—NOTES—

NOTE ON A NON-HOLONOMIC SYSTEM*

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A mechanical system with the coordinates $q_1, q_2, \cdots q_n$ is called non-holonomic if there exist one or more non-integrable kinematical relations of the type

$$A_1\,dq_1 + A_2\,dq_2 + \cdots + A_n\,dq_n = 0, \tag{1}$$

where A_i is a function of $q_1, q_2, \cdots, q_n$. If the number of relations is m the system has $n - m$ degrees of freedom.

In every textbook on mechanics which deals with the matter examples of non-holonomic systems are given. We always meet the sphere moving on a rough plane: the conditions which express that the velocity of the point of contact is zero are non-holonomic ($n = 5, m = 2$). A more simple and very attractive example has been given by Carathéodory[1] in a paper on the sleigh: a line-segment AB is moving on a horizontal plane and one of its points C is subjected to the condition that its velocity has no components perpendicular to $AB (n = 3, m = 1)$. In all these cases the non-holonomic constraint is due to friction.

An example of a non-holonomic system of another type seems the following. Consider a (horizontal) disc D, which is able to rotate without friction about a vertical axis l intersecting D in O, and a particle P which moves without friction on the disc. If there are no external forces acting on the system its moment of momentum about l is constant. We add *the condition that this constant is zero*; the system is now seen to be a non-holonomic one ($n = 3, m = 1$).

If OA is a fixed line on the disc, OX a fixed horizontal line in space, $XOA = \varphi$, $OP = r$, $AOP = \psi$, then r, φ and ψ are coordinates of the system. I being the moment of inertia of D about the axis l, and m the mass of the particle P, the condition reads

$$(I + mr^2)\,d\varphi + mr^2\,d\psi = 0. \tag{2}$$

Obviously this relation is non-integrable. It can also be shown directly that there cannot exist a relation between the three coordinates. Suppose that according to a force acting between the particle and the disc, P describes a curve $r(t)$, $\psi(t)$ on D, starting from rest in $A(r_1, \psi_1)$ and moving to $B(r_2, \psi_2)$. Then during this relative motion the disc rotates about an angle

$$\varphi = -m \int_{t_1}^{t_2} \frac{r^2(t)}{I + mr^2(t)}\,\psi^{\cdot}(t)\,dt.$$

If B coincides with A this value is in general not zero: that means that the coordinate φ is not determined by r and ψ. We have a very simple case if mr^2 can be neglected with respect to I; then $\varphi = -m \int_{t_1}^{t_2} r^2\psi^{\cdot}\,dt = -2mF$, where F is the area enclosed by the curve which P has described on the disc. It is therefore possible to move P in such a way that to given values of r and ψ an arbitrary value of φ belongs. Suppose that the force acting between D and P depends only on r and ψ and that moreover this field

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[1]Carathéodory, *Der Schlitten*, Z. angew. Math. Mech. **13**, 71-76 (1933).

force is conservative, the potential function being $mV(r, \psi)$. Then the Lagrangian function is

$$L = T - V = \tfrac{1}{2}(I + mr^2)\varphi^{\cdot 2} + mr^2\varphi^{\cdot}\psi^{\cdot} + \tfrac{1}{2}mr^2\psi^{\cdot 2} + \tfrac{1}{2}mr^{\cdot 2} - mV(r, \psi).$$

The Lagrange equations of the second kind for φ, ψ and r are respectively

$$\frac{d}{dt}\{(I + mr^2)\varphi^{\cdot} + mr^2\psi^{\cdot}\} = \lambda(I + mr^2),$$

$$4r^{\cdot}\varphi^{\cdot} + 2r\varphi^{\cdot\cdot} + 2r^{\cdot}\psi^{\cdot} + r\psi^{\cdot\cdot} - \frac{1}{r}\frac{\partial V}{\partial \psi} = \lambda r,$$

$$r^{\cdot\cdot} - r\varphi^{\cdot 2} - 2r\varphi^{\cdot}\psi^{\cdot} - r\psi^{\cdot 2} + \frac{\partial V}{\partial r} = 0.$$

From the first of these equations it follows that the multiplier λ is 0. $\varphi^{\cdot}$ and $\varphi^{\cdot\cdot}$ can be eliminated and we have two equations for the relative motion r, ψ of the particle P on the disc. They are rather complicated. Of course we can take $T + V = c$ instead of one of them. If we have the case mentioned above, where mr^2 may be neglected when compared with I, the equations are

$$\varphi^{\cdot} = -\frac{m}{I}r^2\psi^{\cdot}, \qquad r^{\cdot\cdot} - r\psi^{\cdot 2} + \frac{\partial V}{\partial r} = 0, \qquad r^2\psi^{\cdot 2} + r^{\cdot 2} - 2V = c.$$

This means that the relative motion is the same as it would be if the disc were fixed. We give a simple example: $V = k^2/2(r^2 - 2ar\cos\psi)$, the force on P being an attracting force directed to the center $C(r = a, \psi = 0)$, and proportional to the distance PC. If $x = r\cos\psi - a, y = r\sin\psi$, we have $x^{\cdot\cdot} = -k^2x, y^{\cdot\cdot} = -k^2y, x = C_1\cos kt + C_2\sin kt$, $y = C_3\cos kt + C_4\sin kt$, where C_i are constants of integration. Therefore

$$r^2 = (C_1^2 + C_3^2)\cos^2 kt + (C_2^2 + C_4^2)\sin^2 kt + (C_1C_2 + C_3C_4)\sin 2kt$$
$$+ 2a(C_1\cos kt + C_2\sin kt) + a^2,$$

$$\psi = \arctan\frac{C_3\cos kt + C_4\sin kt}{C_1\cos kt + C_2\sin kt + a},$$

$$r^2\psi^{\cdot} = ay^{\cdot} + (xy^{\cdot} - yx^{\cdot}) = ay^{\cdot} + k(C_1C_4 - C_2C_3).$$

Hence

$$\varphi = -\frac{ma}{I}(C_3\cos kt + C_4\sin kt) - \frac{mk}{I}(C_1C_4 - C_2C_3)t + C_5\,.$$

ON LINEAR INSTABILITY*

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1. Let the coefficient function of the linear differential equation

$$x'' + f(t)x = 0 \tag{1}$$

be real-valued and continuous for large positive t. Consider only those solutions $x(t)$ of (1) which are real-valued and distinct from the trivial solution ($\equiv 0$). Then, since

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