STEADY FLOW OF NON-NEWTONIAN FLUIDS THROUGH TUBES*

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1. Introduction. The stress-rate of deformation relation for an isotropic fluid, in which the stress at any point is assumed to depend only on the velocity gradients at that point, has been considered by Reiner [1]** and by Rivlin [2]. If the fluid is incompressible and the stress components are expressible as polynomials in the velocity gradients only, then the stress tensor can be expressed, using Cartesian tensors, in the form

$$t_{ij} = \Theta d_{ij} + \Psi d_{ik}d_{kj} + p\delta_{ij} , \qquad (1.1)$$

where d_{ij} is the rate of deformation tensor defined in terms of the velocity vector v_i by

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \qquad d_{ii} = 0. \qquad (1.2)$$

p is an arbitrary hydrostatic pressure and a comma denotes differentiation with respect to one of the coordinates x_i. Also, Θ and Ψ are polynomials in the invariants II and III defined by

$$II = \tfrac{1}{2}d_{ij}d_{ji} , \qquad III = \tfrac{1}{3}d_{ij}d_{jk}d_{ki} . \qquad (1.3)$$

Rivlin [3] has discussed the steady rectilinear flow of a fluid, for which the stress is given by (1.1), through a uniform tube of circular cross-section and has shown that such a rectilinear flow can be maintained without the application of body forces, without making further assumptions regarding the form of the dependence of Θ and Ψ on II and III. Ericksen† [4] has shown that this is not generally the case when the cross-section of the tube is non-circular and that, in such cases, the maintenance of steady rectilinear flow through the tube will require that, in addition to a uniform "pressure gradient" along the tube, an appropriate distribution of body forces be applied to the fluid. An exception occurs when $\Psi = k\Theta$, where k is a constant. In this case steady rectilinear flow of the fluid through the tube can be maintained by a uniform "pressure gradient" alone.

In the present paper, a suitable distribution of body forces, for the maintenance of steady rectilinear flow through a non-circular tube, is calculated. This distribution has non-zero components only in the planes normal to the direction of flow and these components are independent of position along the tube. This result suggests that if a uniform

**Numbers in square brackets refer to the Bibliography at the end of the paper.

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"pressure gradient" is applied to the fluid, and no body forces are applied, a steady flow will be produced which consists of a rectilinear flow along the tube on which is superposed some flow distribution in planes perpendicular to the length of the tube, independent of position along the tube. This flow distribution is calculated for a tube having an elliptical cross-section, for a fluid whose stress-rate of deformation relation has the form (1.1) with Θ constant and $\Psi = \alpha(c + II)$, where c is a constant and where α is a sufficiently small constant so that the governing equations determining the flow can be linearized with respect to α. This fluid is a nearly Newtonian fluid which departs from the ideal Newtonian law (obtained by taking $\alpha = 0$) in a particular way. The choice of this particular form of departure is governed by considerations of mathematical tractability. It is found that for such a fluid, a uniform pressure gradient will produce in the elliptical tube a rectilinear flow on which is superposed four similar vortex-like flows in planes normal to the length of the tube—one in each quadrant of the elliptical cross-section. These have the same signs in diagonally opposite quadrants and opposite signs in adjacent quadrants. Although the quantitative results have been obtained for a particular type of non-Newtonian fluid, we may expect them to be qualitatively valid much more generally.

2. Rectilinear flow. Assuming zero body forces Ericksen [4] has examined steady flow for which the velocity components are

$$v_1 = v_2 = 0, \qquad v_3 = 2f(x, y), \tag{2.1}$$

in a fluid for which the stress-rate of deformation relation is given by (1.1), where coordinates x_i are replaced by x, y, z. Here we reconsider flow given by (2.1) but retain, at present, body forces. From (1.1) and (2.1) we have

$$d_{ij} = \left\| \begin{array}{ccc} 0, & 0, & f_x \\ 0, & 0, & f_y \\ f_x, & f_y, & 0 \end{array} \right\|, \tag{2.2}$$

$$d_{ir}d_{rj} = \left\| \begin{array}{ccc} f_x^2, & f_xf_y, & 0 \\ f_xf_y, & f_y^2, & 0 \\ 0, & 0, & f_x^2 + f_y^2 \end{array} \right\|, \tag{2.3}$$

where suffixes denote partial differentiation. Also, from (1.1), (1.3), (2.2), (2.3) it follows that

$$II = f_x^2 + f_y^2, \qquad III = 0, \tag{2.4}$$

$$t_{11} = \Psi f_x^2 + p, \qquad t_{12} = \Psi f_xf_y, \qquad t_{22} = \Psi f_y^2 + p,$$

$$t_{13} = \Theta f_x, \qquad t_{23} = \Theta f_y, \qquad t_{33} = \Psi(f_x^2 + f_y^2) + p. \tag{2.5}$$

