of Sec. 6), then *Riemann's integral representation of the solution* (7) *remains valid under the assumptions*, (i) *and* (ii), *of Picard's theorem alone.*

In fact, Riemann's integral representation[6] of the solution (7) contains only the following elements: (α) the boundary data $\varphi(x)$, $\psi(y)$ [which are subject to condition (3)] and their first derivatives and (β) the function $H(x, y; \xi, \eta)$ and their *first* derivatives on the boundary; cf. (5), (5 *bis*). But item (β) does not involve the *second* derivatives of the function H (derivatives which do in general exist). On the other hand, not only the functions $\varphi(x)$, $\psi(y)$ but also their first derivatives, introduced by item (α), are controlled by assumption (ii). Hence it is clear that the italicized assertion, concerning the general validity of Riemann's formula, can be concluded by the same argument (this time by approximating $a(x, y)$, $b(x, y)$ *and* $\varphi(x)$, $\psi(y)$ *as well as* $d\varphi(x)/dx$, $d\psi(y)/dy$ by sequences of smooth functions) which was applied in Sec. 7.

NOTE ON THE AERODYNAMIC HEATING OF AN OSCILLATING INSULATED SURFACE*

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The effect of disturbing the thermal equilibrium of an oscillating conducting surface and its surroundings by changing the isothermal surface temperature at a given time was investigated in Ref. [1]. It was shown therein that the heat transfer and the thermal state of the fluid associated with the oscillating surface can be significantly different from that for conduction from a stationary surface with the same initial temperature difference. To complete this study it is appropriate to investigate the effect of insulating the surface at a given time on the equilibrium state.

Accordingly, consideration is given herein to a doubly infinite plane surface which is oscillating axially (i.e., longitudinally) in a viscous and heat-conducting fluid. It is assumed that sufficient time has elapsed so that an equilibrium state exists in which a periodic motion of the fluid has been established, and the heat obtained by viscous dissipation is all conducted through the surface so that the temperature does not increase indefinitely with time. In this state the fluid velocity is given by [2]

$$u(y, t) = U \exp[-(n/2\nu)^{1/2}y] \cos[nt - (n/2\nu)^{1/2}y] \tag{1}$$

and the temperature is [1]

$$T_e = T_\infty - \frac{U^2 Pr}{4c_p}\left\{\exp[-(2n/\nu)^{1/2}y] - \frac{1}{2 - Pr}[\exp\{-(n/\alpha)^{1/2}y\} \cos\{2nt - (n/\alpha)^{1/2}y\} - \exp\{-(2n/\nu)^{1/2}y\} \cos\{2nt - (2n/\nu)^{1/2}y\}]\right\}, \tag{2}$$

where u is the fluid velocity component parallel to the surface, y is the coordinate normal to the surface, t denotes time, U and n are the amplitude and frequency of the surface

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[6]See, e.g., G. Darboux, *op. cit.*, formula (16) on p. 80

*Received February 20, 1956.

oscillations, ν is the kinematic viscosity coefficient, Pr is the Prandtl number, c_p is the specific heat at constant pressure, T denotes temperature, α is the thermal diffusivity, and the subscripts e and ∞ denote conditions at equilibrium and far from the surface, respectively.

From Eq. (2) it can be seen that the thermal boundary condition at the surface ($y = 0$) for equilibrium is

$$T_e(0, t) \equiv T_{ew} = T_\infty - \frac{U^2 Pr}{4c_p} \,. \tag{3}$$

In the problem treated herein the equilibrium is to be disturbed by replacing, at time t_0, the isothermal condition specified by Eq. (3) by one for an insulated surface

$$T_y(0, t) = 0 \qquad t > t_0 \,, \tag{4}$$

where the subscript denotes partial differentiation. The other boundary conditions are

$$T(\infty, t) = T_\infty \tag{5}$$

$$T(y, t_0) = T_e(y, t_0) \tag{6}$$

and the corresponding differential equation is:

