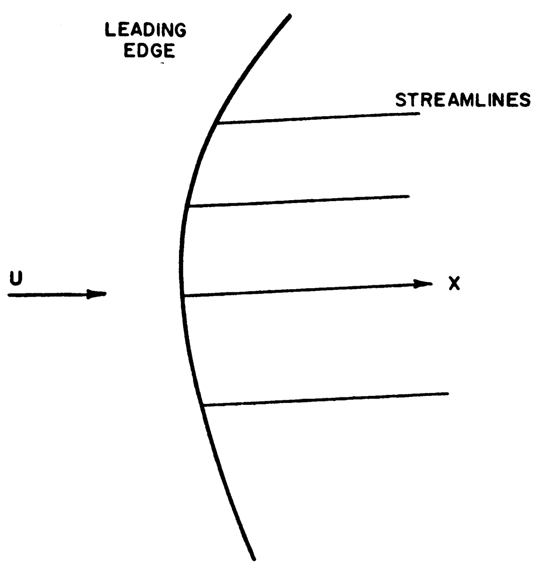

FIG. 1.

secondary flow. (Moore [3], reaches this conclusion by finding a solution of the flow equations. He also indicates how difficulties arise if the leading edge does not have a continuously turning tangent.) A special case of this is the yawed flat plate. However consider yawed infinite cylinders, which many authors have done, [1]. It is easy to see, using the above criterion, that the yawed flat plate is the only case with no secondary flow since for any other cylinder the external flow streamlines cannot be geodesics.

Since $v = 0$ in the streamline coordinates for $k = 0$, the boundary layer flow is essentially two-dimensional. Equations (6), written for a geodesic normal system, become

$$\begin{aligned} uu_x/h_1 + wu_z &= UU_x/h_1 + \nu u_{zz}, \\ \pi_z &= 0, \\ (h_2u)_x/h_1h_2 + w_z &= 0. \end{aligned} \tag{8}$$

A new coordinate, X, can be introduced,

$$X = \int_0^x h_1\, dx$$

and then Eqs. (8) are seen to be in exactly the form of the boundary layer equations over a body of revolution. Therefore the same transformation that Mangler [11] has introduced will reduce equations (8) to the standard two-dimensional equations. Mangler's transformation is

$$\begin{aligned} \xi &= \int_0^X h_2^2\, dX, \\ \eta &= h_2 z, \\ u &= u', \\ w &= h_2 w' - h_{2\xi}\eta u'. \end{aligned} \tag{9}$$

Note that the coordinate transformation here is not of the type (1) and that the transformation to new velocity components (u', w') does not follow a tensor law. Hayes has indicated the possibility of transforming the no-secondary flow equations to two-dimensional form. However his transformation is made by choosing the form of the metric tensor in a suitable manner and it would be very difficult to find out exactly what the transformed coordinates are. From the discussion above it is seen that the transformation is just that of Mangler, except that y appears as a parameter.

Displacement surface. The displacement of the external flow streamlines by the retarding action of the boundary layer is an important effect. In two-dimensional flow the definition of displacement thickness, which is a measure of this effect, is straightforward and there are several equivalent definitions. In three-dimensional flow, if one proceeds in strict analogy to the two-dimensional case, two displacement thicknesses can be defined [9]. These are δ_1 and δ_2, where

$$\begin{aligned} U\delta_1 &= \int_0^h (U - u)\, dx^3, \\ V\delta_2 &= \int_0^h (V - v)\, dx^3, \end{aligned} \tag{10}$$

in which (u, v) and (U, V) are the physical components of the velocity in the x^1 and x^2 directions in the boundary layer and the external flow respectively and h is "some location well outside the boundary layer." (The coordinates x^α are again general coordinates, not the streamline coordinates of the previous section.) Moore refers to these lengths as characterizing mass flow defects and then, using these, defines a displacement surface as follows. In a geodesic normal coordinate system, the displacement surface $x^3 = \Delta(x^1, x^2)$ is an "impermeable surface which would deflect a nonviscous fluid in such a way as to produce a normal velocity (W) satisfying"

$$W = w(x^1, x^2) \quad \text{at} \quad x^3 = h(x^1, x^2),$$

where h has the same meaning as above, and W is the external flow normal velocity. After some approximations Moore gives a differential equation for Δ [9].

