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A NEW APPROACH TO THE NON-LINEAR PROBLEMS OF FM CIRCUITS*

BY

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Abstract. Closed form expressions are developed for the output of a frequency modulation receiver for an arbitrary number of superposed input signals. This corresponds to problems of interference or disturbance due to scatter and multiple reflexions. It is also shown how the Fourier components of the output may be evaluated by methods more direct than the usual Fourier analysis.

Introduction. A frequency modulation receiver is essentially a non-linear device. The input signal is usually fed into two non-linear filters, first into an amplitude limiter which reduces the signal to a constant amplitude, then into a discriminator whose output is a rectified signal with an amplitude proportional to the frequency deviation from the carrier frequency. Sometimes this output of the discriminator is processed through a linear filter which eliminates all but a few frequency components of the modulation.

Because of the non-linearity of the system, special methods must be devised to evaluate the output due to the superposition of input signals. The procedure presented here yields closed form expressions for the output when an arbitrary number of signals or a continuous distribution of them are superposed. The results may be used to predict interference effects or the disturbance due to multiple reflexions or scattering of the main signal. The method makes use of the concept of instantaneous frequency. The limitations of this concept in analyzing the behavior of frequency modulation circuits was discussed extensively by Carson and Fry[1] and Van der Pol[2].

The general theory is developed in Section 1 for the case of an arbitrary input represented by a continuous spectrum. This is applied in Section 2 to the case of an arbitrary number of signals with a single modulation frequency. It is also indicated how the method applies when there is more than one modulation frequency. Section 3 deals with the Fourier analysis of the output. Because of the fact that the expression for the output is in the form of the quotient of two Fourier series, methods more direct than the usual Fourier analysis are applicable. It is shown that the Fourier coefficients

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[1]J. R. Carson and T. C. Fry, *Variable frequency electric circuit theory with application to the theory of frequency modulation*, Bell Syst. Tech. Journal **26**, 513-540 (1937).

[2]B. Van der Pol, *Fundamental principles of frequency modulation*, Journal I.E.E. **111**, 153-158 (1946).

may be evaluated directly by use of the theorem of residues of analytic functions. The method yields directly the Fourier components of the output in both amplitude and phase for any number of superposed input signal. In particular, the input may be represented by the superposition of its spectral components. This has an important bearing on scatter problems since it is then reduced to the calculation of the scatter for each monochromatic component. The procedure has been applied to a number of practical cases and found to be quite satisfactory from the standpoint of simplicity and accuracy. These applications along with some further simplifications will be presented in a subsequent paper.

1. General expressions for the output signal of an FM receiver. We consider an input signal which is both amplitude and frequency modulated,

$$E(t) = \tfrac{1}{2}I(t)[\exp(i\phi) + \exp(-i\phi)] \tag{1.1}$$

with

$$\phi = \omega_c t + \varphi(t),$$

where ω_c represents the carrier frequency, $\varphi(t)$ the frequency modulation and $I(t)$ the amplitude modulation. In the receiver the amplitude is first reduced to a constant value in a limiter circuit. It is then processed through a discriminator. This is usually made of circuits slightly off resonance with the carrier frequency, such that the output is a rectified voltage proportional to the frequency deviation of the signal from the carrier frequency. This frequency deviation being $d\phi/dt - \omega_c$, the ouput of the receiver is then given by

$$K\left(\frac{d\phi}{dt} - \omega_c\right) = K\frac{d\varphi}{dt}. \tag{1.2}$$

Sometimes this output signal is passed through a linear filter so as to extract frequency components in a narrow band.

