—NOTES—

ON SOME CHARACTERISTICS OF THE FREQUENCY EQUATION OF SMALL VIBRATIONS OF HOLONOMIC CONSERVATIVE SYSTEMS WITH STATIC COUPLINGS*

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The frequency equation of a holonomic system which performs vibrations about its configuration of stable equilibrium is

$$\Delta_n(\lambda) = | \mathbf{C} - \lambda \mathbf{A} | = 0, \tag{1}$$

where $\mathbf{A} = (a_{ik})$ the square inertia matrix, $\mathbf{C} = (c_{ik})$ the stiffness matrix, $\lambda = \omega^2$ the characteristic number.

When the system contains only static coupling, $a_{ik} = 0$ for $i \neq k$, and in the special case when the coefficients of matrices are $a_{ii} = a$, $c_{ii} = c$, Eq. (1) becomes

$$\Delta_n(\lambda) = | \lambda \mathbf{I} - \mathbf{P} | = 0, \tag{2}$$

where $\mathbf{P} = \mathbf{A}^{-1}\mathbf{C}$ is the dynamic matrix.

In the three characteristic cases of the torsional vibrations of light shafts with several disks, (Fig. 1), or of the linear oscillations of several coupled masses, (Fig. 2), the matrices $\mathbf{P}$ have the forms:

$$\mathbf{P}_a = \begin{bmatrix} p & -p & \cdot & 0 & 0 \\ -p & 2p & \cdot & 0 & 0 \\ 0 & -p & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & 2p & -p \\ 0 & 0 & \cdot & -p & p \end{bmatrix},$$

$$\mathbf{P}_b = \begin{bmatrix} 2p & -p & \cdot & 0 & 0 \\ -p & 2p & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & 2p & -p \\ 0 & 0 & \cdot & -p & 2p \end{bmatrix}, \quad \mathbf{P}_c = \begin{bmatrix} 2p & -p & \cdot & 0 & 0 \\ -p & 2p & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & 2p & -p \\ 0 & 0 & \cdot & -p & p \end{bmatrix} \tag{3}$$

and the polynomials in λ are

$$f(\lambda) = \lambda^n + B_{n-1}\lambda^{n-1} + \cdots + B_1\lambda + B_0 = 0. \tag{4}$$

As det $\mathbf{P} = 0$ the matrix $\mathbf{P}$ is singular, hence $\lambda = 0$ is not a root of the $f(\lambda) = 0$. This condition shows that the generalized momentum of the system is constant; hence

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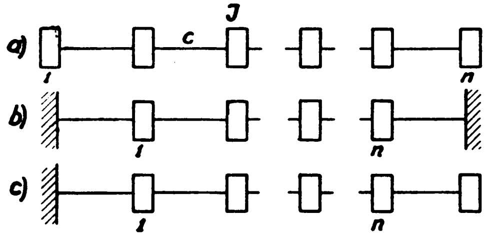

FIG. 1.

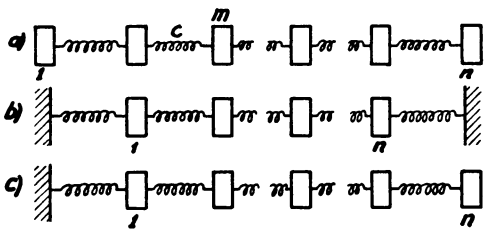

FIG. 2.

it has only $n - 1$ degrees of freedom. In the other two cases **P** is not singular and the systems have n degrees of freedom of vibration.

The Eq. (2) for these cases can be obtained by recurrence formulas:

$$\Delta_n = (\lambda - 2p)\Delta_{n-1} - p^2\Delta_{n-2} = 0 \tag{5}$$

and the determinants yield:

$$\Delta_n^{(a)} = \Delta_{n-1}^{(b)} = 0; \qquad \Delta_n^{(c)} = \Delta_n^{(b)} + p\Delta_{n-1}^{(b)} = \Delta_{n+1}^{(a)} + p\Delta_n^{(a)} = 0;$$
$$\Delta_n^{(c)} = \Delta_n^{(b)} + p\Delta_n^{(a)} = 0. \tag{6}$$

We therefore conclude that clamping of both ends (case b) replaces one disk (mass) of the first case; which is the basic one, since Eq. (4) of the other two cases may be deduced from the equation of the first case.

