INTERMODULATION PRODUCTS FOR ν-LAW BIASED WAVE RECTIFIER FOR MULTIPLE FREQUENCY INPUT*

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Abstract. The intermodulation products obtained by passing the sum of $N + 1$ sinusoids of amplitudes P_1 , $\cdots$ P_N through a power rectifier of characteristic

$$I = \begin{cases} 0 & V < B \\ \alpha(V - B)^\nu & V > B \qquad \nu > 0 \end{cases}$$

have been expressed by S. O. Rice and W. R. Bennett in terms of contour integrals involving products of Bessel functions. In this paper these integrals are rewritten as improper integrals on the real line plus constant terms. These integrals converge fast enough in many cases to be useful in numerical integration.

A number of sub-cases arise, for which formulas for the intermodulation products are listed.

Introduction. The evaluation of the intermodulation products obtained by distorting the time function

$$V(t) = \sum_{r=0}^{N} P_r \cos (p_r t + \gamma_r) \tag{1}$$

through a nonlinear device

$$I = \alpha f(V) \tag{2}$$

has been considered by several authors in recent years [1–10].

The output can be described as†:

$$I(t) = \sum_{m_0=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \tfrac{1}{2} A_{m_0 \cdots m_N} \epsilon_{m_0} \cdots \epsilon_{m_N} \tag{3}$$

$$\cos m_0(p_0 t + \gamma_0) \cos m_1(p_1 t + \gamma_1) \cdots \cos m_N(p_N t + \gamma_N),$$

where

$$\epsilon_0 = 1,$$

$$\epsilon_{m_k} = 2, \qquad m_k = 1, 2, 3 \cdots .$$

A typical term of (3), expressed in terms of sum and difference frequencies, is

$$A_{m_0 \cdots m_N} \cos [m_0(p_0 t + \gamma_0) \pm m_1(p_1 t + \gamma_1) \pm \cdots m_N(p_N t + \gamma_N)].$$

It has been shown‡ that for a wide class of nonlinear functions the intermodulation coefficients are obtainable as contour integrals

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†Reference [3], Eqs. (4.9-15).

‡Reference [3], Eqs. (4.9-17).

$$A_{m_0 \cdots m_N} = \frac{i^M}{\pi} \int_C F(iu) \prod_{r=0}^{N} J_{m_r}(P_r u)\, du, \tag{4}$$

where

$$M = \sum_r m_r \tag{5}$$

and

$$F(iu) = \int_{-\infty}^{\infty} I(V) \exp(-iuV)\, dV, \tag{6a}$$

$$I(V) = \frac{1}{2\pi} \int_C F(iu) \exp(iVu)\, du. \tag{6b}$$

For the biased ν-power rectifier for which

$$I = \begin{cases} 0 & V < B \\ \alpha(V - B)^{\nu} & V > B \end{cases}, \tag{7}$$

$$\nu > 0,$$

$$F(iu) = \alpha\Gamma(\nu + 1)(iu)^{-(\nu+1)} \exp(-iuB) \tag{8}$$

and the path of integration C is the real u axis from $-\infty$ to $+\infty$ with a downward indentation at $u = 0$.

Thus for ν-law rectifiers, (4) specializes to:

$$A^{(\nu)}_{m_0 \cdots m_N} = \alpha\Gamma(\nu + 1) \frac{i^M}{\pi} \int_C (iu)^{-(\nu+1)} \exp(-iuB) \prod_{r=0}^{N} J_{m_r}(P_r u)\, du. \tag{9}$$

In this paper the contour integral (9) is transformed into real converging integrals with bounded integrands plus a constant. Since most of these integrals converge fairly rapidly, numerical desk or machine computation of the coefficients becomes quite practicable, especially when only moderate accuracy is required.

Notation. It should also be noted that the coefficients are functions of ν, m_0, $m_1 \cdots m_N$ and P_0, $P_1 \cdots P_N$, the bias B, and the scale factor α, all together $2N + 5$ degrees of freedom. However, (9) is homogeneous, and can be written

$$A^{(\nu)}_{m_0 \cdots m_N} = \alpha P_0^{\nu}\Gamma(\nu + 1) \frac{i^M}{\pi} \int_C (iu)^{-(\nu+1)} \exp -iu\left(\frac{B}{P_0}\right) \prod_{r=0}^{N} J_{m_r}(P_r u/P_0)\, du. \tag{10}$$

It is convenient to choose as P_0 the largest amplitude of the set of input amplitudes, and to substitute

$$\frac{P_r}{P_0} = k_r \leq 1, \qquad k_0 = 1, \tag{11a}$$

$$\frac{B}{P_0} = h. \tag{11b}$$

The integrand of (10) becomes a function of ν, the degree of the rectifier law; h, the normalized bias; $k_1 \cdots k_N$ the normalized input amplitudes; and $m_0 \cdots m_N$ the order of the harmonics; altogether $2N + 3$ variables.