In practice, the signal is not always given in the form (1.1) so that one cannot use (1.2) to evaluate the receiver output. In particular, we are interested in the calculation of the receiver output when the incoming signal is given by its spectrum

$$E(t) = \int_{-\infty}^{+\infty} G(\omega)e^{i\omega t}\,d\omega. \tag{1.3}$$

We establish a mathematical processing of the expression (1.3) by which it is possible to evaluate a quantity proportional to the rate of change of the phase angle ϕ, hence also proportional to the receiver output. To do this we apply to the signal a linear operator which consists in replacing its spectrum $G(\omega)$ by

$$G(\omega)(1 + K\Omega), \tag{1.4}$$

where

$$\Omega = |\omega| - \omega_c{}^*$$

is the frequency deviation of the spectral component from the carrier frequency. The effect of this linear operation on the signal can be readily evaluated in the form (1.3)

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*If we wanted to restrict ourselves to analytic functions, the same purpose could be accomplished by putting $\Omega = (\omega^2 - \omega_c^2)/2\omega_c$.

by multiplying $G(\omega)$ by $1 + K\Omega$ in the integral. We shall now introduce the basic assumption of the method, namely that this operation is approximately equivalent to multiplying the signal by the factor

$$A = 1 + K\frac{d\varphi}{dt}. \tag{1.5}$$

This assumption is essentially the same as that upon which is based the design of a discriminator circuit, namely that the response of the circuit to the instantaneous amplitude and frequency of the signal is the same as in a steady state. The assumption will, of course, apply if the frequencies at which the amplitude, I, and phase angle, φ, vary are very much smaller than the carrier frequency, ω_c .
If we consider the signal in the form (1.1), it becomes

$$E_1(t) = \tfrac{1}{2}AI(t)[\exp(i\phi) + \exp(-i\phi)]. \tag{1.6}$$

Squaring this quantity we obtain

$$E_1^2(t) = \tfrac{1}{2}A^2I^2 + \tfrac{1}{4}A^2I^2[\exp(2i\phi) + \exp(-2i\phi)]. \tag{1.7}$$

We notice that the first term represents the low frequency components, while the second term represents the high frequency components. In practice, these components are widely separated. We may write

$$\mathfrak{L}E_1^2(t) = \tfrac{1}{2}A^2I^2, \tag{1.8}$$

where the symbol $\mathfrak{L}$ signifies "low frequency part of $\cdots$". Similarly, if we square the original signal $E(t)$ we may write

$$\mathfrak{L}E^2(t) = \tfrac{1}{2}I^2. \tag{1.9}$$

Hence,

$$\frac{\mathfrak{L}E_1^2(t)}{\mathfrak{L}E^2(t)} = A^2 = 1 + 2K\frac{d\varphi}{dt} + K^2\left(\frac{d\varphi}{dt}\right)^2. \tag{1.10}$$

The linear term in K in the expression is proportional to the output signal of the receiver. We may write this output signal as

$$M(t) = K\frac{d\varphi}{dt} = \frac{K}{2\mathfrak{L}E^2(t)}\left[\frac{d}{dK}\,\mathfrak{L}E_1^2(t)\right]_{K=0}. \tag{1.11}$$

This expression will now be evaluated in terms of the representation (1.3) of the input by means of a spectrum. The square of the signal is the double integral.

$$E^2(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(\xi)G(\omega)e^{i(\xi+\omega)t}\,d\xi\,d\omega. \tag{1.12}$$

The integral is extended to the infinite plane. We introduce the following change of variables

$$\xi = \tfrac{1}{2}(\eta + \zeta), \qquad \omega = \tfrac{1}{2}(\eta - \zeta) \tag{1.13}$$

and derive

$$E^2(t) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G\left(\frac{\eta+\zeta}{2}\right)G\left(\frac{\eta-\zeta}{2}\right)e^{i\eta t}\,d\eta\,d\zeta. \tag{1.14}$$

The spectrum of $2E^2(t)$ is therefore

$$R(\eta) = \int_{-\infty}^{+\infty} G\left(\frac{\eta + \zeta}{2}\right)G\left(\frac{\eta - \zeta}{2}\right) d\zeta. \tag{1.15}$$

In evaluating this expression we take into account the fact that $G(\omega)$ is small except in the vicinity of $\omega = \pm \omega_c$. Hence, the contribution to the integral will be only in the vicinity of the four points (see Fig. 1).