Here, using a slight modification of one of the two-dimensional definitions, an equation for Δ is obtained. This approach is quite different from Moore's but, making just the boundary layer approximations, this equation reduces to that of Moore. Analysis of the original equation shows an interesting difference between the three-and two-dimensional cases.

In the following derivation geodesic normal coordinates are used and all vectors are in physical components. It is convenient to use the suffix notation for the range 1, 2 but we set $z = x^3$. First we dispose of some geometrical preliminaries. It can be shown that (see [5], for example) the metric tensor components $g_{\alpha\beta}$ are quadratic functions of z

$$g_{\alpha\beta} = a_{\alpha\beta} + 2b_{\alpha\beta}z + c_{\alpha\beta}z^2, \tag{11}$$

where $a_{\alpha\beta}$ has already been defined and

$$b_{\alpha\beta} = (\partial g_{\alpha\beta}/\partial z)_0/2,$$

$$c_{\alpha\beta} = (\partial^2 g_{\alpha\beta}/\partial z^2)_0/2.$$

Also the following relation holds

$$c_{\alpha\beta} = a^{\sigma\rho}b_{\alpha\sigma}b_{\beta\rho}\ . \tag{12}$$

Specializing to orthogonal surface coordinates, by means of (12), Eqs. (11) can be written as a perfect square. In the notation of (5)

$$H_\alpha = h_\alpha + l_\alpha z, \tag{13}$$

where

$$l_\alpha = b_{\alpha\alpha}/h_\alpha \qquad \text{(no summation)}. \tag{14}$$

If on a surface $z = c =$ constant we draw a simply closed curve, the tangent vector, λ, and the normal vector, n, have the components

$$\begin{aligned} \lambda&: \quad (H_1\, dx^1/ds,\ H_2\, dx^2/ds), \\ n&: \quad (-H_2\, dx^2/ds,\ H_1\, dx^1/ds), \end{aligned} \tag{15}$$

where s is the arc length along the curve.

To define a displacement surface we consider the flux of fluid through a developable surface S formed by the normals to the surface $z = 0$, passing through a simply closed

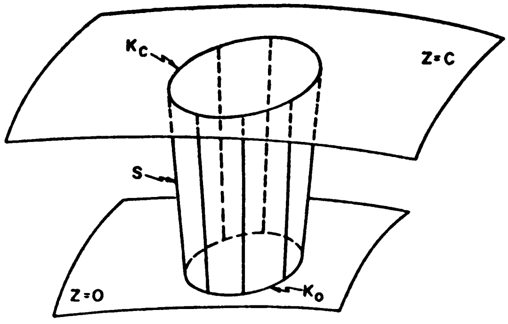

FIG. 2.

curved K_0 on $z = 0$, see Fig. 2. This flux is computed for that part of S between $z = 0$ and $z = h$. For a velocity distribution Q this flux is given by

$$F_Q = \int_S Q\cdot n\, dS = \int_0^h \oint_{K_c} (Q_2 H_1\, dx^1 - Q_1 H_2\, dx^2)\, dz, \tag{16}$$

where K_c is the trace of the developable surface on the surface $z = c$ and $0 < c < h$. The displacement surface

$$z = \Delta(x^1, x^2)$$

is a surface such that the flux through S, for $0 < z < h$, is the same for the following two velocity distributions

$$\begin{aligned}
\text{I.}\quad & Q_1 = Q_2 = 0 && \text{for } 0 < z < \Delta \\
& Q_1 = U && \\
& Q_2 = V && \text{for } \Delta < z < h \\
\text{II.}\quad & Q_1 = u && \\
& Q_2 = v. && \text{for } 0 < z < h
\end{aligned}$$