If we denote body force components by (X, Y, Z) and the (constant) density by ρ, the usual equations of motion reduce to

$$\rho X + \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\Psi f_x^2) + \frac{\partial}{\partial y}(\Psi f_x f_y) = 0,$$

$$\rho Y + \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\Psi f_x f_y) + \frac{\partial}{\partial y}(\Psi f_y^2) = 0, \qquad (2.6)$$

$$\rho Z + \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\Theta f_x) + \frac{\partial}{\partial y}(\Theta f_y) = 0,$$

if we use (2.5) and observe that inertia terms for the flow (2.1) vanish.

We now seek for a solution in which Z is zero and X, Y depend only on x and y. It follows from the third of equations (2.6) that

$$p = p_0(x, y) + \lambda Z, \qquad (2.7)$$

where λ is a constant such that

$$\frac{\partial}{\partial x}(\Theta f_x) + \frac{\partial}{\partial y}(\Theta f_y) + \lambda = 0. \qquad (2.8)$$

The function Ψ depends only on the invariant II which is given by (2.4). Hence, writing

$$q = p_0 + \tfrac{1}{2}\int \Psi \, dII, \qquad (2.9)$$

we have

$$\frac{\partial q}{\partial x} = \frac{\partial p_0}{\partial x} + \Psi(f_x f_{xx} + f_y f_{xy}),$$

$$\frac{\partial q}{\partial y} = \frac{\partial p_0}{\partial y} + \Psi(f_x f_{xy} + f_y f_{yy}).$$

It then follows that the first two of Eqs. (2.6) may be rewritten in the form

$$\rho X + \frac{\partial q}{\partial x} + \left[\frac{\partial}{\partial x}(\Psi f_x) + \frac{\partial}{\partial y}(\Psi f_y)\right]f_x = 0,$$

$$\rho Y + \frac{\partial q}{\partial y} + \left[\frac{\partial}{\partial x}(\Psi f_x) + \frac{\partial}{\partial y}(\Psi f_y)\right]f_y = 0. \qquad (2.10)$$

Ericksen [4] has shown that when X and Y are zero, Eqs. (2.10) imply that q is a function of f only, and, in general, they are incompatible with Eq. (2.8) for f. When X, Y are non-zero a possible solution of Eqs. (2.10) is

$$\rho X = -\left[\frac{\partial}{\partial x}(\Psi f_x) + \frac{\partial}{\partial y}(\Psi f_y) + \frac{dq}{df}\right]f_x,$$

$$\rho Y = -\left[\frac{\partial}{\partial x}(\Psi f_x) + \frac{\partial}{\partial y}(\Psi f_y) + \frac{dq}{df}\right]f_y, \qquad (2.11)$$

where q is now a function of f only. The flow given by (2.1), where f satisfies the differential equation (2.8), together with a suitable boundary condition, can then be maintained

provided we apply body forces whose components (X, Y) along the (x, y) axes respectively, are given by (2.11).

From (2.8) and (2.11) it is seen that the body forces vanish when

(a) $\Psi = k\Theta$, $q = k\lambda f$, where k is a constant, or

(b) f is a function of a linear combination of x and y, or

(c) f is a function of $(x^2 + y^2)^{\frac{1}{2}}$,

in agreement with Ericksen [4].

If the boundary of the cross-section of the tube is symmetrical about both the x- and y-axes, we see that equation (2.8) for f, together with the condition that $f = 0$ on the boundary, implies that f is an even function of x and an even function of y (Θ depends only on $f_x^2 + f_y^2$). Since dq/df in (2.11) is arbitrary we see that, in general, the flow may be maintained by a body force X along the x-axis which is an odd function of x and an even function of y; and by a body force Y which is an even function of x and an odd function of y. This suggests that rectilinear flow down such a tube may be possible without body forces, if in addition there is a flow in a perpendicular direction in each quadrant of the xy plane, the flow in any one quadrant being the mirror image of that in the immediately adjacent quadrants. In Sec. 5 this is confirmed for a special type of fluid flowing in a tube of elliptical cross-section.