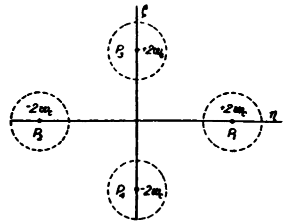

FIG. 1.

$$\begin{array}{lll} P_1: & \eta = 2\omega_c & \zeta = 0 \\ P_2: & \eta = -2\omega_c & \zeta = 0 \\ P_3: & \eta = 0 & \zeta = 2\omega_c \\ P_4: & \eta = 0 & \zeta = -2\omega_c \end{array}$$

The low frequency components of $E^2(t)$ are therefore given by integrating (1.14) in the vicinity of points P_3 and P_4

$$\mathfrak{L}E^2(t) = \frac{1}{2}\int_{P_3,P_4} G\left(\frac{\eta + \zeta}{2}\right)G\left(\frac{\eta - \zeta}{2}\right)e^{i\eta t}\, dS, \tag{1.16}$$

the integral being evaluated over elements of area dS in the vicinity of P_3 and P_4. Similarly, the low frequency components of $E_1^2(t)$ are given by

$$\mathfrak{L}E_1^2(t) = \frac{1}{2}\int_{P_3,P_4} [1 + K(|\,\omega\,| - \omega_c)][1 + K(|\,\xi\,| - \omega_c)]G\left(\frac{\eta + \zeta}{2}\right)G\left(\frac{\eta - \zeta}{2}\right)e^{i\eta t}\, dS. \tag{1.17}$$

Hence,

$$\begin{aligned} &\frac{1}{2}K\left[\frac{d}{dK}\,\mathfrak{L}E_1^2(t)\right]_{K=0} \\ &\quad = \frac{1}{2}K\int_{P_3,P_4}\left(\frac{|\,\eta + \zeta\,|}{4} + \frac{|\,\eta - \zeta\,|}{4} - \omega_c\right)G\left(\frac{\eta + \zeta}{2}\right)G\left(\frac{\eta - \zeta}{2}\right)e^{i\eta t}\, dS. \end{aligned} \tag{1.18}$$

From (1.16), (1.18), and (1.11) we derive the receiver output

$$\begin{aligned} M(t) = K\int_{P_3,P_4}&\left(\frac{|\,\eta + \zeta\,|}{4} + \frac{|\,\eta - \zeta\,|}{4} - \omega_c\right)G\left(\frac{\eta + \zeta}{2}\right)G\left(\frac{\eta - \zeta}{2}\right)e^{i\eta t}\, dS \\ &\cdot\left\{\int_{P_3,P_4} G\left(\frac{\eta + \zeta}{2}\right)G\left(\frac{\eta - \zeta}{2}\right)e^{i\eta t}\, dS\right\}^{-1}. \end{aligned} \tag{1.19}$$

We have thus expressed this output in terms of the input spectrum G. It is seen that, as required by the problem, this expression is independent of the input amplitude. Expression (1.19) may be put in a somewhat different form by introducing

$$F(\eta) = \int_{-\infty}^{+\infty} \left(\frac{|\eta + \zeta|}{4} + \frac{|\eta - \zeta|}{4} - \omega_c\right) G\left(\frac{\eta + \zeta}{2}\right)\left(\frac{\eta - \zeta}{2}\right) d\zeta$$
$$= \int_{-\infty}^{+\infty} \left(\frac{|\eta - \zeta|}{2} - \omega_c\right) G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) d\zeta. \quad (1.20)$$

The equivalence of these two expressions is easily seen if we replace the variable ζ by $-\zeta$. This function constitutes the spectrum of the numerator of expression (1.19). With the spectrum $R(\eta)$ of the denominator, as defined by (1.15), we may write

$$M(t) = K \int_{-\epsilon}^{\epsilon} F(\eta) e^{i\eta t}\, d\eta \left\{\int_{-\epsilon}^{\epsilon} R(\eta) e^{i\eta t}\, d\eta\right\}^{-1}. \quad (1.21)$$

The integrands in the expressions for $F(\eta)$ and $R(\eta)$ are different from zero only in the vicinity of $\zeta = \pm 2\omega_c$ and the variable η is restricted to the vicinity of the origin. The range of integration $-\epsilon < \eta < +\epsilon$ is that of the low frequency portion of the spectral functions $F(\eta)$ and $R(\eta)$. According to the footnote remark ([1]), we could also write for $F(\eta)$

$$F(\eta) = \int_{-\infty}^{\infty} \frac{1}{2\omega_c} \left[\frac{1}{4}(\eta - \zeta)^2 - \omega_c^2\right] G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) d\zeta. \quad (1.22)$$

This expression will be approximately equal to (1.20).