$$T_t - \alpha T_{yy} = \frac{\nu}{c_p} (u_y)^2. \tag{7}$$

Using Laplace transforms, the solution of the problem defined by Eqs. (4) to (7) is found to be:

$$\begin{aligned} T(y, t) = T_e(y, t) &+ \frac{U^2 Pr}{2c_p} \left(\frac{n}{2\nu}\right)^{1/2} \{-y \, erfc(y/2[\alpha(t - t_0)]^{1/2}) \\ &+ 2[\alpha(t - t_0)/\pi]^{1/2} \exp[-y^2/4\alpha(t - t_0)]\} \\ &+ \frac{U^2}{2c_p} \frac{(n/2\pi)^{1/2}}{1 + (2/Pr)^{1/2}} \int_{t_0}^{t} \cos(2n\tau + \pi/4) \exp[-y^2/4\alpha(t - \tau)](t - \tau)^{-1/2} \, d\tau. \end{aligned} \tag{8}$$

The profiles associated with the oscillations are shown in Fig. 1 for two special cases for air. The equilibrium profile (I) is one in which the temperature monotonically increases

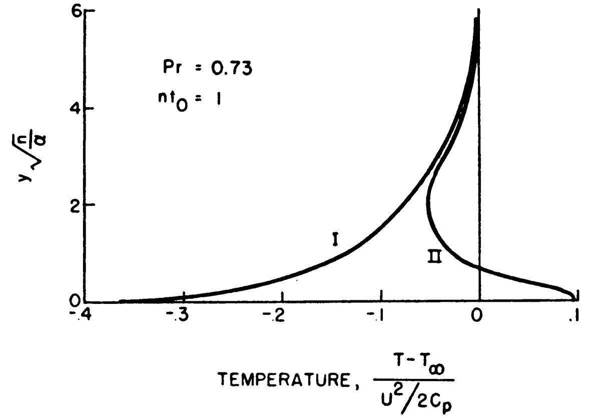

FIG. 1. Temperature profiles.

from the wall to free stream. When the wall is insulated, the wall temperature rises [roughly as $(t - t_0)^{1/2}$] giving a profile (II) which has a temperature minimum in the boundary layer. For larger times the temperature minimum will decrease in magnitude and, presumably move toward the outer edge of the boundary layer. Further, as can be seen from Eq. (8), the thickness of the temperature boundary layer increases linearly with $(t - t_0)$.

Although this solution was developed for an incompressible viscous fluid with constant property values, it is equally valid in the case of a compressible viscous fluid if the boundary-layer assumptions are made, if the Prandtl number and the product of the density, ρ, and absolute viscosity coefficients are assumed constant and if y is replaced by η where

$$\eta = \int_0^y \frac{\rho}{\rho_\infty}\, d\xi.$$

Under these assumptions the compressible boundary-layer equations reduce to those for an incompressible fluid.

The most important property of an insulated surface is perhaps the recovery factor. Using Eqs. (3) and (8), this is

$$r \equiv \frac{T(0, t) - T_\infty}{U^2/2c_p} = -\frac{Pr}{2} + [2Pr\, n(t - t_0)/\pi]^{1/2} + \frac{1}{(\pi)^{1/2}[1 + (2/Pr)^{1/2}]} \int_0^{[2n(t-t_0)]^{1/2}} \cos(2nt + \pi/4 - \beta^2)\, d\beta.$$

Since the time dependence itself is not of primary importance, the recovery factor is averaged over a cycle to yield

$$\begin{aligned} \bar{r} \equiv \frac{n}{\pi}\int_k^{k+\pi/n} r\, dt = \frac{-Pr}{2} &+ \frac{2}{3}(2Pr)^{1/2}\{[1 + n(k - t_0)/\pi]^{3/2} - [n(k - t_0)/\pi]^{3/2}\} \\ &- \frac{\sin(2nt_0 + \pi/4)}{\pi(2)^{1/2}[1 + (2/Pr)^{1/2}]}\{[1 + n(k - t_0)/\pi]^{1/2} - [n(k - t_0)/\pi]^{1/2}\} \\ &+ \frac{1}{2\pi^{3/2}[1 + (2/Pr)^{1/2}]}\int_{[2n(k-t_0)]^{1/2}}^{[2n(k-t_0)+2\pi]^{1/2}} \sin(2nk + \pi/4 - \beta^2)\, d\beta. \end{aligned}$$

