THE EFFECT OF TRANSVERSE SHEAR DEFORMATION ON THE BENDING OF ELASTIC SHELLS OF REVOLUTION*

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1. Introduction. The classical theory of thin elastic shells of revolution with small axisymmetric displacements, due to H. Reissner [1] for spherical shells of uniform thickness, and Meissner [2, 3] for the general shells of revolution, was recently reconsidered by E. Reissner [4], where reference to the historical development of the subject may be found. Although the formulation of the linear theory and the resulting differential equations contained in [4] differ only slightly from those of H. Reissner and Meissner, they offer certain advantages not revealed in earlier formulations.

As has been recently pointed out, the improvement of the linear theory of thin shells, by inclusion of the effects of both transverse shear deformation and normal stress, requires the formulation of suitable stress strain relations and appropriate boundary conditions which, for shells of uniform thickness, have been very recently carried out by E. Reissner [5] and the present author [6]**. The latter also contains explicit stress strain relations when only the effect of transverse shear deformation is fully accounted for but that of normal stress is neglected.

The present paper is concerned with the small axisymmetric deformation of elastic shells of revolution, where only the effect of transverse shear deformation is retained. The basic equations which include the appropriate expression for the transverse (shear) stress resultant due to the variation in thickness, are reduced to two simultaneous second-order differential equations in two suitable dependent variables. These equations are then combined into a single complex differential equation which is valid for shells of uniform thickness, as well as for a large class of variable thickness. Finally, by an extension of the method of asymptotic integration due to Langer [8], the general solution of the complex differential equation is discussed.

2. The basic equations in surface-of-revolution coordinates. The parametric equations of the middle surface of the shell may be written as

$$r = r(\xi), \qquad z = z(\xi), \tag{2.1}$$

where ξ, together with the polar angle θ in the x, y-plane constitute the coordinates of the middle surface. Denoting by ϕ the inclination of the tangent to the meridian of the shell, then

$$\tan \phi = \frac{dz}{dr} \tag{2.2}$$

and

$$r' = \alpha \cos \phi, \qquad z' = \alpha \sin \phi, \tag{2.3}$$

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**The stress strain relations derived in both [5] and [6] were obtained by application of E. Reissner's variational theorem [7]. References to earlier work on the subject appear in [5] and [6].

where

$$\alpha = [(r')^2 + (z')^2]^{1/2} \tag{2.4}$$

and prime denotes differentiation with respect to ξ.

The square of the linear element for the triply orthogonal curvilinear coordinate system ξ, θ, and ζ (measured along the outward normal to the middle surface) is

$$ds^2 = \alpha^2\left(1 + \frac{\zeta}{R_\xi}\right)^2 d\xi^2 + r^2\left(1 + \frac{\zeta}{R_\theta}\right)^2 d\theta^2 + d\zeta^2, \tag{2.5}$$

where

$$R_\xi = -\frac{\alpha}{\phi'}, \qquad R_\theta = -\frac{r}{\sin\phi} \tag{2.6}$$

are the principal radii of the curvature of the middle surface.

For axisymmetric deformation of shells of revolution, the displacements in the tangential and normal directions may be taken in the form:

$$U_\xi = u(\xi) + \zeta\beta(\xi), \qquad W = w(\xi), \tag{2.7}$$

where u and w denote the components of displacements at the middle surface, and β is the change of the slope of the normal to the middle surface of the shell. As pointed out in both [5] and [6], the approximation (2.7) for the displacements is consistent with the neglect of the effect of transverse normal stress in the stress strain relations for a thin shell. It is convenient to express the displacements u and w in terms of the corresponding components u_r and w_z (on the middle surface $\zeta = 0$) along the radial and axial directions, respectively. Thus,

$$u_r = u\cos\phi - w\sin\phi, \qquad w_z = u\sin\phi + w\cos\phi \tag{2.8a}$$

or alternatively,

$$u = u_r\cos\phi + w_z\sin\phi, \qquad w = w_z\cos\phi - u_r\sin\phi. \tag{2.8b}$$

The components of strain, as given in [5, 6], with the aid of (2.8) may be written in the form

$$\epsilon_\xi^0 = \frac{u_r' + z'\omega}{r'}, \qquad \epsilon_\theta^0 = \frac{u_r}{r},$$

$$\gamma_{\xi\zeta}^0 = w_z'\frac{\cos\phi}{\alpha} - u_r'\frac{\sin\phi}{\alpha} + \beta, \tag{2.9a}$$

$$\kappa_\xi = \frac{\beta'}{\alpha}, \qquad \kappa_\theta = \frac{r'}{r}\frac{\beta}{\alpha} \tag{2.9b}$$

and the relevant compatibility equation, obtained from (2.9a) by elimination of u_r is

$$(r'\epsilon_\xi^0) = (r\epsilon_\theta^0)' + z'\omega, \tag{2.10a}$$

where

$$\omega = \gamma_{\xi\zeta}^0 - \beta. \tag{2.10b}$$

It may be noted that, upon the neglect of terms involving $\gamma_{\xi\zeta}^0$ (which represent the effect of transverse shear deformation), (2.9) and (2.10) reduce to the corresponding expressions of the classical theory.