2. Application to the superposition of sinusoidally modulated signals. In certain cases the input signal is made up of the superposition of signals whose frequency modulation is sinusoidal with a common carrier frequency, but with a different phase for each modulation. The signal is then expressed as

$$2E(t) = \sum_j R_j \exp\left[i\omega_c t + i\varphi_j + i\left(\frac{\Delta\omega}{\omega_1}\right)\sin(\omega_1 t + \psi_j)\right]$$
$$+ \sum_j R_j^* \exp\left[-i\omega_c t - i\varphi_j - i\left(\frac{\Delta\omega}{\omega_1}\right)\sin(\omega_1 t + \psi_j)\right], \quad (2.1)$$

where R_j and R_j^* are complex conjugates. Such a signal occurs, for instance, in the case of multiple reflection or scatter. In this case the phase differences are

$$\varphi_j = -\omega_c t_j\,,$$
$$\psi_j = -\omega_1 t_j\,, \quad (2.2)$$

where t_j is the time lag for arrival of the jth component in the receiver. In order to apply the results of the previous section, we must represent the signal by its spectrum. We make use of the identity

$$\exp(i\beta \sin\theta) = \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(ni\theta), \quad (2.3)$$

where J_n is the Bessel function of the first kind of order n. By putting β equal to the modulation index

$$\beta = \frac{\Delta\omega}{\omega_i} \tag{2.4}$$

and

$$\begin{aligned} B_n &= \textstyle\sum_i R_i \exp (i\varphi_i + in\psi_i) \\ B_n^* &= \textstyle\sum_i R_i^* \exp (-i\varphi_i - in\psi_i) \end{aligned} \tag{2.5}$$

we may write the signal as

$$2E(t) = \sum_{n=-\infty}^{\infty} B_n J_n \exp (i\omega_c t + in\omega_1 t) + \sum_{n=-\infty}^{\infty} B_n^* J_n \exp (-i\omega_c t - in\omega_1 t). \tag{2.6}$$

This latter expression constitutes the expansion of the signal into a discrete spectrum of equidistant frequencies. The integration (1.3) is here replaced by a summation. The low frequency components of the square of the signal corresponding to expressions (1.16) are given by

$$\begin{aligned} 4\mathfrak{L}E^2(t) = &\sum_{n=-\infty}^{+\infty} B_n B_n^* J_n^2 \\ &+ \exp (i\omega_1 t) \sum_{n=-\infty}^{+\infty} B_n B_{n-1}^* J_n J_{n-1} + \exp (-i\omega_1 t) \sum_{n=-\infty}^{+\infty} B_n B_{n+1}^* J_n J_{n+1} \\ &+ \exp (2i\omega_1 t) \sum_{n=-\infty}^{+\infty} B_n B_{n-2}^* J_n J_{n-2} + \exp (-2i\omega_1 t) \sum_{n=-\infty}^{+\infty} B_n B_{n+2}^* J_n J_{n+2} \\ &+ \text{etc.} \end{aligned} \tag{2.7}$$

We note that

$$\begin{gathered} \sum_{n=-\infty}^{+\infty} B_n B_{n+1}^* J_n J_{n+1} = \sum_{n=-\infty}^{+\infty} B_{n+1} B_n^* J_{n-1} J_n \\ \sum_{n=-\infty}^{+\infty} B_n B_{n+2}^* J_n J_{n+2} = \sum_{n=-\infty}^{+\infty} B_{n-2} B_n^* J_{n-2} J_n \\ \text{etc.} \end{gathered} \tag{2.8}$$

and put

$$\begin{aligned} C_k &= \sum_{n=-\infty}^{+\infty} B_n B_{n-k}^* J_n J_{n-k} \\ C_k^* &= \sum_{n=-\infty}^{+\infty} B_n^* B_{n-k} J_n J_{n-k} . \end{aligned} \tag{2.9}$$

