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GEOMETRICAL OPTICS OF ANGULAR STRATIFIED MEDIA*

BY
R. SEDNEY
Aberdeen Proving Ground, Maryland

Abstract. Some aspects of the geometrical optics of angular stratified media are considered. A similarity property of the light paths enables one to draw some general conclusions about the behaviour of light rays traversing a cone. An application to the interferometric method of observing supersonic flows shows that a conical flow test is valid including the refraction effect.

Introduction. By the methods of geometrical optics, the paths of rays of light can be found by integrating a system of ordinary differential equations. The integration problem reduces to a quadrature in the two special cases when the isotropic medium is (i) plane stratified and (ii) spherically stratified. In these cases the index of refraction, n, is the function of a single variable, in (i) a Cartesian coordinate and in (ii) the distance from a point. In these cases it is possible to derive a number of properties of the rays without even specifying the functional form of n, [1] and [2]. In particular, the light rays remain in a plane so that these two cases are essentially two-dimensional problems. In this paper we consider media called angular stratified media, with the property that n is a function of the polar angle with respect to some line. Here the two- and three-dimensional problems are essentially different; however they both have a "similarity property". This similarity property was called to the author's attention by Dr. J. H. Giese.

In neither the two- nor three-dimensional cases was it found that the integration could be reduced to a quadrature. But some general conclusions regarding the light paths have been obtained. In the two-dimensional case the problem can be reduced to a quadrature plus a first order differential equation.

Consideration of angular stratified media arises naturally in the study of supersonic flow over a cone or a wedge (Taylor-Maccoll or Prandtl-Meyer flow). Since optical methods of observing flows are quite common, the study of angular stratified media is not entirely academic. The paper concludes with an application to the interferometric method of observing supersonic flow over a cone.

Three-dimensional case. The path of a ray of light in a medium of index of refraction n can be written

$$d(n\, d\mathbf{r}/ds)/ds = \operatorname{grad} n, \tag{1}$$

where $\mathbf{r}$: (x, y, z) is the position vector of a point on the ray and s is the arc length.

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Equations (1) are not independent; in addition

$$(d\mathbf{r}/ds)^2 = 1. \tag{2}$$

Suppose n is a homogeneous function of degree zero

$$n(\lambda\mathbf{r}) = n(\mathbf{r}),$$

i.e., n is constant on radial lines. Then grad n is a vector whose components are homogeneous functions of degree -1. Then it is easy to see that (1) and (2) are invariant under the similarity transformation

$$\begin{aligned} \mathbf{r} &\rightarrow \lambda\mathbf{r}, \\ s &\rightarrow \lambda s. \end{aligned} \tag{3}$$

This is the similarity property of the rays. An angular stratified medium is one in which n is homogeneous and rotationally symmetric, i.e.,

$$n = n[(x^2 + y^2)^{1/2}/z], \tag{4}$$

where z is the axis of symmetry.

For definiteness consider the following optical problem. A cone of half-angle ϕ, vertex at the origin, and axis along z,

$$x^2 + y^2 = z^2 \tan^2 \phi$$

is imbedded in a space of constant index of refraction N, (see Fig. 1). There may be a discontinuity of index across the conical surface. Inside the cone the index is of the form (4).

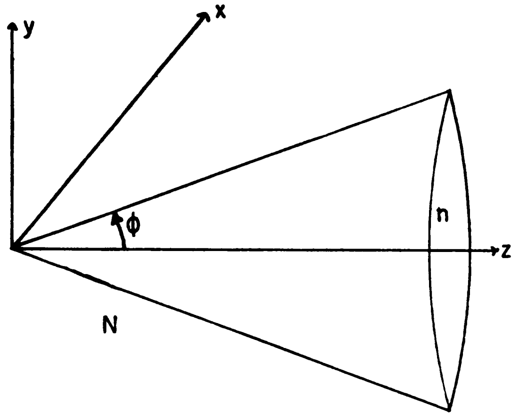

Fig. 1.

A parallel beam of light travelling in the positive x direction strikes the cone. The ray tracing and image problems are to be considered. In the latter the mapping of points in some object plane $x < 0$ onto some image plane $x > 0$ is to be studied.

In the ray tracing problem (1) and (2) must be integrated within the conical region subject to initial conditions for the position and direction of the ray. The initial direction is obtained by applying Snell's law. Consider the points, in an object plane, $x = x_0 < 0$, along a line l_0 through the origin, Fig. 2.

