and

$$D\nabla^4 w - (1 - \lambda\nabla^2)\left(\frac{C}{1-\nu^2}\right)\Big\{\nu(\nabla^2 z)u_{k,k} + \frac{1-\nu}{2}[z_{,i}(\nabla^2 u_i + z_{,i}\nabla^2 w) + z_{,ij}(u_{i,j} + u_{j,i})] + \frac{1+\nu}{2}z_{,i}[(\nabla^2 z)w_{,i} + u_{i,ji} + z_{,ij}w_{,i} + z_{,i}w_{,ij}] + (1-\nu)z_{,i}z_{,ij}w_{,j}\Big\} = [\text{Right side of Eq. (6)}]. \tag{9}$$

In conclusion, it may be mentioned that as in [5] for vibration problems of shallow shells, the effect of rotatory inertia can be easily added to the above equations.

References

1. K. Marguerre, *Zur Theorie der gekrummten Platte Grosser Formanderung*, Proc. 5th Intern. Congr. Appl. Mech., 93-101 (1938)
2. A. E. Green and W. Zerna, *Theoretical elasticity*, Oxford, Clarendon Press, 1954, Chap. XI
3. Eric Reissner, *On some aspects of the theory of thin elastic shells*, J. Boston Soc. Civ. Engrs. **42**, 100-133 (1955)
4. Eric Reissner, *On transverse vibrations of thin shallow elastic shells*, Quart. Appl. Math. **13**, 169-176 (1955)
5. P. M. Naghdi, *On the theory of thin elastic shells*, to appear in Quart. Appl. Math.
6. L. H. Donnell, *Stability of thin-walled tubes under torsion*, Natl. Advisory Comm. Aeronaut., Tech. Rept. No. 479, 1933

ON THE ERRORS IN ANALOGUE SOLUTIONS OF HEAT CONDUCTION PROBLEMS*

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1. Introduction. Many practical heat conduction questions lead to problems not amendable to the methods of classical mathematical physics. Consequently, numerous methods for approximating the solution have been developed; many of these procedures result from the replacement of the differential equation by either finite difference equations, which are usually solved by means of a digital computer, or by systems of ordinary differential equations, which are often treated using an analogue device of some kind. Our interest will be confined to the latter of these types.

Useful analogue machines for the solution of heat conduction problems include differential analyzers [3, 4] and resistance-capacitance circuits [1, 6]. In either case, the derivatives with respect to one variable are retained and all others are replaced by finite differences, and the resulting system of ordinary differential equations is solved. For the heat flow equation in one space variable, it is possible to retain either the space derivative or the time derivative; however, keeping the time derivative is much more desirable from a least work point of view [4]. With more than one space variable, this is the only approach that can treat the space variables in a symmetric manner.

The replacement of the space derivatives by finite differences results in an error in

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the solution of the original problem. Hartree [4] has conjectured that the magnitude of this error is $O(\Delta x^2)$, where Δx is the greatest space increment. The object here is to give a simple proof of this statement. The argument will be an application of a previous result on the backwards difference equation [2]. An interesting point is that the difference equation result was obtained by adapting a convergence proof given by Rothe [7] for the difference-differential equation obtained by replacing the time derivative by a finite difference.

2. Analysis. Consider the boundary value problem

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}, \qquad (x, y) \in R, 0 < t \leq T, \\ &u(x, y, 0) = f(x, y), \\ &u(x, y, t) = g(x, y, t), \qquad (x, y) \in S, 0 < t \leq T, \end{aligned}\right\} \tag{2.1}$$

where R is a bounded, connected set in the plane and S is its boundary. S will be assumed to be sufficiently regular in shape that it can be given by a finite number of polygons through grid points of a lattice of points of the form $(i\Delta x, j\Delta y)$ for some Δx and Δy. Let $z_{ij}(t)$ represent the value of a function of t attached to the point $(i\Delta x, j\Delta y)$. Then, the analogue solution $w_{ij}(t)$ satisfies the following system of ordinary differential equations:

$$\left.\begin{aligned} &\Delta_x^2 w_{ij}(t) + \Delta_y^2 w_{ij}(t) = \frac{dw_{ij}(t)}{dt}, \qquad (i\Delta x, j\Delta y) \in R, 0 < t \leq T, \\ &w_{ij}(t) = g(i\Delta x, j\Delta y, t), \qquad (i\Delta x, j\Delta y) \in S, 0 < t \leq T, \end{aligned}\right\} \tag{2.2}$$
$$w_{ij}(0) = f(i\Delta x, j\Delta y),$$