Suppose now that the fluid is such that the restriction (a) is satisfied approximately. We put

$$\Psi = k\Theta + \alpha\Psi', \qquad q = k\lambda f, \tag{2.12}$$

and Eqs. (2.11) and (2.8) then yield

$$\rho X = -\alpha\left\{\frac{\partial}{\partial x}(\Psi' f_x) + \frac{\partial}{\partial y}(\Psi' f_y)\right\}f_x ,$$

$$\rho Y = -\alpha\left\{\frac{\partial}{\partial x}(\Psi' f_x) + \frac{\partial}{\partial y}(\Psi' f_y)\right\}f_y , \tag{2.13}$$

where Ψ' is a function of the invariant II, and α is a sufficiently small parameter. The value of q can be chosen arbitrarily and we choose to take the particular value in (2.12). In general, therefore, body forces (X, Y) of order α are required to maintain the flow (2.1). For example, if the fluid is nearly Newtonian in such a way that

$$\Theta = 2\mu, \tag{2.14}$$

then X, Y are $0(\alpha)$.

In the next section we examine the modifications which are required in order to obtain steady flow in pipes, when body forces are zero, for a general fluid for which Eq. (1.1) holds.

3. General steady flow in tubes. We seek a solution of the fundamental equations, with zero body forces, in which all the velocity components are independent of time t and the coordinate z. We therefore introduce into (2.1) non-zero values of v_1 , v_2 . In view of the incompressibility condition $d_{ii} = 0$ we may write

$$v_1 = -\psi_y , \qquad v_2 = \psi_x , \qquad v_3 = 2f, \tag{3.1}$$

ψ and f being functions of x, y only. From (1.2) and (3.1) we have

$$d_{ij} = \left\| \begin{matrix} -\psi_{xy}, & \tfrac{1}{2}(\psi_{xx} - \psi_{yy}), & f_x \\ \tfrac{1}{2}(\psi_{xx} - \psi_{yy}), & \psi_{xy}, & f_y \\ f_x, & f_y, & 0 \end{matrix} \right\|. \tag{3.2}$$

$$d_{1m}d_{m1} = \psi_{xy}^2 + \tfrac{1}{4}(\psi_{xx} - \psi_{yy})^2 + f_x^2 ,$$

$$d_{1m}d_{m2} = f_x f_y ,$$

$$d_{1m}d_{m3} = -f_x\psi_{xy} + \tfrac{1}{2}f_y(\psi_{xx} - \psi_{yy}),$$

$$d_{2m}d_{m2} = \psi_{xy}^2 + \tfrac{1}{4}(\psi_{xx} - \psi_{yy})^2 + f_y^2 , \tag{3.3}$$

$$d_{2m}d_{m3} = f_y\psi_{xy} + \tfrac{1}{2}f_x(\psi_{xx} - \psi_{yy}),$$

$$d_{3m}d_{m3} = f_x^2 + f_y^2 ,$$

and the strain invariants (1.3) are then

$$II = f_x^2 + f_y^2 + \psi_{xy}^2 + \tfrac{1}{4}(\psi_{xx} - \psi_{yy})^2,$$

$$III = (f_y^2 - f_x^2)\psi_{xy} + f_x f_y(\psi_{xx} - \psi_{yy}). \tag{3.4}$$

From (1.1), (3.2) and (3.3) we see that the components of stress are

$$t_{11} = -\Theta\psi_{xy} + \Psi\{\psi_{xy}^2 + \tfrac{1}{4}(\psi_{xx} - \psi_{yy})^2 + f_x^2\} + p,$$

$$t_{12} = \tfrac{1}{2}\Theta(\psi_{xx} - \psi_{yy}) + \Psi f_x f_y ,$$

$$t_{22} = \Theta\psi_{xy} + \Psi\{\psi_{xy}^2 + \tfrac{1}{4}(\psi_{xx} - \psi_{yy})^2 + f_y^2\} + p,$$

$$t_{13} = \Theta f_x + \Psi\{-f_x\psi_{xy} + \tfrac{1}{2}f_y(\psi_{xx} - \psi_{yy})\}, \tag{3.5}$$