The rays through the points of l_0 strike the cone along a generator. Thus the initial positions of these rays are related by the similarity transformation $\mathbf{r} \rightarrow \lambda\mathbf{r}$. The initial

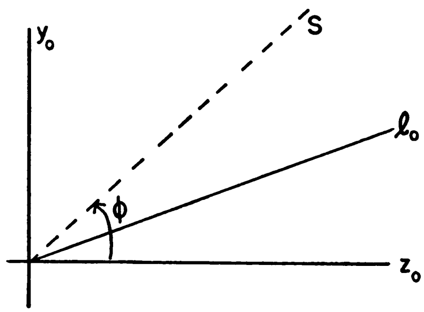

FIG. 2. Object plane.

direction, from Snell's law, is invariant under this transformation since this direction is determined by the direction before refraction and the normal to the cone surface. Therefore, because of the similarity property of the rays and the invariance of initial direction, the light rays are similar curves. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two rays which strike the cone along a generator, then $\mathbf{r}_2 = \lambda\mathbf{r}_1$; for the geometrical path length at corresponding points, $s_2 = \lambda s_1$. Furthermore the optical path lengths $[L_2]$ and $[L_1]$, where

$$[L] = \int_0^s n \, ds,$$

of these two rays are related by

$$[L_2] = \lambda[L_1]. \tag{5}$$

Tracing the paths of all rays which strike along a generator it is seen that these rays lie on a surface. This surface can be generated by the motion of a radial line. In particular, all these rays terminate along a generator and therefore all leave the cone

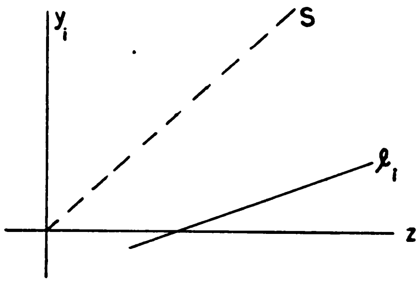

FIG. 3. Image plane.

in the same direction. We have the following picture. The plane sheet of rays defining l_0 in the object plane, after traversing the cone, is still a plane at some angle to the original plane. Thus in an image plane $x = x_i > 0$ the line l_0 is mapped into a straight line l_i , in general, not through the origin, Fig. 3.

Under certain conditions, e.g., $n > N$, this mapping will be singular over certain regions of the image plane. The Jacobian of the transformation will be zero along a certain curve, the envelope of the lines l_i. Between this envelope and the line S no rays are imaged. This accounts for the "shadow of the shock wave" in a shadowgraph of supersonic flow over a cone. Also, for $n > N$, the tangential rays, i.e., when the line l_0 is S, after traversing the cone do not emerge; they are internally reflected. This will also be true of a small bundle of rays in the neighbourhood of the tangential rays. The last two statements concerning the singularity of the mapping and internal reflection are easily proved for the special case n = constant and have been verified by numerical integration of (1) and (2) for a few cases with variable n.

There is another result that can be obtained without resort to numerical integration. A lower bound on the depth of penetration of a ray into the conical region can be obtained. In [2] it is shown that, for any rotationally symmetric medium, the equations of the light rays can be transformed into the equations for a two-dimensional problem provided that the index of refraction is replaced by m,

$$m = [n^2 - (h^2/\rho^2)]^{1/2}$$

where $\rho^2 = x^2 + y^2$ and h is a constant for a given ray. This constant can be written as

$$h = xq - yp,$$

where p and q are the optical direction cosines with respect to the x and y axis. For our parallel bundle of rays, evaluating h before the ray enters the cone

$$h = -y_0 N,$$

where y_0 is the y coordinate at the piercing point. To obtain a real value of m

$$\rho^2 > h^2/n^2 > N^2 y_0^2/n_0^2 ,$$

where n_0 is the maximum value of $n = n(\rho/z)$. For example, in supersonic flow over a cone n is a monotone decreasing function of ρ/z so that n_0 is the index evaluated at the surface of the cone. In the few numerical cases considered this lower bound was found to be quite close to the point of deepest penetration.

