TENSORS ASSOCIATED WITH TIME-DEPENDENT STRESS*

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Abstract. It is assumed that six functional relations exist between the components of stress and their first m material time derivatives and the gradients of displacement, velocity, acceleration, second acceleration, $\cdots$, $(n - 1)$th acceleration. It is shown that these relations may then be expressed as relations between the components of $m + n + 2$ symmetric tensors if $n > m$, and $2m + 2$ symmetric tensors if $m > n$. Expressions for these tensors are obtained.

1. Introduction. It has been shown by Rivlin and Ericksen [1]† that if we assume that the components of stress t_{ij} , in a rectangular Cartesian coordinate system x_i , at any point of a body of isotropic material undergoing deformation, are single-valued functions of the gradients of displacement, velocity, acceleration, $\cdots$, $(n - 1)$th acceleration in the coordinate system x_i at the point of the body considered, then the stress components t_{ij} may be expressed as functions of the components of $(n + 1)$ symmetric tensors defined in terms of these gradients.

We describe the deformation by

$$x_i = x_i(X_j , t), \tag{1.1}$$

where x_i denotes the position, at time t, in the coordinate system x_i , of a material particle which was located at X_i in the same coordinate system at some other instant of time T. Let $v_i^{(1)}, v_i^{(2)}, v_i^{(3)}, \cdots , v_i^{(n)}$ denote the components of velocity, acceleration, second acceleration, $\cdots$, $(n - 1)$th acceleration at time t, in the coordinate system x_i , of a material particle located at x_i . Then, if we assume

$$t_{ij} = t_{ij}\left(\frac{\partial x_p}{\partial X_q}, \frac{\partial v_p^{(1)}}{\partial x_q}, \frac{\partial v_p^{(2)}}{\partial x_q}, \cdots , \frac{\partial v_p^{(n)}}{\partial x_q}\right), \tag{1.2}$$

it follows [1, Sec. 15] that t_{ij} may be expressed as single-valued functions of the components C_{ij} , $A_{ij}^{(r)} (r = 1, 2, \cdots , n)$ of $(n + 1)$ tensors defined by

$$C_{ij} = \frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_k}, \qquad A_{ij}^{(1)} = \frac{\partial v_i^{(1)}}{\partial x_j} + \frac{\partial v_j^{(1)}}{\partial x_i}$$

and

$$A_{ij}^{(r+1)} = \frac{DA_{ij}^{(r)}}{Dt} + A_{mj}^{(r)}\frac{\partial v_m^{(1)}}{\partial x_i} + A_{im}^{(r)}\frac{\partial v_m^{(1)}}{\partial x_j}, \tag{1.3}$$

where D/Dt denotes the material time derivative. This result was obtained from the consideration that the form of the dependence of the stress components t_{ij} on the gradients of the displacement, velocity, acceleration, $\cdots$, $(n - 1)$th acceleration must be independent of the particular choice of the rectangular Cartesian coordinate system x_i .

It will be shown in the present paper, from similar considerations, that if, instead

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*Received Aug. 6, 1954. The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35801 between Brown University and the Office of Naval Research.

†Numbers in square brackets refer to bibliography at the end of the paper.

of describing the dependence of the stress components on the deformation by six functional relations of the type (1.2), we have six independent functional relations of the form

$$f_{ij}\left(\frac{\partial x_p}{\partial X_q}, \frac{\partial v_p^{(1)}}{\partial x_q}, \frac{\partial v_p^{(2)}}{\partial x_q}, \cdots , \frac{\partial v_p^{(n)}}{\partial x_q}, t_{pq} , \frac{Dt_{pq}}{Dt}, \frac{D^2 t_{pq}}{Dt^2}, \cdots , \frac{D^m t_{pq}}{Dt^m}\right) = 0, \tag{1.4}$$

with

$$f_{ij} = f_{ji} , \tag{1.5}$$

then these functional relations must be expressible in the form

$$F_{ij}(C_{pq} , A_{pq}^{(1)}, A_{pq}^{(2)}, \cdots , A_{pq}^{(n)}, t_{pq} , B_{pq}^{(1)}, B_{pq}^{(2)}, \cdots , B_{pq}^{(m)}) = 0 \tag{1.6}$$

if $n > m$, and in the form

$$F_{ij}(C_{pq} , A_{pq}^{(1)}, A_{pq}^{(2)}, \cdots , A_{pq}^{(m)}, t_{pq} , B_{pq}^{(1)}, B_{pq}^{(2)}, \cdots , B_{pq}^{(m)}) = 0, \tag{1.7}$$