The stress differential equations of equilibrium, given in [4], may be written as

$$\begin{aligned} (rP_V)' &= -r\alpha q_V , \\ (rP_H)' - \alpha N_\theta + r\alpha q_H &= 0, \\ (rM_\xi)' - \alpha \cos\phi M_\theta - r\alpha(P_V \cos\phi - P_H \sin\phi) &= 0, \end{aligned} \tag{2.11}$$

where P_H and P_V are "horizontal" and "vertical" stress resultants, respectively; q_H and q_V are the components of the applied load in the horizontal and vertical directions; M_ξ and M_θ denote the stress couples; and the stress resultants N_ξ and N_θ, as well as the transverse stress resultant V (due to $\tau_{\xi\zeta}$), are related to P_H and P_V by

$$\begin{aligned} \alpha N_\xi &= r'P_H + z'P_V , \\ \alpha V &= -z'P_H + r'P_V , \\ \alpha N_\theta &= (rP_H)' + r\alpha q_H . \end{aligned} \tag{2.12}$$

We recall that the stress strain relations (as well as the stress differential equations of equilibrium) which are employed in the classical theory of shells (and plates) of variable thickness, are those for shells of uniform thickness, although the resulting differential equations take into account the effect of thickness variation. As the present analysis is concerned mainly with the effect of transverse shear deformation, it becomes necessary to modify slightly the available stress strain relations of uniform shells [5, 6] by incorporating the effect of thickness variation into the expression for the transverse shear stress $\tau_{\xi\zeta}$. To this end, we replace in Eqs. (2.9) of Ref. [6] the expression corresponding to $\tau_{\xi\zeta}$ by

$$\left(1 + \frac{\zeta}{R_\theta}\right)\tau_{\xi\zeta} = \frac{3}{2h} V\left[1 - \left(\frac{\zeta}{h/2}\right)^2\right] - 3\left(\frac{h'}{2\alpha}\right)\left(\frac{M_\xi}{h^2}\right)\left[1 - 3\left(\frac{\zeta}{h/2}\right)^2\right] + \left(\frac{h'}{2\alpha}\right)\left(\frac{N_\xi}{h}\right)\left(\frac{\zeta}{h/2}\right), \tag{2.13}$$

which, together with σ_ξ, satisfies the stress boundary conditions on $\zeta = \pm h/2$, where the direction cosines of the outward normal to the middle surface are $\{-h'/2\alpha, 0, 1\}/[1 + (h'/2\alpha)^2]^{\frac{1}{2}}$. While expression (2.13) will not change the differential equations of equilibrium (2.11), it does contribute to the stress strain relations, as may be seen from the variational equation employed in [5, 6]. However, in view of the neglect of the transverse normal stress, together with neglect of second-order corrections in h/R in comparison with first-order corrections, the resulting stress strain relations for a uniform shell (in the form of (3.6) of Ref. [6]), except for a modification in the transverse shear stress strain relation, remain unaltered.

We close this section by recording the stress strain relations in a manner suitable for subsequent analysis. Thus,

$$\begin{aligned} \epsilon_\xi^0 &= \frac{1}{C}(N_\xi - \nu N_\theta) + k\lambda\left[\frac{1}{\alpha}\left(\beta' + \nu\frac{r'}{r}\beta\right)\right], \\ \epsilon_\theta^0 &= \frac{1}{C}(N_\theta - \nu N_\xi) - k\lambda\left[\frac{1}{\alpha}\left(\frac{r'}{r}\beta + \nu\beta'\right)\right], \end{aligned} \tag{2.14a}$$

$$\begin{aligned}
\gamma^0_{\xi\zeta} &= \frac{6}{5Gh}\left[V - \frac{1}{2}\left(\frac{h'}{h}\right)\frac{M_\xi}{\alpha}\right],\\
\alpha M_\xi &= D\left[\left(\beta' + \nu\frac{r'}{r}\beta\right) - \alpha\lambda\left(\frac{u_r' + z'\omega}{r'}\right)\right],\\
\alpha M_\theta &= D\left[\left(\frac{r'}{r}\beta + \nu\beta'\right) + \alpha\lambda\left(\frac{u_r}{r}\right)\right],
\end{aligned} \tag{2.14b}$$

where

$$C = Eh, \qquad G = \frac{E}{2(1+\nu)}, \qquad D = \frac{Eh^3}{12(1-\nu^2)},$$

E and ν are Young's modulus and Poisson's ratio, respectively, and

$$k = \frac{h^2}{12(1-\nu^2)}, \qquad \lambda = \left(\frac{1}{R_\xi} - \frac{1}{R_\theta}\right) = \frac{1}{\alpha}\left(\frac{z'}{r} - \phi'\right). \tag{2.15}$$