Equation (3) may be rewritten as

$$I(t) = \alpha P_0^{\nu} \sum_{m_0=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \tfrac{1}{2} A_{m_0 \cdots m_N}^{(\nu)}(h, k_1, \cdots k_N) \prod_{r=0}^{N} \epsilon_{m_r} \cos m_r(p_r t + \gamma_r), \qquad (12)$$

where the coefficients

$$\alpha P^{\nu} A_{m_0, \cdots m_N}^{(\nu)}(h, k_1 \cdots k_N) = A_{m_0, \cdots m_N}^{(\nu)} . \qquad (13)$$

The coefficients on the left [with the variables $(h, k_1 \cdots k_N)$ spelled out] correspond to the notation of Sternberg, Kaufman, Shipman and Thurston, who have called the coefficients "Bennett functions," while the coefficients on the right are given in the notation of Rice, provided that the amplitudes have been normalized. This paper will use the notation of Sternberg, Kaufman, Shipman and Thurston, and concern itself with the transformation of

$$A_{m_0 \cdots m_N}^{(\nu)}(h, k_1 \cdots k_N) = \Gamma(\nu + 1) \frac{i^M}{\pi} \int_C (iu)^{-(\nu+1)} \exp(-iuh) \prod_{r=0}^{N} J_{m_r}(k_r u)\, du, \qquad (14)$$

$$M = \sum_{r=0}^{N} m_r , \qquad k_0 = 1,$$

into real integrals. The reader is advised that for integer ν, especially of low order, the coefficients (14) are available in terms of published tables [6, 8, 9] computed by numerical evaluation of certain multiple integrals, and may be extended to other values of the parameters by recurrence relations [2, 4, 5]. The utility of this paper rests partly on the independent evaluation of coefficients without using multiple integration, partly on the evaluation of coefficients for non-integer power laws for which the multiple integral formulation does not apply directly, and partly on the function theoretical perspective concerning Bennett functions, obtainable from the integral formulation.

Evaluation of the contour integral. The integral (14) is broken up into three ranges of integration: over the negative real u axis, $\infty < u < -\delta$; over the positive u axis, $\delta < u < \infty$; and a semicircular contour with radius δ indented into the negative imaginary half plane. By a change of variable from u to $-u$ the negative u axis integral is reflected onto the positive half axis. One obtains:

$$\begin{aligned} A_{m_0 \cdots m_N}^{(\nu)}(h, k_1, \cdots k_N) = \frac{1}{\pi} \Gamma(\nu + 1) i^M \Big\{ & \oint_\delta (iu)^{-(\nu+1)} \exp(-iuh) \prod_{r=0}^{N} J_{m_r}(k_r u)\, du \\ & + 2 \cos \frac{\pi}{2} (\nu + 1) \int_\delta^\infty u^{-(\nu+1)} \cos uh \prod_{r=0}^{N} J_{m_r}(k_r u)\, du \qquad (15a) \\ & - 2 \sin \frac{\pi}{2} (\nu + 1) \int_\delta^\infty u^{-(\nu+1)} \sin uh \prod_{r=0}^{N} J_{m_r}(k_r u)\, du \Big\} \end{aligned}$$

if

$$\sum_{r=0}^{N} m_r = M \quad \text{even}$$

or

$$A^{(\nu)}_{m_0,\cdots m_N}(h, k_1, \cdots k_N) = \frac{1}{\pi}\Gamma(\nu+1)i^{M-1}\left\{i \oint_\delta (iu)^{-(\nu+1)} \exp(-iuh) \prod_{r=0}^{N} J_{m_r}(k_r u)\, du \right.$$
$$+ 2\cos\frac{\pi}{2}(\nu+1)\int_\delta^\infty u^{-(\nu+1)} \sin uh \prod_{r=0}^{N} J_{m_r}(k_r u)\, du \tag{15b}$$
$$\left. + 2\sin\frac{\pi}{2}(\nu+1)\int_\delta^\infty u^{-(\nu+1)} \cos uh \prod_{r=0}^{N} J_{m_r}(k_r u)\, du\right\}$$

if

$$\sum_{r=0}^{N} m_r = M \quad \text{odd.}$$

The semicircular contour integral in (15) is evaluated by a) replacing the integrand by a power series in u, b) replacing u by its complex polar form

$$u = \delta \exp(i\theta) \qquad du = iu\, d\theta \tag{16}$$

and c) integrating term by term from $\theta = \pi$ to $\theta = 2\pi$.