$$t_{23} = \Theta f_y + \Psi\{f_y\psi_{xy} + \tfrac{1}{2}f_x(\psi_{xx} - \psi_{yy})\},$$

$$t_{33} = \Psi(f_x^2 + f_y^2) + p.$$

Using (3.1) and (3.5) we may now write down the equations of motion. A more convenient form of these equations may, however, be obtained by introducing the complex variables ζ, ζ^* where

$$\zeta = x + iy, \qquad \zeta^* = x - iy. \tag{3.6}$$

Then, observing that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \zeta^*} , \qquad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \zeta^*}\right), \tag{3.7}$$

we find, from (3.5), that

$$t_{11} + t_{22} = 2p + 4\Psi(f_\zeta f_{\zeta^*} + 2\psi_{\zeta\zeta}\psi_{\zeta^*\zeta^*}),$$

$$t_{11} - t_{22} + 2it_{12} = 4i\Theta\psi_{\zeta^*\zeta^*} + 4\Psi f_{\zeta^*}^2 ,$$

$$t_{13} + it_{23} = 2\Theta f_{\zeta^*} + 4i\Psi f_\zeta \psi_{\zeta^*\zeta^*} ,$$

$$t_{33} = p + 4\Psi f_\zeta f_{\zeta^*} .$$

(3.8)

Also, using (3.1) and (3.6), the equations of motion become

$$\frac{\partial}{\partial \zeta^*}(t_{11} + t_{22}) + \frac{\partial}{\partial \zeta}(t_{11} - t_{22} + 2it_{12}) = 4\rho(\psi_\zeta \psi_{\zeta^*\zeta^*} - \psi_{\zeta^*}\psi_{\zeta\zeta^*}),$$

$$\frac{\partial t_{33}}{\partial z} + \frac{\partial}{\partial \zeta}(t_{13} + it_{23}) + \frac{\partial}{\partial \zeta^*}(t_{13} - it_{23}) = 4i\rho(f_\zeta \psi_{\zeta^*} - f_{\zeta^*}\psi_\zeta),$$

(3.9)

so that with (3.8) we then have

$$\frac{\partial p}{\partial \zeta^*} + 2\frac{\partial}{\partial \zeta^*}\{\Psi(f_\zeta f_{\zeta^*} + 2\psi_{\zeta\zeta}\psi_{\zeta^*\zeta^*})\}$$

$$+ 2\frac{\partial}{\partial \zeta}\{i\Theta\psi_{\zeta^*\zeta^*} + \Psi f_{\zeta^*}^2\} = 2\rho(\psi_\zeta \psi_{\zeta^*\zeta^*} - \psi_{\zeta^*}\psi_{\zeta\zeta^*}),$$

(3.10)

$$\frac{\partial p}{\partial z} + 2\frac{\partial}{\partial \zeta}\{\Theta f_{\zeta^*} + 2i\Psi f_\zeta \psi_{\zeta^*\zeta^*}\}$$

$$+ 2\frac{\partial}{\partial \zeta^*}\{\Theta f_\zeta - 2i\Psi f_{\zeta^*}\psi_{\zeta\zeta}\} = 4i\rho(f_\zeta \psi_{\zeta^*} - f_{\zeta^*}\psi_\zeta).$$

(3.11)

From these equations we see that

$$p = \lambda z + p_0(\zeta, \zeta^*),$$

(3.12)

where λ is a constant and p_0 is a function of ζ, ζ^*. Hence

$$\lambda + 2\frac{\partial}{\partial \zeta}\{\Theta f_{\zeta^*} + 2i\Psi f_\zeta \psi_{\zeta^*\zeta^*}\} + 2\frac{\partial}{\partial \zeta^*}\{\Theta f_\zeta - 2i\Psi f_{\zeta^*}\psi_{\zeta\zeta}\}$$
$$= 4i\rho(f_\zeta \psi_{\zeta^*} - f_{\zeta^*}\psi_\zeta), \quad (3.13)$$

$$\frac{\partial p_0}{\partial \zeta^*} + 2\frac{\partial}{\partial \zeta^*}\{\Psi(f_\zeta f_{\zeta^*} + 2\psi_{\zeta\zeta}\psi_{\zeta^*\zeta^*})\}$$

$$+ 2\frac{\partial}{\partial \zeta}\{i\Theta\psi_{\zeta^*\zeta^*} + \Psi f_{\zeta^*}^2\} = 2\rho(\psi_\zeta \psi_{\zeta^*\zeta^*} - \psi_{\zeta^*}\psi_{\zeta\zeta^*}).$$