Then (2.7) may be written

$$4\mathfrak{L}E^2(t) = C_0 + \sum_{k=1}^{+\infty} [C_k e^{ik\omega_1 t} + C_k^* e^{-ik\omega_1 t}]. \tag{2.10}$$

We must also evaluate $E_1(t)$. This is obtained by multiplying by $1 + K\Omega$ each frequency component in the expansion (2.6) of $E(t)$. In this case

$$\begin{aligned} \Omega &= |\,\omega\,| - \omega_c \\ \omega &= \pm(\omega_c + n\omega_1). \end{aligned} \tag{2.11}$$

We note that in practice the terms in the series (2.6) are vanishingly small for $|n| > \beta$ because, in that case, $J_n \cong 0$. It is therefore legitimate to write

$$\Omega = n\omega_1 . \tag{2.12}$$

Putting

$$A_n = 1 + nK\omega_1 \tag{2.13}$$

we find

$$2E_1(t) = \sum_{n=-\infty}^{+\infty} A_nB_nJ_n \exp(i\omega_c t + in\omega_1 t) + \sum_{n=-\infty}^{+\infty} A_nB_n^*J_n \exp(-i\omega_c t - in\omega_1 t). \tag{2.14}$$

Proceeding as before

$$4\mathfrak{L}E_1^2(t) = F_0 + \sum_{k=1}^{+\infty} [F_k \exp(ik\omega_1 t) + F_k^* \exp(-ik\omega_1 t)] \tag{2.15}$$

with

$$\begin{aligned} F_k &= \sum_{n=-\infty}^{+\infty} A_nA_{n-k}B_nB_{n-k}^*J_nJ_{n-k} \\ F_k^* &= \sum_{n=-\infty}^{+\infty} A_nA_{n-k}B_n^*B_{n-k}J_nJ_{n-k} . \end{aligned} \tag{2.16}$$

In order to obtain $d/dK\ \mathfrak{L}E_1^2(t)$ we consider the factor A_nA_{n-k} which is the only one to contain K. We have

$$A_nA_{n-k} = [1 - nK\omega_1][1 + (n - k)K\omega_1] \tag{2.17}$$

and

$$\left[\frac{d}{dK} A_nA_{n-k}\right]_{K=0} = (2n - k)\omega_1 . \tag{2.18}$$

Hence,

$$4\left[\frac{d}{dK} \mathfrak{L}E_1^2(t)\right]_{K=0} = \omega_1H_0 + \omega_1 \sum_{k=1}^{+\infty} [H_k \exp(ik\omega_1 t) + H_k^* \exp(-ik\omega_1 t)] \tag{2.19}$$

with

$$\begin{aligned} H_k &= \sum_{n=-\infty}^{+\infty} (2n - k)B_nB_{n-k}^*J_nJ_{n-k} \\ H_k^* &= \sum_{n=-\infty}^{+\infty} (2n - k)B_n^*B_{n-k}J_nJ_{n-k} . \end{aligned} \tag{2.20}$$

The output signal due to the input $E(t)$ given by (2.1) is

$$\begin{aligned} M(t) &= \frac{K}{2\mathfrak{L}E^2(t)} \left[\frac{d}{dK} \mathfrak{L}E_1^2(t)\right]_{K=0} \\ &= \frac{K\omega_1}{2} \frac{H_0 + \sum_{k=-\infty}^{+\infty} [H_k \exp(ik\omega_1 t) + H_k^* \exp(-i\omega_1 kt)]}{C_0 + \sum_{k=-\infty}^{+\infty} [C_k \exp(ik\omega_1 t) + C_k^* \exp(-i\omega_1 kt)]} . \end{aligned} \tag{2.21}$$

The same method may be applied if the superposed signals are not modulated by a single sinusoidal component. Consider, for instance, a signal component $E_i(t)$ containing two simultaneous modulation frequencies ω_1 and ω_2.