Two-dimensional case. A two-dimensional problem is obtained as a special case of the three-dimensional problem by considering a parallel sheet of rays in the $x - z$ plane. In this case introduce polar coordinate r, θ. Then for an angular stratified medium $n = n(\theta)$. It is evident that the similarity property holds also in this case. Using θ as the independent variable instead of arc length, the path of a light ray is described by the equation

$$[nr'/(r'^2 + r^2)^{1/2}]' - nr/(r'^2 + r^2)^{1/2} = 0, \tag{6}$$

where $' = d/d\theta$. Let ψ be the angle between a radial line and the tangent to the path. Then

$$\cos \psi = r'/(r'^2 + r^2)^{1/2},$$

$$\sin \psi = \pm r/(r'^2 + r^2)^{1/2}.$$

Then (6) can be written

$$\psi' - (n'/n) \cos \psi = \pm 1, \tag{7}$$

$$r'/r = \cot \psi. \tag{8}$$

The integration problem reduces to solving (7) plus the quadrature (8). It does not appear possible to solve (7) without specifying n. (The integration problem for (7) would be simple for special functions n, e.g., $n = ae^{b\theta}$.)

An application. In the interferometric method of observing supersonic flow, one tries to relate the density in the flow field to the shift in a fringe pattern. Since this is a tedius data-handling problem, it is useful to devise methods of getting partial information from a photograph of the fringe shifts. One such method is a test for conical flow, [3], which is a result of showing that the fringe shift is a homogeneous function of degree one in y and z. In [3] this was shown under the assumption that the light rays are not refracted as they pass through the conical flow region. The same result can be obtained allowing refraction.

In the usual experimental arrangement a lens is placed in the one beam of light that passes through the disturbed region. The lens is placed so that the image of the median plane, $x = 0$, is photographed. Straight lines through the origin of an object plane, l_0, are mapped into straight lines through the origin of this image plane.

The fringe shift is proportional to the difference in optical path lengths of two interfering rays, one of which passes through the disturbance, the other does not. It has already been shown that for a ray through an angular stratified medium the optical path length has the similarity property (5). It remains to show that the same holds for the optical path length of the interfering, undisturbed ray. The interfering ray can be obtained as follows. The parallel bundle of undeviated rays can be projected back as a virtual beam (the interferometer actually separates this beam from the deviated rays). Taking account of the lens, it is necessary to identify an undeviated virtual ray with a deviated one that appears to come from the same point in the median plane (see Fig. 4) which is a projection of the three-dimensional phenomena onto a plane.

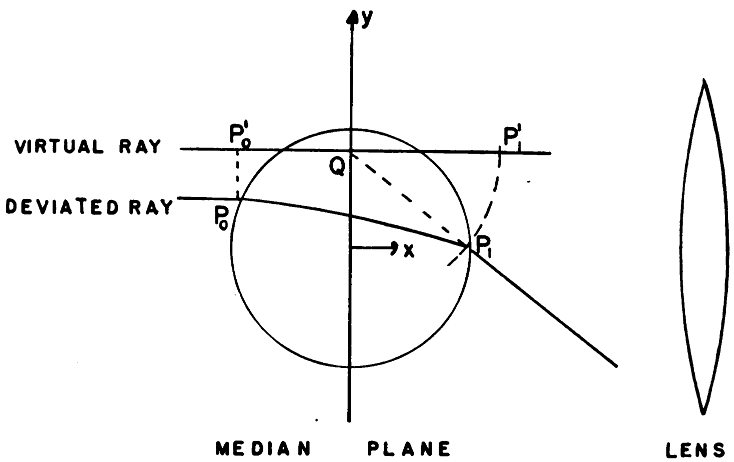

FIG. 4.

The appropriate undeviated ray is the one that passes through Q. Lay off a spherical segment with center Q and radius QP_1. By Fermat's principle the optical path lengths from P_1 and P_1' to the point at which interference takes place are the same and also the optical path lengths from the light source to P_0 and P_0' are the same. (Actually the interferometer is usually arranged so that there is a constant difference between these optical path lengths but this is of no consequence in our considerations.) The difference in optical path lengths between P_0P_1 and P_0P_1' must be shown to have the similarity property. From (5), $[P_0P_1]$ has this property. Since P_0' and P_1' are determined by the initial and terminal points of the ray P_0P_1, the coordinates of P_0' and P_1' also have the similarity property. Since the virtual beam passes through only a region of constant index N

$$[P_0'P_1'] = N\,\overline{P_0'P_1'}\,.$$

If $\delta = [P_0P_1] - [P_0'P_1']$ then δ has the similarity property. Finally the coordinates in the median plane (y_m, z_m) also have the similarity property. Therefore,

$$\delta(\lambda y_m, \lambda z_m) = \lambda\delta(y_m, z_m). \tag{9}$$

This is the basis for the conical flow test of [3]. Note that it is essential in proving (9) that the lens be used to focus on the median plane. For any other plane (9) would no longer be true.

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