3. Differential equations of shells of revolution. With β and (rP_H) as basic dependent variables, proper elimination between Eqs. (2.12), (2.14), (2.11), and (2.10) leads to the following two second-order differential equations:

$$\begin{aligned}
&\beta'' + \left[(1 - k\lambda^2)\frac{(rD/\alpha)'}{(rD/\alpha)} - k\lambda\left(2\lambda' + \frac{k'}{k}\lambda\right)\right]\beta'\\
&\qquad - \left\{(1-k\lambda^2)\left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'D/\alpha)'}{(rD/\alpha)}\right] + \nu k\lambda\left(2\lambda' + \frac{k'}{k}\lambda\right)\left(\frac{r'}{r}\right)\right\}\beta\\
&\qquad + \frac{z'}{(rD/\alpha)}(rP_H) + \left(\frac{\lambda}{C}\right)\left\{-[(r'P_H)' - \nu(rP_H)'']\right.\\
&\qquad - \left[\frac{(\lambda/C)'}{(\lambda/C)} + \frac{(rD/\alpha)'}{(rD/\alpha)} - \nu\left(\frac{r'}{r}\right)\right](r'P_H)\\
&\qquad \left. + \left[\nu\frac{(\lambda/C)'}{(\lambda/C)} + \nu\frac{(rD/\alpha)'}{(rD/\alpha)} + \left(\frac{r'}{r}\right)\right](rP_H)'\right\}\\
&= \frac{r'}{(rD/\alpha)}(rP_V) + \left(\frac{\lambda}{C}\right)\left\{[(z'P_V)' - \nu(r\alpha q_H)']\right.\\
&\qquad + \left[\frac{(\lambda/C)'}{(\lambda/C)} + \frac{(rD/\alpha)'}{(rD/\alpha)} - \nu\left(\frac{r'}{r}\right)\right](z'P_V)\\
&\qquad \left. - \left[\nu\frac{(\lambda/C)'}{(\lambda/C)} + \nu\frac{(rD/\alpha)'}{(rD/\alpha)} - \left(\frac{r'}{r}\right)\right](r\alpha q_H)\right\},
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&(rP_H)'' + \frac{(r/\alpha C)'}{(r/\alpha C)}(rP_H)' - \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/\alpha C)'}{(r/\alpha C)}\right](rP_H)\\
&\qquad - \frac{z'}{(r/\alpha C)}\beta - \frac{k\lambda}{(r/\alpha C)}\left\{\nu\left(\frac{r}{\alpha}\right)\beta'' + \left[2\left(\frac{r'}{\alpha}\right) + \nu\left(\frac{r}{\alpha}\right)' + \nu\left(\frac{r}{\alpha}\right)\frac{(k\lambda)'}{(k\lambda)}\right]\beta'\right.\\
&\qquad \left. + \left[\left(\frac{r}{\alpha}\right)\left(\frac{r'}{r}\right)' + \left(\frac{r}{\alpha}\right)'\left(\frac{r'}{r}\right) + \nu\left(\frac{r'}{\alpha}\right)\left(\frac{r'}{r}\right) + \left(\frac{r'}{\alpha}\right)\frac{(k\lambda)'}{(k\lambda)}\right]\beta\right\}\\
&= \left[\frac{r'z'}{r^2} + \nu\frac{(z'/\alpha C)'}{(r/\alpha C)}\right](rP_V) + \nu\frac{z'}{r}(rP_V)'\\
&\qquad - \left[\frac{(r/\alpha C)'}{(r/\alpha C)} + \nu\frac{r'}{r}\right](r\alpha q_H) - (r\alpha q_H)' - \frac{z'}{(r/\alpha C)}\gamma^0_{\xi\zeta},
\end{aligned} \tag{3.2}$$

where k and λ are given by (2.15), and $(rP_V) = -\int r\, \alpha q_V\, d\xi$. With β and (rP_H) known, the radial and axial displacements may be determined from the second of (2.14a) and the equation

$$w_z = \int (z'\epsilon_\xi^0 + r'\omega)\, d\xi \tag{3.3}$$

which, with the aid of (2.3), is deduced from (2.9a).

When the effect of transverse shear deformation is neglected (by setting $\lambda = \gamma_{\xi\zeta}^0 = 0$), (3.1) and (3.2) reduce to the corresponding equations of the classical theory given by E. Reissner [4], and which were first derived in a slightly different form by H. Reissner [1] and Meissner [2, 3].