where $\Delta_x^2 w_{ij}(t)$ is the divided second difference $(w_{i+1,j} - 2w_{ij} + w_{i-1,j})/(\Delta x)^2$. Provided $g(x, y, t)$ is a continuous function of t for each $(x, y) \in S$, it is well known [5, p. 71] that (2.2) possesses a unique solution. Moreover, if g_t exists and is bounded uniformly on $(x, y) \in S$ for $0 < t \leq T$, then $w_{ij}(t)$ possesses a bounded second derivative for $0 < t \leq T$. In this case,

$$\Delta_x^2 w_{ij}(t) + \Delta_y^2 w_{ij}(t) = \frac{w_{ij}(t) - w_{ij}(t - \Delta t)}{\Delta t} + \frac{1}{2}\frac{d^2 w_{ij}}{dt^2}\Delta t, \tag{2.3}$$

with $d^2 w_{ij}/dt^2$ being evaluated at some point in $(t - \Delta t, t)$. Let (2.1) have a solution u such that u_{tt}, u_{xxxx}, and u_{yyyy} exist and are bounded for $(x, y) \in R$ and $0 < t \leq T$; then

$$\begin{aligned} \Delta_x^2 u_{ij}(t) + \Delta_y^2 u_{ij}(t) = \frac{u_{ij}(t) - u_{ij}(t - \Delta t)}{\Delta t} &+ \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t \\ &+ \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(\Delta x)^2 + \frac{1}{12}\frac{\partial^4 u}{\partial y^4}(\Delta y)^4. \end{aligned} \tag{2.4}$$

Let

$$v_{ij}(t) = u_{ij}(t) - w_{ij}(t) \tag{2.5}$$

denote the truncation error incurred by the use of (2.2). Then, assuming $\Delta y = 0(\Delta x)$,

$$\left.\begin{aligned} \Delta_x^2 v_{ij}(t) + \Delta_y^2 v_{ij}(t) &= \frac{v_{ij}(t) - v_{ij}(t - \Delta t)}{\Delta t} + A_{ij}(t)\Delta t \\ &\qquad + B_{ij}(t)(\Delta x)^2, \qquad (i\Delta x, j\Delta y) \varepsilon R, \\ v_{ij}(t) &= 0, \qquad (i\Delta x, j\Delta y) \varepsilon S, \\ v_{ij}(0) &= 0, \qquad (i\Delta x, j\Delta y) \varepsilon R, \end{aligned}\right\} \qquad (2.6)$$

where $A_{ij}(t)$ and $B_{ij}(t)$ are bounded. If $A = max \mid A_{ij}(t) \mid$ and $B = max \mid B_{ij}(t) \mid$, then

$$\mid v_{ij}(t) \mid \leq AT\Delta t + BT(\Delta x)^2, \qquad (2.7)$$

as

$$\max_{i,j} \mid v_{ij}(t) \mid \leq \max_{i,j} \mid v_{ij}(t - \Delta t) \mid + A(\Delta t)^2 + B(\Delta x)^2 \Delta t \qquad (2.8)$$

by the growth lemma of [2]. As $v_{ij}(t)$ is independent of Δt, let Δt diminish to zero, holding Δx and Δy constant. Thus,

$$\mid v_{ij}(t) \mid \leq BT(\Delta x)^2, \qquad (i\Delta x, j\Delta y) \varepsilon R, 0 \leq t \leq T. \qquad (2.9)$$

This completes the proof of Hartree's conjecture for the cases satisfying the hypotheses made above. While for simplicity the proof has been given for the linear equation (2.1), it is apparent from [2] that the same argument can be applied to the quasi-linear equation

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2} = F(x_1, \cdots, x_m, t, u) \frac{\partial u}{\partial t} + G(x_1, \cdots, x_m, t, u) \qquad (2.10)$$

if it can be shown that $d^2 w_{i_1, \cdots, i_m}/dt^2$ is bounded as needed above.

References

1. W. A. Bruce, *An electrical device for analyzing oil-reservoir behavior*, Trans. AIME **151**, 112-124 (1943)
2. J. Douglas, *On the numerical integration of quasi-linear parabolic differential equations*, to appear in the Pacific Journal of Mathematics
3. N. R. Eyres, D. R. Hartree, J. Ingham, R. Jackson, R. J. Sarjant, and J. B. Wagstaff, *The calculation of variable heat flow in solids*, Phil. Trans. Roy. Soc. London, Ser. **A**, **240**, 1-57 (1946-8)
4. D. R. Hartree, *The application of the differential analyser to the evaluation of solutions of partial differential equations*, Proc. First Canadian Math. Congress, 327-337 (1945)
5. E. L. Ince, *Ordinary differential equations*, New York
6. V. Paschkis and H. D. Baker, *A method for determining unsteady-state heat transfer by means of an electrical analogy*, Trans. ASME, **64**, 105-110 (1942)
7. E. Rothe, *Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben*, Math. Ann. **102**, 650-670 (1930)