if $n < m$, where $F_{ij} = F_{ji}$ in both cases, C_{pq} and $A_{pq}^{(r)} (r = 1, 2, \cdots , n)$ are defined by (1.3) and $B_{pq}^{(r)} (r = 1, 2, \cdots , m)$ are the components of symmetric tensors, defined by

$$B_{ij}^{(r)} = \frac{DB_{ij}^{(r-1)}}{Dt} + B_{lj}^{(r-1)} \frac{\partial v_l^{(1)}}{\partial x_i} + B_{il}^{(r-1)} \frac{\partial v_l^{(1)}}{\partial x_j}$$

and

$$B_{ij}^{(0)} = t_{ij} . \tag{1.8}$$

Zaremba [2] introduced a rate of change of stress tensor, which is given in terms of the tensors t_{ij} , $A_{ij}^{(1)}$ and $B_{ij}^{(1)}$ by $B_{ij}^{(1)} - \frac{1}{2}t_{kj}A_{ki}^{(1)} - \frac{1}{2}t_{ik}A_{kj}^{(1)}$.

It may be remarked that Eqs. (1.4) and (1.5) are not, in general, sufficient for the determination of the stress resulting from the subjection of the material to a specified deformation history. They may be regarded as a set of six independent differential equations in six dependent variables $t_{ij} (t_{ij} = t_{ji})$ and one independent variable t. Suitable "initial" conditions at specified values of t must be chosen if the equations are to have a solution. However, we are concerned here only with the limitations which must exist on the form of the relations (1.4), as a result of the necessity that they are invariant under a transformation from one orthogonal coordinate system to another, quite apart from any question of the sufficiency of the equations for the determination of the stress components.

2. The deformation tensors. It is well known that if ds is the distance at time t between two material particles of a body, undergoing a deformation described by (1.1), which are located at x_i and $x_i + dx_i$ in the rectangular Cartesian coordinate system x_i , then

$$(ds)^2 = dx_k \, dx_k \tag{2.1}$$

$$= \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} dX_i \, dX_j , \tag{2.2}$$

where X_i and $X_i + dX_i$ are the positions of the particles at a previous instant of time T. Differentiating $(ds)^2$ r times with respect to t, we have, from (2.1),

$$\frac{D^r(ds)^2}{Dt^r} = A_{ij}^{(r)} \, dx_i \, dx_j , \tag{2.3}$$

where $A_{ij}^{(r)}$ is a symmetric tensor given by (1.3). A corresponding result in a convected coordinate system was obtained by Oldroyd [3].

Equations (2.2) and (2.3), with the left-hand sides given constant values, describe the deformation quadrics at the point of the body considered.

It has been seen [1, Sec. 10] that $A_{ij}^{(r)}$ may also be expressed as

$$A_{ij}^{(r)} = \frac{\partial v_i^{(r)}}{\partial x_j} + \frac{\partial v_j^{(r)}}{\partial x_i} + \sum_{p=1}^{r-1} \binom{r}{p} \frac{\partial v_k^{(r-p)}}{\partial x_i} \frac{\partial v_k^{(p)}}{\partial x_j}. \tag{2.4}$$

3. The stress tensors. If we define a quantity $\mathcal{B}$ by

$$\mathcal{B} = t_{ij}\, dx_i dx_j\ , \tag{3.1}$$

then, since t_{ij} transforms as a tensor from the rectangular Cartesian coordinate system x_i to any other, irrespective of any relative motion of the two coordinate systems, it is apparent that $\mathcal{B}$ is a scalar, invariant under such transformations of the reference system. From (3.1), we have

$$\begin{aligned} \frac{D^s\mathcal{B}}{Dt^s} = \frac{D^s(t_{ij}\, dx_i\, dx_j)}{Dt^s} &= \sum_{q=0}^{s} \binom{s}{q} \frac{D^{s-q}t_{ij}}{Dt^{s-q}} \frac{D^q(dx_i\, dx_j)}{Dt^q} \\ &= \sum_{q=0}^{s} \left[\binom{s}{q} \frac{D^{s-q}t_{ij}}{Dt^{s-q}} \sum_{l=0}^{q} \binom{q}{l} \frac{D^{q-l}(dx_i)}{Dt^{q-l}} \frac{D^l(dx_j)}{Dt^l} \right]. \end{aligned} \tag{3.2}$$