Introduction of the function ψ defined by

$$\psi = \frac{m}{Eh^2}(rP_H), \qquad m = [12(1-\nu^2)]^{1/2}, \tag{3.4}$$

into (3.1) and (3.2) and some rearrangement of terms results in*

$$\begin{aligned}
\beta'' &+ \left[(1-k\lambda^2)\left(\frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\right) - 2k\lambda^2\left(\frac{\lambda'}{\lambda} + \frac{h'}{h}\right)\right]\beta' \\
&- (1-k\lambda^2)\left[\left(\frac{r'}{r}\right)^2 - \nu\left(\frac{(r'/\alpha)'}{(r/\alpha)} + 3\frac{r'h'}{rh}\right)\right]\beta - \frac{\alpha^2 m}{R_\theta h_0}\left(\frac{h_0}{h}\right)\psi \\
&+ \nu k^{1/2}\lambda\Bigg\{\psi'' + \left[\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)} - 2\nu\frac{r'}{r} + 6\frac{h'}{h}\right]\psi' \\
&- \frac{1}{\nu}\left[\left(\frac{r'}{r}\right)' + \frac{r'}{r}\left(\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)}\right)\right. \\
&\left. - \left(\frac{r'}{r}\right)^2 - 2\nu\frac{h''}{h} - 2\nu\frac{h'}{h}\left(3\frac{h'}{h} - \frac{1}{\nu}\frac{r'}{r} + \frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)}\right)\right]\psi\Bigg\} \\
&= F_1,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\psi'' &+ \left[\frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h} - \nu[[k^{1/2}\lambda T]]\right]\psi' \\
&- \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/\alpha)'}{(r/\alpha)} + \frac{12(1+\nu)}{5}\left(\frac{z'}{r}\right)^2 - 2\frac{h''}{h} - 2\frac{h'}{h}\left(\frac{(r/\alpha)'}{(r/\alpha)} + \frac{\nu}{2}\frac{r'}{r}\right)\right. \\
&\left. - [[k^{1/2}\lambda T\left(\frac{r'}{r} - 2\nu\frac{h'}{h}\right)]]\right]\psi + \frac{\alpha^2 m}{R_\theta h_0}\left(\frac{h_0}{h}\right)\beta \\
&- \nu k^{1/2}\lambda\Bigg\{\beta'' + \left[\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)} + \frac{2}{\nu}\frac{r'}{r} + 2\frac{h'}{h} + [[\frac{m}{\nu\lambda h}(1-k\lambda^2)T]]\right]\beta' \\
&+ \frac{1}{\nu}\left[\left(\frac{r'}{r}\right)' + \frac{r'}{r}\left(\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)}\right) + \nu\left(\frac{r'}{r}\right)^2 + 2\frac{r'h'}{rh}\right. \\
&\left. + [[\frac{m}{\lambda h}\left(\frac{r'}{r}\right)(1-k\lambda^2)T]]\right]\beta\Bigg\} \\
&= F_2,
\end{aligned} \tag{3.6}$$

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*In Eqs. (3.5) and (3.6), as well as in the subsequent analysis, the effect of thickness variation due to shear stress strain relation is placed in double-squared brackets, i.e., [[ ]].

where h_0 is the value of the thickness h at some reference section (say $\xi = \xi_0$); F_1 denotes the right-hand side of (3.1); and F_2 and T are given by

$$F_2 = \frac{m}{Eh^2}\left\{\left[\frac{r'z'}{r^2}\left(1 - \frac{12(1+\nu)}{5}\right) + \nu\frac{(z'/\alpha C)'}{(r/\alpha C)} + [[\frac{\alpha}{R_\theta}k^{1/2}\lambda T]]\right](rP_V) + \nu\frac{z'}{r}(rP_V)' - \left[\nu\frac{r'}{r} + \frac{(r/\alpha C)'}{(r/\alpha C)} - \nu[[k^{1/2}\lambda T]]\right](r\alpha q_H) - (r\alpha q_H)'\right\}, \tag{3.7a}$$

$$T = -\frac{1}{2}\left(\frac{h}{R_\theta}\right)\left(\frac{h'}{h}\right)\left[\frac{12(1+\nu)}{5m}\right]. \tag{3.7b}$$

Since $h/R \ll 1$, then

$$k\lambda^2 = \frac{h^2}{m^2}\left(\frac{1}{R_\xi} - \frac{1}{R_\theta}\right) \ll 1;$$

hence, in what follows, terms of the order $O(k\lambda^2)$ will be neglected in comparison with unity. With

$$X_1 = \beta + \nu k^{1/2}\lambda\psi, \qquad X_2 = \psi - \nu k^{1/2}\lambda\beta, \tag{3.8a}$$

from which

$$\beta = (X_1 - \nu k^{1/2}\lambda X_2) + O(k\lambda^2),$$
$$\psi = (X_2 + \nu k^{1/2}\lambda X_1) + O(k\lambda^2), \tag{3.8b}$$

and with the aid of the relations

$$\frac{(k^{1/2}\lambda)'}{(k^{1/2}\lambda)} = \frac{\lambda'}{\lambda} + \frac{h'}{h},$$
$$\frac{(k^{1/2}\lambda)''}{(k^{1/2}\lambda)} = \frac{\lambda''}{\lambda} + 2\frac{\lambda' h'}{\lambda h} + \frac{h''}{h}, \tag{3.9}$$