(3.14)

We also note that the invariants in (3.4) may now be written in the form

$$II = 4(f_\zeta f_{\zeta^*} + \psi_{\zeta\zeta}\psi_{\zeta^*\zeta^*}),$$

$$III = 4i(f_{\zeta^*}^2 \psi_{\zeta^*\zeta^*} - f_{\zeta^*}^2 \psi_{\zeta\zeta}).$$

(3.15)

Equations (3.13) and (3.14) are three equations for the three unknown functions p_0, f, ψ. In addition, if we impose the condition that the velocity is zero on some closed

curve $G(\zeta, \zeta^*) = 0$ in the (x, y) plane then

$$f = 0, \qquad \psi_\zeta = 0 \qquad [G(\zeta, \zeta^*) = 0]. \tag{3.16}$$

In the next section we consider the solution of Eqs. (3.13) and (3.14), subject to boundary conditions (3.16), for a fluid for which

$$\Psi = k\Theta + \alpha\Psi'. \tag{3.17}$$

This fluid was discussed in Sec. 2 where it was seen that, when $\alpha = 0$, a solution of the problem was possible with $\psi = 0$.

4. Solution by successive approximations. We suppose that Ψ is given by (3.17). We seek a solution of Eqs. (3.13) and (3.14) in power series in α, based on the solution $\psi = 0$ when $\alpha = 0$, assuming that α is a parameter which is sufficiently small, but we only carry out detailed calculations for terms of $0(\alpha)$. Thus, we assume that

$$\psi = \alpha\varphi + \cdots, \qquad f = h + \alpha g + \cdots, \tag{4.1}$$

so that, from (3.15)

$$II = II_0 + \alpha II_1 + \cdots,$$
$$III = \alpha III_1 + \cdots, \tag{4.2}$$

where

$$II_0 = 4h_\zeta h_{\zeta*}, \qquad II_1 = 4(h_\zeta g_{\zeta*} + h_{\zeta*} g_\zeta),$$
$$III_1 = 4i(h_\zeta^2 \varphi_{\zeta*\zeta*} - h_{\zeta*}^2 \varphi_{\zeta\zeta}). \tag{4.3}$$

Assuming that Θ and Ψ' are analytic functions of α, we then have

$$\Theta(II, III) = \Theta_0 + \alpha\Theta_1 + \cdots,$$
$$\Psi'(II, III) = \Psi'_0 + \cdots, \tag{4.4}$$

where

$$\Theta_0 = \Theta(II_0, 0), \qquad \Psi'_0 = \Psi'(II_0, 0),$$
$$\Theta_1 = II_1 \frac{\partial\Theta(II_0, 0)}{\partial II} + III_1 \frac{\partial\Theta(II_0, 0)}{\partial III}. \tag{4.5}$$

Hence, from (3.17) and (4.4),

$$\Psi = k\Theta_0 + \alpha(k\Theta_1 + \Psi'_0) + \cdots. \tag{4.6}$$

We now substitute (4.1), (4.4) and (4.6) into Eqs. (3.13) and (3.14), and equate coefficients of powers of α. It follows that the constant λ has the form

$$\lambda = \lambda_0 + \alpha\lambda_1 + \cdots, \tag{4.7}$$

where $\lambda_0, \lambda_1, \cdots$ are constants. Also, if we recall (2.9) and result (a) in Sec. 2, we see that the pressure may be written

$$p_0 = -\frac{k}{2} \int \Theta_0 \, dII_0 + k\lambda_0 h + \alpha p_1 + \cdots, \tag{4.8}$$

where p_1 is a function of ζ, ζ^* only. In addition, using (4.7) and (4.8), we see that the terms independent of α in (3.13) yield the equation

$$2\left\{\frac{\partial}{\partial\zeta}(\Theta_0 h_{\zeta^*}) + \frac{\partial}{\partial\zeta^*}(\Theta_0 h_\zeta)\right\} + \lambda_0 = 0, \tag{4.9}$$

whilst the terms independent of α in (3.14) now cancel. If we consider terms in α in (3.13) and (3.14) we obtain