The resulting series in ascending powers of δ will contain negative powers of δ (the principal part) provided $\nu > M$; a series in ascending positive powers of δ, and, if ν is an integer, a term free of δ (the zero power of δ) equal to $\pi i^{-\nu} a_\nu$, where

$$a_\nu = \frac{1}{\nu!}\left[\frac{d^\nu}{du^\nu}\left[\exp(-iuh) \prod_{r=0}^{N} J_{m_r}(k_r u)\right]\right]_{u=0}. \tag{17}$$

a_ν can be recognized as the residue of the integrand of (14) and is obtained for computational purposes most easily by identification with the νth coefficient of one of the power series expansions

$$F_1(u) = \cos uh \prod_{r=0}^{N} J_{m_r}(k_r u) = \sum_{l=0}^{\infty} a_l u^l; \quad \text{if} \quad M + \nu \quad \text{odd} \tag{18a}$$

$$F_2(u) = -i \sin uh \prod_{r=0}^{N} J_{m_r}(k_r u) = \sum_{l=0}^{\infty} a_l u^l; \quad \text{if} \quad M + \nu \quad \text{even.} \tag{18b}$$

Since all powers of u have zero coefficients for

$0 \le l < M$ in the case of (18a) and

$0 \le l < M + 1$ in the case of (18b),

the residue a_ν vanishes for $M \ge \nu + 1$ if $M + \nu$ is odd and for $M \ge \nu$ if $M + \nu$ is even.

The real integrals of (15), containing δ as the lower limit, can be transformed so as to bring into evidence the principal part of the Laurent series in δ contained in them. If $F(u)$ and its derivatives $F'(u)$, $F''(u)$, $\cdots$ $F^{(p)}(u)$ are bounded and p is an integer so that

$$p - 1 < \nu \le p \tag{19}$$

then, by repeated application of integration by parts, one obtains the identity

$$\int_\delta^b \frac{F(u)}{u^{\nu+1}}\,du = -\left[\frac{F(b)}{\nu b^\nu} + \frac{F'(b)}{\nu(\nu-1)b^{\nu-1}} + \cdots + \frac{F^{(p-1)}(b)}{\nu(\nu-1)\cdots(\nu-p+1)b^{\nu-p+1}}\right]$$
$$+\left[\frac{F(\delta)}{\nu\delta^\nu} + \frac{F'(\delta)}{\nu(\nu-1)\delta^{\nu-1}} + \cdots + \frac{F^{(p-1)}(\delta)}{\nu(\nu-1)\cdots(\nu-p+1)\delta^{\nu-p+1}}\right] \tag{20}$$
$$+\frac{1}{\nu(\nu-1)\cdots(\nu-p+1)}\int_\delta^b u^{-(\nu-p+1)}F^{(p)}(u)\,du.$$

Substituting for $F(u)$ $F_1(u)$ or $F_2(u)$ (see 18), one sees that the first bracket of (20) vanishes as $b \to \infty$. Replacing each term of the second bracket by its power series expansion, one obtains a Laurent series in δ for the second bracket collectively.

From (15a) and (15b) it is clear that for ν integer $u^{-(\nu+1)}F(u)$ in (20) must be even for which $u^{-\nu}F(u)$ is an odd function. Hence, for all cases of interest in which ν is an integer, each term of the brackets of (20) will be an odd function and it will be free of the zero power of δ. If ν is not an integer, all powers of the Laurent series in δ of the second bracket of (20) are not integers; therefore a possible term constant in δ is excluded.

Finally it can be shown that the integrals on the right hand side of (20) contain, as far as terms of interest for substitution into (15) are concerned, δ only as a power series in positive terms of δ. Thus, as δ becomes small, these real integrals converge to their value with the lower limit equal to zero.