(3.5) and (3.6) may be written as

$$L(X_1) + \mu^2 f\left(\frac{h_0}{h}\right)\left[1 + \nu k^{1/2}\lambda(g_1 + G_1)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]X_2 - \nu k^{1/2}\lambda\left(\frac{2}{\nu}\frac{r'}{r} + \frac{\lambda'}{\lambda} - \frac{h'}{h}\right)X_2' = F_1, \tag{3.10}$$

$$L(X_2) - (\gamma - \kappa)X_2 - \mu^2 f\left(\frac{h_0}{h}\right)\left[1 + \left(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\right)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]X_1 - \left[\nu k^{1/2}\lambda\left(\frac{2}{\nu}\frac{r'}{r} - \frac{\lambda'}{\lambda} - 3\frac{h'}{h}\right) + [[T]]\right]X_1' = F_2, \tag{3.11}$$

where

$$L(\ ) \equiv (\ )'' + \left[\frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\right](\ )' - \left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'/\alpha)'}{(r/\alpha)} - \nu k^{1/2}\lambda\mu^2 f\left(\frac{h_0}{h}\right) - 3\nu\frac{r'h'}{rh}\right](\ ),$$

$$g_1 = \frac{1+\nu}{\nu}\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'/\alpha)'}{(r/\alpha)} - \frac{1}{\nu}\left(\frac{r'}{r}\right)' - \frac{1}{\nu}\left(\frac{r'}{r}\right)\frac{(r/\alpha)'}{(r/\alpha)} - \frac{\lambda''}{\lambda} - \frac{\lambda'}{\lambda}\left[\frac{(r/\alpha)'}{(r/\alpha)} + \frac{1}{\nu}\frac{r'}{r}\right],$$

$$g_2 = 2\left(\frac{r'}{r}\right)^2 + \frac{1}{\nu}\left(\frac{r'}{r}\right)' + \nu\frac{(r'/\alpha)'}{(r/\alpha)} + \frac{12(1+\nu)}{5}\left(\frac{z'}{r}\right)^2 + \frac{1}{\nu}\left(\frac{r'}{r}\right)\frac{(r/\alpha)'}{(r/\alpha)} - \frac{\lambda''}{\lambda} - \frac{\lambda'}{\lambda}\left[\frac{(r/\alpha)'}{(r/\alpha)} - \frac{1}{\nu}\frac{r'}{r}\right],$$

$$G_1 = \frac{h''}{h} + \frac{h'}{h}\left[4\frac{h'}{h} - 3\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)} - \frac{4+3\nu^2}{\nu}\left(\frac{r'}{r}\right)\right],$$

$$G_2 = -3\frac{h''}{h} - \frac{h'}{h}\left[3\frac{h'}{h} + 3\frac{(r/\alpha)'}{(r/\alpha)} + 5\frac{\lambda'}{\lambda} - \frac{2-\nu^2}{\nu}\left(\frac{r'}{r}\right)\right],$$

$$\gamma = 2\left[\nu\frac{(r'/\alpha)'}{(r/\alpha)} + \frac{6(1+\nu)}{5}\left(\frac{z'}{r}\right)^2\right], \tag{3.12}$$

$$\kappa = 2\left[\frac{h''}{h} + \frac{h'}{h}\left(\frac{(r/\alpha)'}{(r/\alpha)} - \nu\frac{r'}{r}\right)\right],$$

$$\mu^2 f(\xi) = -\frac{\alpha^2 m}{R_\theta h_0},$$

and it is to be noted that in the last of (3.12), μ is a constant and $f(\xi)$ is independent of thickness $h(\xi)$.

Multiplying (3.11) by $i\delta$ (where δ is a function of ξ), adding the resulting equation to (3.10), and observing that

$$X_1' = (X_1 + i\delta X_2)' - i\delta X_2' - i\delta' X_2, \qquad i = (-1)^{1/2}, \tag{3.13}$$

will result in a fairly intricate complex differential equation. By taking δ in the form

$$\delta\left[1 + \left(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\right)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]$$
$$= -\frac{1}{2}i(\gamma - \kappa)\left(\mu^2 f\frac{h_0}{h}\right)^{-1} + \left\{\left[1 + \nu k^{1/2}\lambda(g_1 + G_1)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]\cdot\left[1 + \left(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\right)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right] - \left[\frac{1}{2}(\gamma - \kappa)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]^2\right\}^{1/2} \tag{3.14}$$

with the restriction that

$$\delta' = \delta'' = 0, \tag{3.15}$$

the complex differential equation mentioned may be reduced to

$$
\begin{aligned}
L(X_1 + i\delta X_2) &+ i\delta\left[\nu k^{1/2}\lambda\left(\frac{(k^{1/2}\lambda)'}{(k^{1/2}\lambda)} - \frac{2}{\nu}\frac{r'}{r} + 2\frac{h'}{h}\right) - [[T]]\right](X_1 + i\delta X_2)' \\
&- i\delta\mu^2 f\left(\frac{h_0}{h}\right)\left[1 + \left(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\right)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right](X_1 + i\delta X_2) \\
&= F_1 + i\delta F_2 \\
&+ \left[\nu k^{1/2}\lambda\left(\frac{2}{\nu}\frac{r'}{r}(1 + \delta^2) + \frac{\lambda'}{\lambda}(1 - \delta^2) - \frac{h'}{h}(1 + 3\delta^2)\right) + [[\delta^2 T]]\right]X_2' .
\end{aligned}
\tag{3.16}
$$