The coefficients B_r of (4) are connected by the relations

$$B_r^{(n)} = B_r^{(n-1)} - 2pB_{r-1}^{(n-1)} - p^2B_{r-2}^{(n-2)}, \qquad r = 0, 1, \cdots$$

They form a diagonal series of numbers whose differences are $\Delta^r = (-1)^r(2p)^r$, $\Delta^{r+1} = 0$, where r is the index of the row. In the first case (a) the index r should be replaced by $r - 1$. They also satisfy Newton's interpolation formula (for $h = 1$, $x_0 = r + 1$ or $x_0 = r$) and may be computed by the formula:

$$B_r = (1 + \Delta)^{n-r} = B_0^{(r)} + \binom{n-r}{1}\Delta B_0^{(r)} + \cdots + \binom{n-(r+1)}{r}\Delta^r B_0^{(r)}. \tag{7}$$

From (7) we obtain the polynomials (4) in the following forms:

a) $$\lambda^{n-1} - \binom{2n-2}{1}p\lambda^{n-2} + \cdots$$
$$\pm \binom{2n-(r+1)}{r}p^r\lambda^{n-(r+1)} + \cdots + (-1)^{n-1}np^{n-1} = 0,$$

(8)

c) $$\lambda^n - \binom{2n-1}{1}p\lambda^{n-1} + \cdots$$
$$\pm \binom{2n-r}{r}p^r\lambda^{n-r} + \cdots + (-1)^n p^n = 0.$$

Putting in the first equation, instead of the index n the index $n + 1$, one obtains the formula for the second case, [1].

Contrary to the method of finite-difference equations of second order, [2], the characteristic numbers in all three cases can be obtained this way. From Eq. (2), by the substitution $\lambda - 2p = \xi = (p^2/\eta) + \eta$, we obtain $\Delta_n - \eta\Delta_{n-1} = (p^2/\eta)^{n-1}$ because $\Delta_1 = \xi$, $\Delta_0 = 1$. Since $\eta^{-n}\Delta_n - \eta^{1-n}\Delta_{n-1} = q^{2n}$, $q = p/\eta$, summing from n to 1, while $\Delta_n = 0$, one obtains:

$$\eta^{-n}\Delta_n - 1 = q(q^n - 1)/(q - 1), \qquad \eta = p \exp [\nu\pi i/(n + 1)],$$

and the characteristic numbers are:

$$\lambda_\nu = 4p \cos^2 \frac{\nu\pi}{2n}; \qquad \lambda_\nu = 4p \cos^2 \frac{\nu\pi}{2(n+1)}; \qquad \lambda_\nu = 4p \cos^2 \frac{(2\nu - 1)\pi}{2(2n+1)} \tag{9}$$

$$\nu = 1, 2, \cdots, n - 1 \qquad \nu = 1, 2, \cdots, n \qquad \nu = 1, 2, \cdots, n$$

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n .$$

Taking into consideration the relations between the roots (9) and the coefficients of polynomials (4) we obtain several trigonometric formulas (Table 1).

TABLE 1

Case	a	b	c
Value of x	$\frac{\nu\pi}{2n}$; $m = n - 1$	$\frac{\nu\pi}{2(n+1)}$; $m = n$	$\frac{(2\nu - 1)\pi}{2(2n+1)}$; $m = n$
$\sum_{\nu=1}^{m} C_r^m \sin^2 x$	$\frac{1}{4^r}\binom{2n - (r+1)}{r}$	$\frac{1}{4^r}\binom{2n - (r-1)}{r}$	$\frac{1}{4^r}\binom{2n - r}{r}$
$\prod_{\nu=1}^{m} \sin^2 x$	$\frac{1}{4^{n-1}}\binom{n}{n-1} = \frac{n}{4^{n-1}}$	$\frac{1}{4^n}\binom{n+1}{n} = \frac{n+1}{4^n}$	$\frac{1}{4^n}\binom{n}{n} = \frac{1}{4^n}$
$\sum_{\nu=1}^{m} \cos 2x$	0	0	$\frac{1}{2}$

The symbol ΣC_r^m represents the sum of the combinations of the rth class with m members, whereas x are the special values of the number π.

These relations can be mathematically proved by means of the gamma-functions and the geometric series.

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