From (16) we set

$$F_{\text{I}} = F_{\text{II}}$$

and this is the condition imposed to find Δ. Interchanging the order of integration and making use of (13) and (14), after which the line integrals are calculated for K_0, this requirement yields

$$\oint_{K_0} (G_1 h_2\, dx^2 - G_2 h_1\, dx^1) = 0, \tag{17}$$

where

$$\begin{aligned}
G_1 &= U\Delta[1 + (l_2\Delta/2h_2)] + \int_0^h (u - U)[1 + (zl_2/h_2)]\, dz, \\
G_2 &= V\Delta[1 + (l_1\Delta/2h_1)] + \int_0^h (v - V)[1 + (zl_1/h_1)]\, dz.
\end{aligned} \tag{18}$$

The integral in (17) can be written as an integral over the area bounded by K_0 using the "surface divergence theorem" [12]. Since K_0 is arbitrary we then obtain the result

$$\operatorname{div} G = 0 \tag{19}$$

for the vector G: (G_1, G_2) where the operator div is the "surface divergence" [12], i.e.

$$\operatorname{div} G = [\partial(h_2 G_1)/\partial x^1 + \partial(h_1 G_2)/\partial x^2]/h_1 h_2$$

or in the subtensor notation (19) can be written

$$G^{\alpha}_{;\alpha} = 0.$$

For given velocity distributions I and II and a value of h, (19) is a differential equation for Δ. However, in boundary layer theory, h cannot be given a definite value. In fact, in accordance with the requirements of this theory, we must let $h \to \infty$. (Extrapolating from the two-dimensional case, the integrals should converge since $u \to U$, $v \to V$ exponentially as $x \to \infty$.)

Thus far we have made no approximations. We take (U, V) to be the (physical) velocity components, in the x^1 and x^2 directions, of the external flow evaluated on the surface $z = 0$ and (u, v) the corresponding velocity components throughout the boundary layer. Applying a transformation of the type (3) to z and Δ and keeping only the lowest order terms in (18) gives the boundary layer approximation to G_1 and G_2

$$\begin{aligned} G_1 &= U\Delta + \int_0^\infty (u - U)\, dz = U(\Delta - \delta_1), \\ G_2 &= V\Delta + \int_0^\infty (v - V)\, dz = V(\Delta - \delta_2), \end{aligned} \tag{20}$$

using the definitions (10). With these expressions (20) for the vector G, (19) is essentially Moore's equation [9] for the displacement surface. Moore gives some examples of the computation of Δ for special kinds of flow. Here we give a simple example by specializing only the coordinate system. For the streamline coordinates of the previous section $V = 0$, therefore $G_2 = \int_0^\infty v\,dz$ and (19) becomes

$$\partial[h_2 U(\Delta - \delta_1)]/\partial x^1 = -\partial(h_1 G_2)/\partial x^2 = F(x^1, x^2).$$

Thus

$$h_2 U(\Delta - \delta_1) = \int F(x^1, x^2)\, dx^1 + f(x^2),$$

where $f(x^2)$ is a constant of the integration with respect to x^1 and must be evaluated from the conditions of the flow. Note that in the streamline coordinates the product $V\delta_2$ is well defined but δ_2 alone is undefined.

The method used above to obtain Δ is a modification of the corresponding two-dimensional definition of displacement thickness. In two dimensions the developable surface S is taken to be a plane perpendicular to the plane of flow and as a result no differential equation need be solved for the displacement thickness. The Δ determined from (19) using (20) differs from the δ of (10) by a constant of integration which, in most cases of interest, is zero. This is discussed by Moore [9].

However, it is interesting to compare the general expressions (18) for the vector G

in the two- and three-dimensional cases. Let $x^2 = 0$ be the plane of flow for a two-dimensional flow. Then $v = V = 0$ and it is easy to show that $l_2 = 0$. Thus the terms that are neglected after making the boundary layer approximations to obtain (20) disappear automatically for the special case of two-dimensional flow. This would be important if one wanted to calculate approximations to a viscous flow beyond the classical boundary layer theory as Kuo [13] has done for two-dimensional flow past a flat plate. For two-dimensional flow the displacement thickness expression does not change for the higher order approximations whereas for three-dimensional flow, including axially symmetric flow, an expansion of the expressions (18) in powers of $\nu^{1/2}$ would be necessary.

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