The coefficient of $(X_1 + i\delta X_2)'$ in the above equation (say B) is

$$
B = \frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h} + i\delta\left[\nu k^{1/2}\lambda\left(\frac{(k^{1/2}\lambda)'}{(k^{1/2}\lambda)} - \frac{2}{\nu}\frac{r'}{r} + 2\frac{h'}{h}\right) - [[T]]\right]
\tag{3.17}
$$

and since

$$
\begin{aligned}
\Theta = \exp\left\{\frac{1}{2}\int B\,d\xi\right\} = \left(\frac{r}{\alpha}\right)^{1/2} h^{3/2} \exp\left\{\frac{1}{2}\,i\delta(\nu k^{1/2}\lambda)\right. \\
\left. - i\delta\nu\int\left[k^{1/2}\lambda\left(\frac{1}{\nu}\frac{r'}{r} - \frac{h'}{h}\right) + [[\frac{1}{2\nu}T]]\right]d\xi\right\}
\end{aligned}
\tag{3.18}
$$

then, by means of the transformation

$$
Y = \Theta(X_1 + i\delta X_2)
\tag{3.19}
$$

we finally obtain

$$
Y'' + [i^3\mu^2\Psi^2(\xi) + \Lambda(\xi)]Y - \Theta\Phi X_2' = \Theta(F_1 + i\delta F_2)
\tag{3.20}
$$

which, in view of the presence of the function X_2' on the left-hand side, may be called the "quasi-normal" form of (3.13). In (3.20), the functions Φ, Ψ^2, and Λ are given by

$$
\Phi(\xi) = (\nu k^{1/2}\lambda)\left[\frac{2}{\nu}\frac{r'}{r}(1 + \delta^2) + \frac{\lambda'}{\lambda}(1 - \delta^2) - \frac{h'}{h}(1 + 3\delta^2)\right] + [[\delta^2 T]]
\tag{3.21}
$$

and

$$
\begin{aligned}
\Psi^2(\xi) = \left\{\delta\left[1 + \left(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\right)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]\right. \\
\left. + \frac{1}{2}\,i\gamma\left(\mu^2 f\frac{h_0}{h}\right)^{-1} + i\nu k^{1/2}\lambda\right\}\left(\frac{h_0}{h}\right)f
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
\Lambda(\xi) = &-\left(\frac{r'}{r}\right)^2 - \frac{1}{2}\frac{(r/\alpha)''}{(r/\alpha)} + \frac{1}{4}\left[\frac{(r/\alpha)'}{(r/\alpha)}\right]^2 - \frac{6(1 + \nu)}{5}\left(\frac{z'}{r}\right)^2 \\
&- \frac{3}{2}\frac{h''}{h} - \frac{3}{4}\left(\frac{h'}{h}\right)^2 - \frac{3}{2}\frac{(r/\alpha)'}{(r/\alpha)}\frac{h'}{h} + 3\nu\frac{r'h'}{rh} \\
&+ i\tilde{\delta}\nu k^{1/2}\lambda\left\{\frac{1}{\nu}\left(\frac{r'}{r}\right)' + \frac{1}{\nu}\left(\frac{r'}{r}\right)\frac{(r/\alpha)'}{(r/\alpha)} - \frac{1}{2}\frac{\lambda''}{\lambda} + \frac{\lambda'}{\lambda}\left(\frac{1}{\nu}\frac{r'}{r} - \frac{1}{2}\frac{(r/\alpha)'}{(r/\alpha)}\right)\right. \\
&\left. - \frac{5}{2}\frac{h''}{h} + \frac{h'}{h}\left[\frac{4}{\nu}\frac{r'}{r} - \frac{5}{2}\frac{(r/\alpha)'}{(r/\alpha)} - \frac{9}{2}\frac{\lambda'}{\lambda} - \frac{15}{2}\frac{h'}{h}\right]\right\}.
\end{aligned}
\tag{3.23}
$$

4. Approximation of δ. Since δ, as given by (3.14) is fairly complicated, we impose a further restriction on the order of magnitude of $(\mu^2)^{-1}$, i.e.,

$$(\mu^2)^{-1} = O(k^{1/2}\lambda). \tag{4.1}$$

Thus, by (4.1), $\nu k^{\frac{1}{2}}\lambda\mu^{-2} = O(k\lambda^2)$ which, as in the previous section, may be neglected in comparison with unity. With this approximation, (3.14) simplifies considerably and reads as follows:

$$\delta = -\frac{1}{2} i(\gamma - \kappa)\left(\mu^2 f \frac{h_0}{h}\right)^{-1} + \left\{1 - \left[\frac{1}{2}(\gamma - \kappa)\left(\mu^2 f \frac{h_0}{h}\right)^{-1}\right]^2\right\}^{1/2}. \tag{4.2}$$

It should be noted that the restriction (4.1) is physically plausible, as it holds true for many cases of shells of revolution, e.g., ellipsoidal, paraboloidal and toroidal.

