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LIMITS ON THE ZEROS OF A NETWORK DETERMINANT*

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The properties of a network are determined from the location of the zeros of the network matrix and its minors; the exact determination of these zeros often involves the solution of a high order equation. In many applications, e.g. in questions of stability, one is interested not in the exact determination of these zeros but in their exclusion from certain parts of the plane. In this note, limits will be given on the location of the roots of the matrix of a network consisting of two kinds of elements.

The matrix of a network containing active sources and only two kinds of elements is given by

$$M(z) = \begin{bmatrix} a_{11}z + b_{11}, & a_{12}z + b_{12}, & \cdots, & a_{1n}z + b_{1n} \\ a_{21}z + b_{21}, & a_{22}z + b_{22}, & \cdots, & a_{2n}z + b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}z + b_{n1}, & a_{n2}z + b_{n2}, & \cdots, & a_{nn}z + b_{nn} \end{bmatrix}, \tag{1}$$

where

$$a_{kk} \geq 0, \qquad b_{kk} \geq 0. \tag{2}$$

In this matrix, $z = p$ for the R-L and R-C case, $z = p^2$ for the L-C case.

We introduce the constants[1]:

$$R_k = \sum_r{}' \mid a_{kr} \mid, \qquad P_k = \sum_r{}' \mid b_{kr} \mid,$$

$$\Theta_k = \text{Arcsin}\,\frac{R_k}{a_{kk}}, \qquad \varphi_k = \text{Arcsin}\,\frac{P_k}{b_{kk}}, \tag{3}$$

and

$$A = \max_k.\ \frac{b_{kk} + P_k}{a_{kk} - R_k}, \qquad a = \min_k.\ \frac{b_{kk} - P_k}{a_{kk} + R_k},$$

$$\alpha = \max_k.\ (\Theta_k + \varphi_k). \tag{4}$$

Suppose that z_i is a root of the determinant of $M(z)$. We shall show that z_i satisfies the following inequalities

$$a \leq \mid z_i \mid \leq A, \qquad \mid \arg(-z_i) \mid \leq \alpha. \tag{5}$$

Consider the system

$$\sum_{r=1}^{n} (a_{kr}z_i + b_{kr})w_r = 0 \qquad k = 1, 2, \cdots, n. \tag{6}$$

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[1]In this paper we shall denote by $\sum_r' a_{kr}$ the summation for $r = 1, 2, \cdots, n$ $r \neq k$.

The determinant of its coefficients is by assumption zero, therefore it has a non-trivial solution $w_1, w_2, \cdots, w_n$. If w_k is the variable with the greatest absolute value, then with

$$\frac{w_r}{w_k} = t_r ,$$

we have

$$| t_r | \leq 1, \qquad r = 1, 2, \cdots, n. \tag{7}$$

Dividing the kth equation of (6) by w_k and solving for z_i we obtain

$$-z_1 = \frac{b_{kk} + \sum_r' b_{kr} t_r}{a_{kk} + \sum_r' a_{kr} t_r}. \tag{8}$$

Therefore [see (2) and (7)]

$$\frac{b_{kk} - P_k}{a_{kk} + R_k} \leq | z_i | \leq \frac{b_{kk} + P_k}{a_{kk} - R_k} ,$$

$$| \arg (-z_i) | \leq \Theta_k + \varphi_k. \tag{9}$$

The relations (9) are true for some k, hence (5) follows. Interchanging rows and columns of $M(z)$ we obtain results similar to (5). Combining the two sets of inequalities for z_i and arg $(-z_i)$ we might thus improve our limits.

From the above it follows that if

$$\alpha < \frac{\pi}{2} \tag{10}$$

then

$$| \arg (-z_i) | < \frac{\pi}{2}$$

hence z_i has a negative real part. This is true if

$$R_k < \frac{a_{kk}}{(2)^{1/2}} , \qquad P_k < \frac{b_{kk}}{(2)^{1/2}} \qquad k = 1, 2, \cdots, n. \tag{11}$$

because then

$$\Theta_k < \frac{\pi}{4} , \qquad \varphi_k < \frac{\pi}{4}.$$

These conditions can readily be verified for a given matrix; one thus establishes simple sufficient conditions for the negativeness of the real part of the zeros of $M(z)$, which is useful information in network analysis. Suppose that a network contains R-C elements, control sources and feedback, depending on a parameter k. The matrix of such a network is given by (1) and the system is stable if the real part of its zeros is negative. To determine the values of k for which this is true, it would be necessary to find the zeros of the determinant in terms of k; this is not easy. The above discussion offers a simple method of establishing sufficient conditions on k, for the stability of the system; clearly beyond these limits the system is not necessarily unstable.