Since

$$\frac{D^l(dx_j)}{Dt^l} = dv_j^{(l)} = \frac{\partial v_j^{(l)}}{\partial x_g} dx_g$$

and

$$\frac{D^{q-l}(dx_i)}{Dt^{q-l}} = dv_i^{(q-l)} = \frac{\partial v_i^{(q-l)}}{\partial x_h} dx_h\ , \tag{3.3}$$

where $v_i^{(0)} = x_i$, so that $\partial v_i^{(0)}/\partial x_j = \delta_{ij}$, we obtain from (3.2),

$$\frac{D^s\mathcal{B}}{Dt^s} = \sum_{q=0}^{s} \left[\binom{s}{q} \frac{D^{s-q}t_{ij}}{Dt^{s-q}} \sum_{l=0}^{q} \binom{q}{l} \frac{\partial v_i^{(q-l)}}{\partial x_h} \frac{\partial v_j^{(l)}}{\partial x_g} dx_g\, dx_h \right]. \tag{3.4}$$

With the definition

$$B_{ij}^{(s)} = \sum_{q=0}^{s} \left[\binom{s}{q} \frac{D^{s-q}t_{gh}}{Dt^{s-q}} \sum_{l=0}^{q} \binom{q}{l} \frac{\partial v_g^{(q-l)}}{\partial x_j} \frac{\partial v_h^{(l)}}{\partial x_i} \right], \tag{3.5}$$

we can re-write (3.4) as

$$\frac{D^s\mathcal{B}}{Dt^s} = B_{ij}^{(s)}\, dx_i\, dx_j\ , \tag{3.6}$$

with $B_{ij}^{(s)} = B_{ji}^{(s)}$. Since $D^s\mathcal{B}/Dt^s$ transforms as a scalar, between two rectangular Cartesian coordinate systems with arbitrary relative motion, $B_{ij}^{(s)}$ transforms as a tensor between such coordinate systems.

We note, from (3.6) and (3.3), that

$$\begin{aligned} B_{ij}^{(s+1)}\, dx_i\, dx_j &= \frac{D^{s+1}\mathcal{B}}{Dt^{s+1}} = \frac{D}{Dt}(B_{ij}^{(s)}\, dx_i\, dx_j) \\ &= \left(\frac{DB_{ij}^{(s)}}{Dt} + B_{lj}^{(s)} \frac{\partial v_l^{(1)}}{\partial x_i} + B_{il}^{(s)} \frac{\partial v_l^{(1)}}{\partial x_j} \right) dx_i\, dx_j\ . \end{aligned} \tag{3.7}$$

Whence,

$$B_{ij}^{(s+1)} = \frac{DB_{ij}^{(s)}}{Dt} + B_{lj}^{(s)} \frac{\partial v_l^{(1)}}{\partial x_i} + B_{il}^{(s)} \frac{\partial v_l^{(1)}}{\partial x_j}. \tag{3.8}$$

4. The stress-deformation relations. We assume that the dependence of the stress components on the deformation is described by the six functional relations (1.4) and the forms of the functions f_{ij} are independent of the rectangular Cartesian coordinate system in which Eqs. (1.4) are expressed. Let x_i^* be a rectangular Cartesian coordinate system moving in an arbitrary manner with respect to x_i and related to x_i by

$$x_i^* = a_{ij}(x_j - b_j) \qquad a_{ij}a_{ik} = \delta_{jk} , \tag{4.1}$$

where a_{ij} and b_j are, in general, functions of time. Let X_i^* denote the coordinates in the system x_i^* of a point located at X_i in the coordinate system x_i and let $v_i^{*(1)}$, $v_i^{*(2)}$, $\cdots$, $v_i^{*(n)}$ be the components of the velocity, acceleration, $\cdots$, $(n - 1)$th acceleration respectively in the coordinate system x_i^* . Then, if t_{ij}^* are the components of the stress in the coordinate system x_i^* , we have

$$f_{ij}\left(\frac{\partial x_p^*}{\partial X_q^*}, \frac{\partial v_p^{*(1)}}{\partial x_q^*}, \frac{\partial v_p^{*(2)}}{\partial x_q^*}, \cdots , \frac{\partial v_p^{*(n)}}{\partial x_q^*}, t_{pq}^* , \frac{Dt_{pq}^*}{Dt}, \cdots , \frac{D^m t_{pq}^*}{Dt^m}\right) = 0. \tag{4.2}$$

It has been shown in a previous paper [1, Secs. 5 and 15] that we can choose the coordinate system x_i^* in such a way that:

(i) instantaneously at time t, the directions of the axes of x_i^* are parallel to those of x_i , so that

$$a_{ij} = \delta_{ij} ; \tag{4.3}$$

(ii) instantaneously at time t, the angular velocity, angular acceleration, angular second acceleration, $\cdots$, angular $(n - 1)$th acceleration of the coordinate system x_i^* relative to x_i are such that the velocity, acceleration, $\cdots$, $(n - 1)$th acceleration fields relative to the coordinate system x_i^* , in the immediate neighborhood of the material particle considered, are irrotational;

(iii) instantaneously at time T, the axes of the coordinate system x_i^* have directions relative to the coordinate system x_i defined by

$$a_{ij} = \frac{\partial x_k}{\partial X_j} (\mathbf{c}^{-1})_{ik} , \tag{4.4}$$

where

$$\mathbf{c}^2 = \left\| \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k} \right\| \tag{4.5}$$

and $\mathbf{c}(= \| c_{ij} \|)$ is the matrix satisfying this equation which has all its eigenvalues positive.

The choice of the coordinate system x_i^* in accordance with the condition (ii) implies that, at time t,

$$\partial v_i^{*(r)}/\partial x_j^* = \partial v_j^{*(r)}/\partial x_i^* \qquad (r = 1, 2, \cdots , n). \tag{4.6}$$

Also, the choice of the coordinate system x_i^* in accordance with conditions (i) and (iii) implies that, at time t,

$$\partial x_i^*/\partial X_j^* = c_{ij} . \tag{4.7}$$

With the notation

$$d_{ij}^{*(r)} = \frac{1}{2}\left(\frac{\partial v_i^{*(r)}}{\partial x_j^*} + \frac{\partial v_j^{*(r)}}{\partial x_i^*}\right), \tag{4.8}$$

it follows from (4.6) and (4.7) that Eq. (4.2) can be re-written as

$$f_{ij}\left(c_{pq}, d_{pq}^{*(1)}, d_{pq}^{*(2)}, \cdots, d_{pq}^{*(n)}, t_{pq}^*, \frac{Dt_{pq}^*}{Dt}, \cdots, \frac{D^m t_{pq}^*}{Dt^m}\right) = 0 \tag{4.9}$$

at time t.

From (3.5), (4.6), and (4.8), bearing in mind that $v_i^{*(0)} = x_i^*$ and $t_{ij}^* = t_{ji}^*$, we see that if $B_{ij}^{*(s)}$ are the components of the tensor $B_{ij}^{(s)}$ in the coordinate system x_i^*,

$$\begin{aligned} B_{ij}^{*(s)} &= \sum_{q=0}^{s}\left[\binom{s}{q}\frac{D^{s-q}t_{gh}^*}{Dt^{s-q}}\sum_{l=0}^{q}\binom{q}{l}\frac{\partial v_g^{*(q-l)}}{\partial x_i^*}\frac{\partial v_h^{*(l)}}{\partial x_j^*}\right] \\ &= \frac{D^s t_{ij}^*}{Dt^s} + \sum_{q=1}^{s}\left[\binom{s}{q}\frac{D^{s-q}t_{gh}^*}{Dt^{s-q}}\sum_{l=0}^{q}\binom{q}{l}d_{gi}^{*(q-l)}d_{hi}^{*(l)}\right]. \end{aligned} \tag{4.10}$$

Also, at time t, from (2.4), (4.6) and (4.8), we see that if $A_{ij}^{*(s)}$ are the components of the tensor $A_{ij}^{(s)}$ in the coordinate system x_i^*, then at time t,

$$A_{ij}^{*(r)} = 2d_{ij}^{*(r)} + \sum_{p=1}^{r-1}\binom{r}{p}d_{ki}^{*(r-p)}d_{kj}^{*(p)}. \tag{4.11}$$

Since,

$$A_{ij}^{*(1)} = 2d_{ij}^{*(1)}, \tag{4.12}$$

we see, from (4.11), that $d_{ij}^{*(r)}$ can be expressed as a polynomial in the quantities $A_{pq}^{*(1)}$, $A_{pq}^{*(2)}, \cdots, A_{pq}^{*(r)}$. We also see, from (4.10), that $D^s t_{ij}^*/Dt^s$ can be expressed as a polynomial in the quantities $A_{pq}^{*(1)}, A_{pq}^{*(2)}, \cdots, A_{pq}^{*(s)}, t_{pq}^*, B_{pq}^{*(1)}, B_{pq}^{*(2)}, \cdots, B_{pq}^{*(s)}$. Consequently, Eqs. (4.9) may be re-written in the form