$$E_i(t) = \exp[i\omega_c t + i\varphi_1 + i\varphi_2] + \exp[-i\omega_c t - i\varphi_1 - i\varphi_2] \tag{2.22}$$

with

$$\begin{aligned} \varphi_1 &= \beta_1 \sin \omega_1 t \\ \varphi_2 &= \beta_2 \sin \omega_2 t. \end{aligned} \tag{2.23}$$

Using the identity (2.3) the spectrum of $E_i(t)$ is obtained by writing

$$\begin{aligned} E_i(t) &= \exp i\omega_c t \exp i\varphi_1 \exp i\varphi_2 \\ &= \exp i\omega_c t\left[\sum_{n=+\infty}^{+\infty} J_n(\beta_1) \exp(in\omega_1 t)\right]\left[\sum_{n=-\infty}^{+\infty} J_n(\beta_2) \exp(in\omega_2 t)\right]. \end{aligned} \tag{2.24}$$

Performing the multiplication yields a discrete spectrum. However, this time the frequency intervals are not equal. From (2.24) we derive the spectrum due to the superposition of signals of the type (2.22) with individual time lags and amplitude factors as in (2.1)

$$E(t) = \sum_i R_i E_i(t - t_i). \tag{2.25}$$

Proceeding as above, the spectrum may be used to evaluate the receiver output due to this superposition.

3. Fourier analysis of the receiver output. We consider the case of the superposition of input signals modulated by the same modulation frequency ω_1. We have shown that the output is given by expression (2.21) which is the quotient of two Fourier series. It is, itself, a periodic function of frequency ω_1, which we may expand into a Fourier series. We omit the constant factor in expression (2.21) and write, putting $\omega_1 t = \tau$

$$\begin{aligned} M(t) &= \frac{H_0 + \sum_{k=1}^{+\infty} [H_k \exp(ik\tau) + H_k^* \exp(-ik\tau)]}{C_0 + \sum_{k=1}^{+\infty} [C_k \exp(ik\tau) + C_k^* \exp(-ik\tau)]} \\ &= M_0 + \sum_{k=1}^{+\infty} [M_k \exp(ik\tau) + M_k^* \exp(-ik\tau)]. \end{aligned} \tag{3.1}$$

If the receiver output is filtered through a lowpass filter so that only the fundamental component of the Fourier series of M is observed, we must evaluate the coefficients M_1 and M_1^*. We have

$$\begin{aligned} 2\pi M_1 &= \int_0^{2\pi} M(\tau) \exp(-i\tau)\, d\tau \\ 2\pi M_1^* &= \int_0^{2\pi} M(\tau) \exp(i\tau)\, d\tau. \end{aligned} \tag{3.2}$$

The integrals may be evaluated by a method which takes advantage of the particular form of the function $M(t)$ in the present case. Consider the second integral and change the variable of integration τ to a complex variable p

$$p = \exp(i\tau) \tag{3.3}$$

and express $M(\tau)$ in terms of p

$$M(\tau) = \frac{H_0 + \sum_{k=1}^{+\infty} [H_k p^k + H_k^* p^{-k}]}{C_0 + \sum_{k=1}^{+\infty} [C_k p^k + C_k^* p^{-k}]}. \tag{3.4}$$

The value of M_1^* is then given by the contour integral on the unit circle

$$M_1^* = \frac{1}{2\pi i} \oint \frac{H_0 + \sum_{k=1}^{+\infty} [H_k p^k + H_k^* p^{-k}]}{C_0 + \sum_{k=1}^{+\infty} [C_k p^k + C_k^* p^{-k}]} dp. \tag{3.5}$$

The value of this integral is equal to the sum of the residues and depends on the poles contained within the unit circle. There are multiple poles at $p = 0$ and poles corresponding to the roots of the denominator. These are the roots of the equation

$$C_0 + \sum_{k=1}^{\infty} [C_k p^k + C_k^* p^{-k}] = 0. \tag{3.6}$$

If

$$p_1 = r_1 \exp(i\theta_1) \tag{3.7}$$

is a root of this equation, then

$$C_0 + \sum_{k=1}^{+\infty} [C_k r_1^k \exp(ik\theta_1) + C_k^* r_1^{-k} \exp(-ik\theta_1)] = 0. \tag{3.8}$$