$$\lambda_1 + 2\frac{\partial}{\partial\zeta}\{\Theta_0 g_{\zeta^*} + \Theta_1 h_{\zeta^*} + 2ik\Theta_0 h_\zeta \varphi_{\zeta^*\zeta^*}\}$$

$$+ 2\frac{\partial}{\partial\zeta^*}\{\Theta_0 g_\zeta + \Theta_1 h_\zeta - 2ik\Theta_0 h_{\zeta^*}\varphi_{\zeta\zeta}\} = 4i\rho(h_\zeta\varphi_{\zeta^*} - h_{\zeta^*}\varphi_\zeta), \tag{4.10}$$

$$\frac{\partial p_1}{\partial\zeta^*} + 2\frac{\partial}{\partial\zeta^*}\{k\Theta_0(h_\zeta g_{\zeta^*} + h_{\zeta^*}g_\zeta) + (k\Theta_1 + \Psi_0')h_\zeta h_{\zeta^*}\}$$

$$+ 2\frac{\partial}{\partial\zeta}\{i\Theta_0\varphi_{\zeta^*\zeta^*} + 2k\Theta_0 h_{\zeta^*}g_{\zeta^*} + (k\Theta_1 + \Psi_0')h_{\zeta^*}^2\} = 0. \tag{4.11}$$

We continue discussion of these equations for the special fluid which is defined by

$$\Theta = 2\mu, \qquad \Psi' = c + II, \qquad k = 0, \tag{4.12}$$

where μ and c constants, so that

$$\Theta_0 = 2\mu, \qquad \Psi_0' = c + II_0, \qquad k = 0, \qquad \Theta_1 = 0. \tag{4.13}$$

Equations (4.9), (4.11) and (4.10) respectively then reduce to

$$8\mu h_{\zeta\zeta^*} + \lambda_0 = 0, \tag{4.14}$$

$$\frac{\partial p_1}{\partial\zeta^*} + 4i\mu\varphi_{\zeta\zeta^*\zeta^*} + 8\frac{\partial}{\partial\zeta^*}(h_\zeta^2 h_{\zeta^*}^2) + 8\frac{\partial}{\partial\zeta}(h_\zeta h_{\zeta^*}^3) + 2c\left\{\frac{\partial}{\partial\zeta^*}(h_\zeta h_{\zeta^*}) + \frac{\partial}{\partial\zeta}(h_{\zeta^*}^2)\right\} = 0, \tag{4.15}$$

$$8\mu g_{\zeta\zeta^*} + \lambda_1 = 4i\rho(h_\zeta\varphi_{\zeta^*} - h_{\zeta^*}\varphi_\zeta). \tag{4.16}$$

If we differentiate Eq. (4.15) with respect to ζ, subtract the result from its complex conjugate, and use (4.14), we obtain the following equation for φ:

$$i\mu\varphi_{\zeta\zeta\zeta^*\zeta^*} + \frac{\partial^2}{\partial\zeta^2}(h_\zeta h_{\zeta^*}^3) - \frac{\partial^2}{\partial\zeta^{*2}}(h_{\zeta^*}h_\zeta^3) = 0,$$

or

$$i\mu\varphi_{\zeta\zeta\zeta^*\zeta^*} + 6h_{\zeta\zeta^*}(h_{\zeta^*}^2 h_{\zeta\zeta} - h_\zeta^2 h_{\zeta^*\zeta^*})$$

$$+ h_{\zeta^*}^3 h_{\zeta\zeta\zeta} - h_\zeta^3 h_{\zeta^*\zeta^*\zeta^*}$$

$$+ 3h_\zeta h_{\zeta^*}(h_{\zeta^*}h_{\zeta\zeta\zeta^*} - h_\zeta h_{\zeta\zeta^*\zeta^*}) = 0. \tag{4.17}$$

5. Flow through elliptical pipe. In this section we consider the solution of Eqs. (4.14), (4.16) and (4.17) for the special case of flow through a pipe whose cross-section is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{5.1}$$

In complex coordinates ζ, ζ^* this equation may be re-written in the form

$$(a^2 + b^2)\zeta\zeta^* - \tfrac{1}{2}(a^2 - b^2)(\zeta^2 + \zeta^{*2}) - 2a^2b^2 = 0. \tag{5.2}$$

At this boundary we assume that the velocity of the fluid is zero so that

$$h = 0, \qquad \varphi_\zeta = 0, \qquad g = 0, \tag{5.3}$$

when ζ, ζ^* are related by Eq. (5.2).