Comparing this result with that for a semi-infinite plate in steady flow (for which $r_s = Pr^{1/2}$) and writing the integral in terms of Fresnel integrals yields

$$\begin{aligned} r_1 \equiv \frac{\bar{r}}{r_s} = -\frac{1}{2}(Pr)^{1/2} &+ \frac{(2)^{3/2}}{3}\{[1 + n(k - t_0)/\pi]^{3/2} - [n(k - t_0)/\pi]^{3/2}\} \\ &- \frac{\sin(2nt_0 + \pi/4)}{2\pi[1 + (Pr/2)^{1/2}]}\{[1 + n(k - t_0)/\pi]^{1/2} - [n(k - t_0)/\pi]^{1/2}\} \\ &+ \frac{1}{4\pi[1 + (Pr/2)^{1/2}]}\left\{\sin(2nk + \pi/4)\int_{2[n(k-t_0)/\pi]^{1/2}}^{2[1+n(k-t_0)/\pi]^{1/2}} \cos\frac{\pi}{2}\beta^2\, d\beta \right. \\ &\left. - \cos(2nk + \pi/4)\int_{2[n(k-t_0)/\pi]^{1/2}}^{2[1+n(k-t_0)/\pi]^{1/2}} \sin\pi\beta^2/2\, d\beta\right\}. \qquad (9) \end{aligned}$$

Obviously, the recovery factor with oscillations is increased ($r_1 > 1$) or decreased ($r_1 < 1$) with respect to that for steady flows depending on the relative orders of magnitude of the various terms in Eq. (9). Let us consider the first cycle (i.e., $k = t_0$). Then Eq. (9) can be written, after some simplification, as

$$r_1 = \left[\frac{2^{3/2}}{3} - \frac{1}{2}(Pr)^{1/2}\right] - \frac{0.1235}{1 + (Pr/2)^{1/2}} \sin(2nt_0 - 0.2233). \qquad (10)$$

On the other hand, if $n(k - t_0)/\pi$ is appreciable (over 1/2, say) then the first two terms of Eq. (9) give r_1 with an error of less than 5 per cent for air.

For air, ($Pr = 0.73$) Eq. (10) shows that, for $k = 0$,

$$0.439 \leq r_1 \leq 0.593,$$

depending on the value of nt_0. On the other hand, for large k, the recovery factor increases as $(k - t_0)^{1/2}$, which increase is directly due to the energy added to the boundary layer by the plate oscillations. This variation is shown in Fig. 2. It is seen that advantageous recovery factors (i.e., $r_1 < 1$) will only persist for a short time, until the initially cold boundary layer (Fig. 1, I) is heated by the plate oscillations.

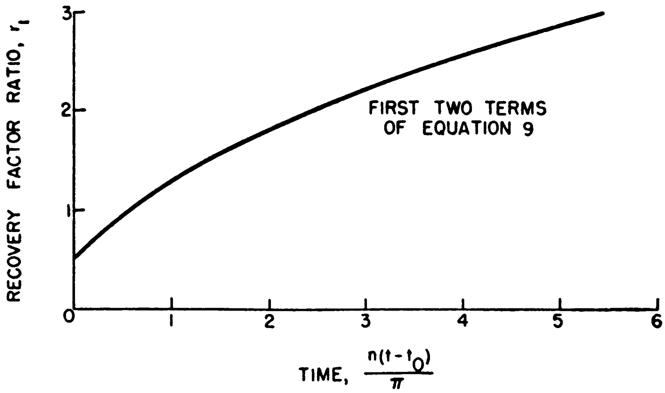

FIG. 2. Recovery factor ratio.

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