The conjugate of this expression must also be zero, hence,

$$C_0 + \sum_{k=1}^{+\infty} [C_k^* r_1^k \exp(-ik\theta_1) + C_k r_1^{-k} \exp(ik\theta_1)] = 0. \tag{3.9}$$

But this expresses that

$$p_2 = \frac{1}{r_1} \exp(i\theta_1) \tag{3.10}$$

is also a root of equation (3.6). Therefore, the roots are grouped in pairs of the same argument θ_1 and such that the product of their moduli is unity. Half the roots will be inside the unit circle and to each of these roots corresponds an outside root on the same radius from the origin. If there are roots on the circle, the above conclusion does not hold. However, since the expression (3.6) on the circle represents the square of the amplitude of the input signal $I^2(t)$, this can only happen if the input vanishes. As an example, let us consider the case when the denominator contains only C_0 and C_1 . This will often be the case in practical applications when higher order terms are negligible. Denote by $N(p)$ the numerator of the integrand in (3.5). This integral becomes

$$M_1^* = \frac{1}{2\pi i} \oint \frac{N(p)\, dp}{C_0 + C_1 p + C_1^* p^{-1}}. \tag{3.11}$$

The roots of the denominator are

$$\begin{aligned} p_1 &= \frac{-C_0 + (C_0^2 - 4C_1C_1^*)^{1/2}}{2C_1} \\ p_2 &= \frac{-C_0 - (C_0^2 - 4C_1C_1^*)^{1/2}}{2C_1}. \end{aligned} \tag{3.12}$$

We shall assume the radical is real, in which case the roots are not on the unit circle. Moreover, if p_1 is inside the unit circle, the other root p_2 is outside.

The integrand of (3.11) is

$$\frac{pN(p)}{C_1p^2 + C_0p + C_1^*} = \frac{H_0p + \sum_{k=1}^{\infty} [H_kp^{k+1} + H_k^*p^{-k+1}]}{C_1(p - p_1)(p - p_2)}. \tag{3.13}$$

There is a residue due to the root p_1 and residues due to the terms $H_k^*p^{-k+1}$ for $k > 1$ corresponding to poles of order $k - 1$ at the origin. The sum of residues inside the unit circle is

$$M_1^* = \frac{p_1N(p_1)}{C_1(p_1 - p_2)} + \sum_{k=2}^{\infty} \frac{H_k^*}{(k-2)!} \left[\frac{d^{k-2}}{dp^{k-2}} \left(\frac{1}{C_1p^2 + C_0p + C_1^*}\right)\right]_{p=0}. \tag{3.14}$$

This expression gives the phase and amplitude of the fundamental Fourier component of the output signal. If the denominator contains more terms than assumed here, we must solve a complex algebraic equation of higher degree and similarly evaluate the residue for these roots inside the circle. If the numerator $N(p)$ contains only one Fourier component

$$N(p) = H_0 + H_1p + H_1^*p^{-1} \tag{3.15}$$

then (3.14) reduces to the very simple form

$$M_1^* = \frac{H_1^* + H_0p_1 + H_1p_1^2}{(C_0^2 - 4C_1C_1^*)^{1/2}}. \tag{3.16}$$

Instead of using the theory of residues, another method of computing the coefficients M_k of the Fourier expansion (3.1) is to write

$$\frac{H_0 + \sum_{k=1}^{\infty} [H_kp^k + H_k^*p^{-k}]}{C_0 + \sum_{k=1}^{\infty} [C_kp^k + C_k^*p^{-k}]} = M_0 + \sum_{k=1}^{\infty} [M_kp^k + M_k^*p^{-k}] \tag{3.17}$$

considering M_k as undetermined, then to multiply both sides of this equation by the denominator and equating coefficients of the same power of p.