$$\varphi_{ij}(c_{pq}, A_{pq}^{*(1)}, A_{pq}^{*(2)}, \cdots, A_{pq}^{*(n)}, t_{pq}^*, B_{pq}^{*(1)}, B_{pq}^{*(2)}, \cdots, B_{pq}^{*(m)}) = 0, \tag{4.13}$$

if $n > m$ and in the form

$$\varphi_{ij}(c_{pq}, A_{pq}^{*(1)}, A_{pq}^{*(2)}, \cdots, A_{pq}^{*(m)}, t_{pq}^*, B_{pq}^{*(1)}, B_{pq}^{*(2)}, \cdots, B_{pq}^{*(m)}) = 0, \tag{4.14}$$

if $m > n$. It may be noted that if, in (4.9), f_{ij} is a polynomial function of c_{pq}, $d_{pq}^{*(1)}$, $\cdots$, $D^m t_{pq}^*/Dt^m$, then in (4.13) and (4.14), φ_{ij} is a polynomial in the dependent variables.

Since $B_{ij}^{(s)}$ and $B_{ij}^{*(s)}$ are the components of the same tensor in the coordinate systems x_i and x_i^* respectively, we have

$$B_{pq}^{*(s)} = B_{ij}^{(s)}a_{pi}a_{qj} \qquad \text{and} \qquad t_{pq}^* = t_{ij}a_{pi}a_{qj}. \tag{4.15}$$

Since $A_{ij}^{(r)}$ and $A_{ij}^{*(r)}$ are the components of the same tensor in the coordinate systems x_i and x_i^* respectively, we have

$$A_{pq}^{*(r)} = A_{ij}^{(r)}a_{pi}a_{qj}. \tag{4.16}$$

Since the coordinate system is chosen in accordance with condition (i) we see that, at the instant of time t, a_{ij} is given by (4.3) and Eqs. (4.15) and (4.16) yield

$$B_{pq}^{*(s)} = B_{pq}^{(s)} \qquad \text{and} \qquad A_{pq}^{*(r)} = A_{pq}^{(r)}. \tag{4.17}$$

Introducing the results (4.17) into Eqs. (4.13) and (4.14), we have

$$\varphi_{ij}(c_{pq}, A_{pq}^{(1)}, A_{pq}^{(2)}, \cdots, A_{pq}^{(n)}, t_{pq}, B_{pq}^{(1)}, B_{pq}^{(2)}, \cdots, B_{pq}^{(m)}) = 0, \tag{4.18}$$

if $n > m$ and

$$\varphi_{ij}(c_{pq}, A_{pq}^{(1)}, A_{pq}^{(2)}, \cdots, A_{pq}^{(m)}, t_{pq}, B_{pq}^{(1)}, B_{pq}^{(2)}, \cdots, B_{pq}^{(m)}) = 0, \tag{4.19}$$

if $m > n$.

Following the method adopted by Rivlin and Ericksen [1, Secs. 7 and 15] and employing the notation

$$C_{ij} = c_{ik}c_{kj} = \frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_k}, \tag{4.20}$$

the relations (4.18) and (4.19) may be written as

$$F_{ij}(C_{pq}, A_{pq}^{(1)}, A_{pq}^{(2)}, \cdots, A_{pq}^{(n)}, t_{pq}, B_{pq}^{(1)}, B_{pq}^{(2)}, \cdots, B_{pq}^{(m)}) = 0, \tag{4.21}$$

if $n > m$ and

$$F_{ij}(C_{pq}, A_{pq}^{(1)}, A_{pq}^{(2)}, \cdots, A_{pq}^{(m)}, t_{pq}, B_{pq}^{(1)}, B_{pq}^{(2)}, \cdots, B_{pq}^{(m)}) = 0, \tag{4.22}$$

if $m > n$, where F_{ij} is a single-valued function of the independent variables.

If we assume in Eq. (1.4) that the functions f_{ij} are polynomials in the variables, then it follows from the manner in which Eqs. (4.21) and (4.22) are derived that F_{ij} are polynomials in the variables. It is also readily seen that if we assume the functions f_{ij} are single-valued functions of $\partial v_p^{(1)}/\partial x_q$, $\partial v_p^{(2)}/\partial x_q$, $\cdots$, $\partial v_p^{(n)}/\partial x_q$, t_{pq}, Dt_{pq}/Dt, D^2t_{pq}/Dt^2, $\cdots$, $D^m t_{pq}/Dt^m$ then they may be expressed in the form (4.21) or (4.22) with C_{pq} omitted.

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