Since h vanishes on the boundary (5.2) we try a solution of (4.14) of the form

$$h = A\{(a^2 + b^2)\zeta\zeta^* - \tfrac{1}{2}(a^2 - b^2)(\zeta^2 + \zeta^{*2}) - 2a^2b^2\}. \tag{5.4}$$

This satisfies Eq. (4.14) if

$$8\mu(a^2 + b^2)A = -\lambda_0 . \tag{5.5}$$

If we substitute h from (5.4) in the right hand side of (4.17) we have

$$i\mu\varphi_{\zeta\zeta^*\zeta^*} - 24A^4a^2b^2(a^4 - b^4)(\zeta^2 - \zeta^{*2}) = 0. \tag{5.6}$$

We require a solution of (5.6) such that φ_ζ (and therefore φ_{ζ^*}) vanishes on the boundary (5.2). We therefore assume that

$$i\mu\varphi = B\{(a^2 + b^2)\zeta\zeta^* - \tfrac{1}{2}(a^2 - b^2)(\zeta^2 + \zeta^{*2}) - 2a^2b^2\}^2(\zeta^2 - \zeta^{*2}). \tag{5.7}$$

This satisfies the boundary conditions and will also satisfy the differential equation (5.6) if

$$B\{4(a^2 + b^2)^2 + (a^2 - b^2)^2\} - 4A^4a^2b^2(a^4 - b^4) = 0. \tag{5.8}$$

Using (5.8), and expressing φ in terms of the coordinates x, y, we have

$$\varphi = \frac{64A^4a^2b^2(a^4 - b^4)(b^2x^2 + a^2y^2 - a^2b^2)^2xy}{\mu\{4(a^2 + b^2)^2 + (a^2 - b^2)^2\}} . \tag{5.9}$$

This represents a flow in the x, y plane whose streamlines are the curves $\varphi = $ constant. The flow is illustrated in Fig. 1.
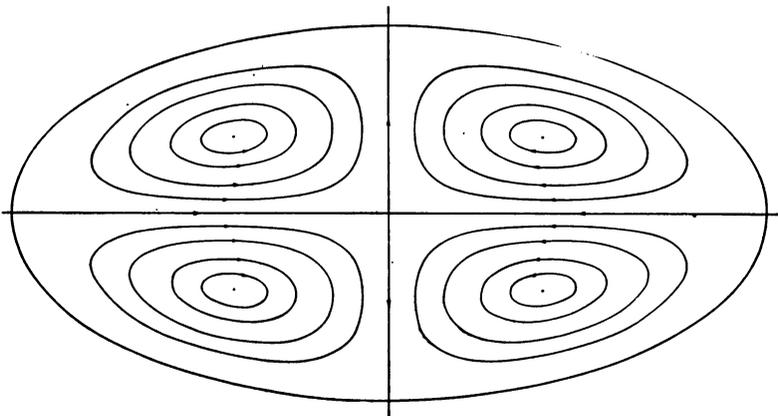

FIG. 1. Secondary flow in a tube with elliptical cross-section ($\alpha > 0$).

To complete the solution of the problem as far as terms of order α are concerned we must now solve Eq. (4.16). In this equation we may put $\lambda_1 = 0$ since this only contributes terms of the same type as those in the original flow down the tube. The remaining terms, which arise solely from the inertia of the material, are found from (4.16), where h, φ are given by (5.4) and (5.7) respectively. Thus

$$8\mu g_{\zeta\zeta^*} = \frac{8\rho AB}{\mu} \{(a^2 + b^2)\zeta\zeta^* - \tfrac{1}{2}(a^2 - b^2)(\zeta^2 + \zeta^{*2})$$

$$- 2a^2 b^2\}^2 \{2(a^2 - b^2)\zeta\zeta^*$$

$$- (a^2 + b^2)(\zeta^2 + \zeta^{*2})\}, \qquad (5.10)$$

together with the boundary condition

$$g = 0, \qquad (5.11)$$

on the boundary (5.2). The solution of this boundary value problem is straightforward, but we omit details since the resulting flow will only modify slightly the original flow down the tube.

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