We now return to (3.14) and observe that condition (3.15) is fulfilled if δ is a constant, and this may be achieved by proper choice of γ and κ.

For shells of variable thickness, as seen from (4.2), condition (3.15) is satisfied provided $(\gamma - \kappa)$ $(\mu^2 f h_0/h)^{-1}$ is constant. Hence, by (3.12),

$$\left[\left(\frac{r}{\alpha}\right)h'\right]' - \nu\left[\left(\frac{r'}{\alpha}\right)h\right]' = mK\frac{\alpha r}{R_\theta} - \frac{6(1+\nu)}{5}\left(\frac{z'}{R_\theta}\right)h \tag{4.3}$$

which resembles the corresponding equation of the classical theory, first given by Meissner [3] and derived in a different manner recently by Naghdi and DeSilva [9]; in (4.3), K is a constant and the effect of the transverse shear deformation is represented by the second term on the right-hand side.

For shells of uniform thickness, κ vanishes identically and γ in general will not be constant. However, for numerous shell configurations,

$$\delta = 1 - \frac{1}{2} i\gamma\left(\mu^2 f \frac{h_0}{h}\right)^{-1} \tag{4.4}$$

may be approximated to a constant, in which case the coefficient function Ψ^2 reads

$$\Psi^2 = [1 + i\nu k^{1/2}\lambda]f \tag{4.5}$$

and (4.3) may be reduced to

$$h = \left(\frac{mK}{\nu} R_\xi\right)\left[1 - \frac{6(1+\nu)}{5\nu}\frac{R_\xi}{R_\theta}\right]^{-1}. \tag{4.6}$$

It follows from (4.6) that γ and δ will in fact be constant if both R_ξ and R_θ are constant. It is clear that this requirement is more restrictive than the corresponding result of the classical theory [9] where $h = (mK/\nu)R_\xi$.

5. Formal solution of Eq. (3.20) by asymptotic integration. We conclude the present paper by discussing formally the solution of the homogeneous differential equation associated with (3.20), namely

$$Y'' + [i^3\mu^2\Psi^2 + \Lambda]Y = \Theta\Phi X_2' , \tag{5.1}$$

where μ is a large parameter.

It follows from the work of Langer [8] that corresponding to the homogeneous equation associated with (5.1), that is, the equation

$$Y'' + [i^3\mu^2\Psi^2 + \Lambda]Y = 0, \tag{5.2}$$

there exists a related differential equation whose solution is asymptotic with respect to μ to the solution of (5.2) and that its domain of validity is dependent upon the character of the coefficient functions of Y. If ξ ranges over an interval I_ξ which includes a point ξ_0 at which (i) the function Λ may admit a pole of first or second order*, (ii) Ψ^2 may contain as a factor the quantity $(\xi - \xi_0)^{a-2}$, a being any real positive constant, and (iii) both Λ and Ψ^2 are analytic bounded elsewhere in I_ξ, then the asymptotic solution mentioned will be valid in the entire interval I_ξ including ξ_0. If, on the other hand, the behavior of Λ and Ψ^2 at ξ_0 does not meet the required specifications, then the solution will be valid in a sub-interval of I_ξ.

Thus, if we write Λ and Ψ^2 in the form

$$\Lambda = \frac{A_1}{(\xi - \xi_0)^2} + \frac{B_1}{(\xi - \xi_0)} + \Lambda_1(\xi), \tag{5.3a}$$

$$\Psi^2 = (\xi - \xi_0)^{a-2}\Psi_1^2(\xi), \tag{5.3b}$$

where A_1 and B_1 are constants, Λ_1 is analytic and bounded with respect to μ in I_ξ, and Ψ_1^2 is a non-vanishing single-valued analytic function in I_ξ including ξ_0, then according to Langer [8] the functions

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = (\xi - \xi_0)^{-1/2(a/2-1)}\Psi_1^{-1/2}\varphi^{1/2}\begin{Bmatrix} J_\rho(\varphi) \\ Y_\rho(\varphi) \end{Bmatrix} \tag{5.4}$$

are the solutions of the related differential equation

$$y'' + \left[i^3\mu^2\Psi^2 + \frac{A_1}{(\xi - \xi_0)^2} + \frac{B_1}{(\xi - \xi_0)} + \Omega(\xi)\right]y = 0, \tag{5.5}$$

where Ω is analytic and bounded with respect to μ in I_ξ, J_ρ and Y_ρ are Bessel functions of the first and second kinds, and

$$\rho = \frac{b}{a}, \qquad b = (1 - 4A_1)^{1/2},$$
$$\varphi = i^{3/2}\mu \int_{\xi_0}^{\xi} \Psi(\eta)\, d\eta. \tag{5.6}$$