In the case of integer ν, $\nu = p$ and the integral on the right of (20) becomes:

$$\frac{1}{\nu!}\int_\delta^\infty u^{-1}F^{(\nu)}(u)\,du.$$

From (18) $u^\nu F^{(\nu-1)}(u)$ is an odd function; it follows that $u^{-1}F^{(\nu)}(u)$ is an even function, and its integrand becomes an odd function of the limits of integration. On the other hand, since $F^{(\nu)}(u)$ exists for $u = 0$, its power series expansion involves only non-negative odd terms, the lowest of which is u^1. Hence, the power series expansion of $u^{-1}F^{(\nu)}(u)$ contains only non-negative integral powers of u. Integrating term by term and introducing the lower limit one obtains only odd integer positive powers of δ.

If ν is non-integral, the integrals on the right side of (20) involve the lower limit δ only in positive non-integral powers of δ. Since $0 < \nu - p + 1 < 1$ and $F^{(p)}(u)$ is expressible in a power series in non-negative powers of (u), a term-by-term integration leads to positive powers of the lower limit only. We note in passing that, for the special case of $M = 0$,

$$\lim_{\delta\to 0}\int_\delta^\infty u^{-(\nu-p+1)}\cos uh J_0(u)J_0(k_1u)\cdots J_0(k_Nu)\,du$$

is an improper integral which for machine purposes may be transformed into a proper form with bounded integrand by one further integration by parts.

Since the integrand of (14) is regular for all finite u, except at the origin, (14) must be independent of the value of δ. This implies that the principal parts of the Laurent series in δ of the expressions in (15) cancel exactly. The contribution of the positive powers of δ in each expression of (15) can be made as small as desired by making δ small.

If (20) has been substituted into (15), the integrals on the right side of (20) go over into integrals with lower limit of integration zero.

The only possible contribution of the semicircular contour is the residue term appearing for integer ν.

The integrals (15) may be written:

$$A^{(\nu)}_{m_0,\cdots m_N}(h, k_1, \cdots k_N) = \frac{\Gamma(\nu+1)}{\nu(\nu-1)\cdots(\nu-p+1)} i^M$$
$$\cdot\left\{\left[\cos\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{p-\nu-1}\frac{d^p}{d^p u}\left[\cos uh\prod_{r=0}^N J_{m_r}(k_r u)\right]du\right.$$
$$-\left[\sin\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{p-\nu-1}\frac{d^p}{d^p u}\left[\sin uh\prod_{r=0}^N J_{m_r}(k_r u)\right]du \tag{21a}$$
$$\left.+\,p!\,i^{-\nu}a_\nu\right\} \quad \text{if} \quad M \quad \text{even},$$

$$A^{(\nu)}_{m_0,\cdots m_N}(h, k_1, \cdots k_N) = \frac{\Gamma(\nu+1)}{\nu(\nu-1)\cdots(\nu-p+1)} i^{M-1}$$
$$\cdot\left\{\left[\cos\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{p-\nu-1}\frac{d^p}{d^p u}\left[\sin uh\prod_{r=0}^N J_{m_r}(k_r u)\right]du\right.$$
$$+\left[\sin\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{p-\nu-1}\frac{d^p}{d^p u}\left[\cos uh\prod_{r=0}^N J_{m_r}(k_r u)\right]du \tag{21b}$$
$$\left.+\,p!\,i^{1-\nu}a_\nu\right\} \quad \text{if} \quad M \quad \text{odd}$$

and for both (21a) and (21b)

$$a_\nu = \begin{cases} 0 & \nu \text{ non-integer} \\ \dfrac{d}{du}\left|\exp - iuh\prod_{r=0}^N J_{m_r}(k_r u)\right|_{u=0} & \nu \text{ integer} \end{cases} \tag{21c}$$

as well as:

$$p-1 < \nu \le p,$$
$$M = \sum_{r=0}^N m_r\,,$$
$$k_0 = 1, \qquad k_r \le 1.$$

Equation (21) is more complicated than necessary since all possible cases are accounted for in two possible forms. Specializing further, one obtains simpler equations which are listed in Table 1.

If

$$h \ge \sum_{r=0}^N k_r: \qquad A^{(\nu)}_{m_0,\cdots m_N}(h, k_1, \cdots k_N) = 0 \tag{22a}$$

If

$$h \leq -\sum_{r=0}^{N} k_r : \qquad A^{(\nu)}_{m_0, \cdots m_N}(h, k_1, \cdots k_N) = 2\nu!\, i^{(M-\nu)} a_\nu \tag{22b}$$

and ν integer where a_ν is given by (17) or (18).

The above results were obtained by observing that if ν is an integer one may, without contributing to the integral (14), close the contour over the negative imaginary halfplane if $h \geq \sum_{r=0}^{N} k_r$; and close the contour over the positive imaginary halfplane if $h \leq -\sum_{r=0}^{N} k_r$. These forms express the output when the bias is either so large as to prevent all output, or so negative as to be free of cut-off regions.

Concluding remarks. This paper has been concerned with the analytic formulation in terms of integrals with infinite upper limit. A glance at the original integral (10) should convince the reader that the integrals resulting from it must be bounded for all portions of the path C except possibly in the neighborhood of the origin, and it has been the burden of this paper to show that the contribution of the integral (10) over a portion of the path near the origin is also bounded.

For numerical evaluation of the integrals in Table 1 one must either find closed form solutions in terms of other transcendental functions for which tabulated results exist, or resort to numerical integration. For the zero bias case with two frequencies one can indeed identify integrals (21a, b) with Weber-Schafheitlin integrals [1, 11, 12] provided that $m_0 + m_1 > +1$. These integrals are expressible in hypergeometric series. For other two-frequency coefficients and integer power laws, the evaluation of the intermodulation coefficients might best be performed in terms of the published Tables of Bennett functions, and recursion relations listed in [4, 8]. (These relations and others can, of course, also be obtained by application of the standard recursion formulas of Bessel functions to the integrals in Table 1).

In all other cases the integrals (22) in Table 1 must be evaluated numerically after truncation at an upper value of u. From the point of view of precision computation it is imperative that the incomplete (truncated) integrals converge to their limit rapidly as the truncation value is increased. Therefore, for large u, the power of u in the integrands (22) should be as negative as possible.

For integrals (22c to 22h) the identity (20) can be applied in the form suggested in note 4 of Table 1. The integrals (22) are split into two regions at $u = 1$ and the slowly converging integral in the region of large u is replaced, by virtue of (20), by an integral with faster rate of convergence plus an expression in terms of the derivatives of the integrand evaluated at $u = 1$. (See note 4 of Table 1). The problem of truncation still remains. Dr. Sternberg has pointed out to the author that, for $|x| \gg 1$ and $|x| \gg m$, the Bessel functions may be replaced by one of their asymptotic representations, (11, 13). The simplest representation becomes

$$J_m(x) = (\tfrac{1}{2}\pi x)^{-1/2} \cos\left[x + (m - .5)\frac{\pi}{2}\right]. \tag{23}$$

The above approximation can be applied to the estimation of the integrals (22) beyond their truncation value. Using $x_r = k_r u$, one obtains integrals involving trigonometric functions and negative powers of u. If it happens that the negative power of u is an integer, one can, by repeated integration by parts, obtain integrals with integrands

TABLE 1.

$$A^{(\nu)}_{m_0,\cdots m_N}(h, k_1, \cdots, k_N) = \frac{\Gamma(\nu+1)i^M}{\pi}\int_C u^{-(\nu+1)} \exp(-iuh) \prod_{m_r=0}^{N} J_{m_r}(k_r u)\, du$$

	Conditions	$A^{(\nu)}_{m_0,\cdots m_N}(h, k_1, \cdots, k_N)$	
(22,a)	ν integer $M \geq \nu + 1$ $M + \nu$ odd	$\nu! i^{M-\nu-1} \frac{2}{\pi}\int_0^\infty u^{-(\nu+1)} \cos uh \prod_{r=0}^{N} J_{m_r}(k_r u)\, du$	notes 1, 2
(22,b)	ν integer $M \geq \nu$ $M + \nu$ even	$\nu! i^{M-\nu-2} \frac{2}{\pi}\int_0^\infty u^{-(\nu+1)} \sin uh \prod_{r=0}^{N} J_{m_r}(k_r u)\, du$	
(22,c)	ν integer $M < \nu + 1$ $M + \nu$ odd	$i^{M-\nu-1}\left\{\frac{2}{\pi}\int_0^\infty u^{-1} \frac{d^\nu}{du^\nu}\left[\cos uh \prod_{r=0}^{N} J_{m_r}(k_r u)\right] du + \operatorname{Lim}_{u\to 0} \frac{d^\nu}{du^\nu}\left[\sin uh \prod_{r=0}^{N} J_{m_r}(k_r u)\right]\right\}$	notes 3, 4
(22,d)	ν integer $M < \nu$ $M + \nu$ even	$i^{M-\nu-2}\left\{\frac{2}{\pi}\int_0^\infty u^{-1} \frac{d^\nu}{du^\nu}\left[\sin uh \prod_{r=0}^{N} J_{m_r}(k_r u)\right] du - \operatorname{Lim}_{u\to 0} \frac{d^\nu}{du^\nu}\left[\cos uh \prod_{r=0}^{N} J_{m_r}(k_r u)\right]\right\}$	
(22,e)	ν non-integer $M > \nu + 1$ M even	$\Gamma(\nu+1) i^M \left\{\left[\cos \frac{\pi}{2}(\nu+1)\right] \frac{2}{\pi}\int_0^\infty u^{-(\nu+1)} \cos uh \prod_{r=0}^{N} J_{m_r}(k_r u)\, du - \left[\sin \frac{\pi}{2}(\nu+1)\right] \frac{2}{\pi}\int_0^\infty u^{-(\nu+1)} \sin uh \prod_{r=0}^{N} J_{m_r}(k_r u)\, du\right\}$	