Writing Λ_1 as

$$\Lambda_1 = \Omega(\xi) + \Delta(\xi), \tag{5.7}$$

where Δ is also analytic and bounded with respect to μ in I_ξ, then (5.2) may be written as

$$Y'' + \left[i^3\mu^2\Psi^2(\xi) + \frac{A_1}{(\xi - \xi_0)^2} + \frac{B_1}{(\xi - \xi_0)} + \Omega(\xi)\right]Y = -\Delta(\xi)Y. \tag{5.8}$$

Since the left-hand side of (5.8) is identical with the left-hand side of the related differential equation (5.5), then, by the method of variation of parameters, there is obtained

$$Y_{H_j} = y_j + \int_{\xi_0}^{\xi} D_1^*(\xi, \eta)[\Delta(\eta)Y_{H_j}(\eta)]\, d\eta; \qquad j = 1, 2, \tag{5.9a}$$

*In cases of ellipsoidal and paraboloidal shells of revolution of uniform thickness, Λ contains a pole of second order at $\xi = 0$.

where

$$D_1^*(\xi, \eta) = \Gamma_1^{-1}(y_1, y_2)[y_1(\xi)y_2(\eta) - y_2(\xi)y_1(\eta)] \tag{5.9b}$$

and Γ_1 denotes the Wronskian of y_1 and y_2.

The integral equation (5.9) by the familiar process of successive iteration leads formally to the relation

$$Y_{H_i} = y_i(\xi) + \sum_{n=1}^{\infty} y_i^{(n)}(\xi), \tag{5.10}$$

where

$$y_i^{(n+1)}(\xi) = \int_{\xi_0}^{\xi} D_1^*(\xi, \eta)\Delta(\eta)y_i^{(n)}(\eta)\, d\eta,$$
$$y_i^{(0)}(\xi) = y_i(\xi). \tag{5.11}$$

The proof for the uniform convergence of $\sum_{n=1}^{\infty} y_i^{(n)}$ in I_ξ is given by Langer [8] and he has further shown that y_i is dominant in (5.10) and thus, y_i is asymptotic with respect to μ to the solution Y_{H_i} of (5.2).

Recalling that δ as given by (4.2) is a complex constant, i.e.,

$$\delta = \delta_1 + i\delta_2 \tag{5.12}$$

then, by (3.19) and (3.18),

$$X_2' = \frac{1}{\delta_1} \mathscr{I}m \left[\frac{Y'}{\Theta} - \frac{\Theta'}{\Theta^2} Y\right], \tag{5.13}$$

where $\mathscr{I}m$ denotes "imaginary part of". Treating (5.1) as a non-homogeneous differential equation whose homogeneous solution is given by (5.10), then again by variation of parameters, there results

$$Y_j = Y_{H_j} - \int_{\xi_0}^{\xi} D_2^*(\xi, \eta)\left\{\frac{1}{\delta_1} \Theta(\eta)\Phi(\eta)\, \mathscr{I}m \left[\frac{Y_j'}{\Theta}(\eta) - \frac{\Theta'}{\Theta^2}(\eta) Y_j(\eta)\right]\right\} d\eta; \qquad j = 1, 2, \tag{5.14a}$$

where

$$D_2^*(\xi, \eta) = \Gamma_2^{-1}(Y_{H_1}, Y_{H_2})[Y_{H_1}(\xi) Y_{H_2}(\eta) - Y_{H_2}(\xi) Y_{H_1}(\eta)] \tag{5.14b}$$

and Γ_2 is the Wronskian of Y_{H_1} and Y_{H_2}.

As in (5.10), by successive iteration, the integral equation (5.14) can be written in the form

$$Y_j = Y_{H_j} - \sum_{n=1}^{\infty} [\chi(\xi)]_j^{(n)}, \tag{5.15}$$

where

$$[\chi(\xi)]_j^{(n+1)} = \int_{\xi_0}^{\xi} D_3(\xi, \eta)\Theta(\eta)\Phi(\eta)[X_2'(\eta)]_j^{(n)}\, d\eta,$$
$$[X_2'(\xi)]_j^{(0)} = [X_2'(\xi)]_j, \tag{5.16}$$

and X_2' is given by (5.13).

The above solution is formally valid if $\sum_{n=1}^{\infty} [\chi]_i^{(n)}$ is uniformly convergent in I_ξ. On account of the interdependence on λ of the various functions involved in (5.16), the proof for the uniform convergence of the series in (5.15) for general shells of revolution appears to be difficult and requires further investigation. Nevertheless, from the comparison of (5.11) and (5.16), it appears reasonable to except that $\sum [\chi]_i^{(n)}$ has in general the same behavior as $\sum y_i^{(n)}$, so that

$$Y_i = Y_{H_i} + O\left(\frac{1}{\mu}\right) \tag{5.17}$$

and hence, y_i is asymptotic with respect to μ to Y_i.

Finally, it may be noted that whenever the right-hand side of (5.1) vanishes identically, as in the case of spherical shells where $\lambda = 0$, the differential equation is of the same form as the corresponding equation of the classical theory (Ref. [9]), although the coefficient functions are not the same.

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