TABLE 1 (*cont'd*)

(22,f)	ν non-integer $M > \nu + 1$ M odd	$\Gamma(\nu+1)i^{M-1}\left\{\left[\cos\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{-(\nu+1)}\sin uh\prod_{r=0}^{N}J_{m_r}(k_ru)\,du + \left[\sin\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{-(\nu+1)}\cos uh\prod_{r=0}^{N}J_{m_r}(k_ru)\,du\right\}$
(22,g)	ν non-integer $M < \nu + 1$ M even $p - 1 < \nu < p$	$\frac{\Gamma(\nu+1)i^M}{\nu(\nu+1)\cdots(\nu-p+1)}\left\{\left[\cos\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{-(\nu-p+1)}\frac{d^p}{du^p}\left[\cos uh\prod_{r=0}^{N}J_{m_r}(k_ru)\right]du - \left[\sin\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{-(\nu-p+1)}\frac{d^p}{du^p}\left[\sin uh\prod_{r=0}^{N}J_{m_r}(k_ru)\right]du\right\}$
(22,h)	ν non-integer $M < \nu + 1$ M odd $p - 1 < \nu < p$	$\frac{\Gamma(\nu+1)i^M}{\nu(\nu+1)\cdots(\nu-p+1)}\left\{\left[\cos\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{-(\nu-p+1)}\frac{d^p}{du^p}\left[\sin uh\prod_{r=0}^{N}J_{m_r}(k_ru)\right]du + \left[\sin\frac{\pi}{2}(\nu+1)\right]\frac{2}{\pi}\int_0^\infty u^{-(\nu-p+1)}\frac{d^p}{du^p}\left[\sin uh\prod_{r=0}^{N}J_{m_r}(k_ru)\right]du\right\}$

Note 1. $k_0 = 1 \geq k_r$
k_0 is the normalized largest amplitude.

Note 2. $M = \sum^{N} m_r$.

Note 3. $\lim_{u\to 0}\frac{d^\nu}{du^\nu}F(u) = \nu!\,a_\nu$
a_ν is the residue of $u^{-(\nu+1)}F(u)$.

Note 4.
for $p - 1 < \nu \leq p$
p integer

$$\int_0^\infty u^{-(\nu-p+1)}\frac{d^pF(u)}{du^p}\,du = \int_0^1 u^{-(\nu-p+1)}\frac{d^pF(u)}{du^p}\,du + [\nu(\nu-1)\cdots(\nu-p+1)]\left\{\int_1^\infty u^{-(\nu+1)}F(u)\,du - \left[\frac{F(1)}{\nu} + \frac{F^{(1)}(1)}{\nu(\nu-1)} + \cdots \frac{F^{(p-1)}(1)}{\nu(\nu-1)\cdots(\nu-p+1)}\right]\right\}.$$

$u^{-1} \sin u$ or $u^{-1} \cos u$. Thus estimation of error resulting from truncation at U amounts to table look-up of $Si(U)$ and $Ci(U)$. In general, an upper bound to the approximation is found by neglecting the oscillating cosine term. Unfortunately this bound is too large to be of much use in accurate computation. In general, the error in the truncation estimate, which results from a substitution of the trigonometric functions for Bessel functions, can also be estimated from consideration of the error in the first neglected term of the semi-convergent series (23). It should also be noted that, for small k_r, u must be large to make $k_r u \gg 1$. The latter fact precludes the direct use of the asymptotic approximation for the Bessel function in those cases where the input amplitudes P_r [Eq. (1)] differ greatly; however, in this case one may use the power series expansion of the Bessel function for the intermediate region:

$$U < u < \frac{A}{k_r}, \qquad A \gg 1.$$

As for the numerical integration of (22) between fixed limits, the reader is referred to books on numerical analysis for a discussion of methods and error bounds.

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