SEICHE IN RECTANGULAR PORTS*

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Introduction. In an extension of a recent publication by one of the writers [1],† the excitation by an external wave of the water within a harbor is studied for the case of a harbor which is rectangular in plan and for which the walls are vertical and totally reflecting and the depth is constant. The pattern of seiche, or mass oscillation, within the rectangular basin is predictable for a given position and size of the entrance and for known characteristics of the external wave. The result is useful in the interpretation of occurrences both in actual harbors and in laboratory models.

The goal of the analysis is the derivation in series form of a velocity potential from which the periodic velocities and surface displacements can be calculated, the period of the motion being that of the exciting wave. A special boundary condition for the entrance must be devised from observations in the laboratory. A solution to the Laplace equation satisfying the mixed boundary condition is obtained from an expansion in terms of eigen-functions and auxiliary functions, as outlined previously in note form [2]. Solutions for two distinct cases must be obtained depending upon whether or not the imposed period coincides with that for one of the eigen-functions; these are designated as resonant and non-resonant, respectively. Similar results were found in [1] for a harbor of circular shape. From that study, a few elements essential to the following development are repeated herein. The authors wish to acknowledge the useful suggestions during the preparation of the manuscript made by Professors C. S. Yih and E. N. Oberg of the State University of Iowa, Professor J. Kuntzman of the University of Grenoble, and Mr. A. Apté of the Poona Laboratory in India, at present a graduate student at the University of Grenoble.

Statement of the problem. Cartesian coordinates are utilized, with the origin in the undisturbed free surface and the z-axis directed vertically upward. The depth of water is h and the dimensions of the harbor are a and b, so that the undisturbed liquid occupies the space $-h \leq z \leq 0$; $0 \leq x \leq a$; and $0 \leq y \leq b$. The opening or entrance to the harbor, which extends over the entire depth h in one of the walls of length b, is defined by $-h \leq z$; $0 < \alpha \leq x \leq \beta < a$; $y = 0$. The free surface is denoted for convenience as D and its periphery as Γ.

The periodic wave causing the mass oscillation within the harbor is taken to be a plane Stokian wave of period T; it approaches the harbor over a horizontal bottom which is at the same elevation ($z = -h$) as that of the harbor. The mechanism whereby the excitation within the harbor is caused was described in the first paragraph of [1]. To complete the statement of the problem, the velocity component $V_n(x, z, t)$ normal to the plane of the entrance must be defined, t denoting time as usual. The variation of V_n can be determined only from experiment.

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†References indicated by numbers in brackets are collected at the end of the paper.

Fortunately, for the occurrence being studied, the variation of V_n can be closely approximated from the assumption that a clapotis, or standing wave, forms in the immediate vicinity of the entrance (see Chap. III of [1]), the amplitude being twice that of the generating wave. Hence, from the classical theory of Stokes,

$$V_n = \frac{A V_0}{k} \sin \frac{2\pi t}{T} \frac{\cosh [k(z + h)]}{\cosh (kh)} , \tag{1}$$

where k is the reciprocal of a reference length, V_0 is the amplitude of the velocity variation, and A is a constant which depends upon the phase of the clapotis at the entrance. It is evident that the velocity is discontinuous at either end of the entrance.

Within the harbor the motion is restricted to sinusoidal oscillations of the same period as the external wave and is assumed to be irrotational. The liquid is, of course, taken to be incompressible. Definition of a velocity potential $\varphi(x, y, z, t)$ is the goal of the analysis. As shown in [1],

$$\varphi = (V_n/A)F(x, y) , \tag{2}$$

where $F(x, y)$ is an analytic function appropriate for the harbor plan in question. As $\nabla^2\varphi = 0$,

$$\nabla^2 F + k^2 F = 0. \tag{3}$$

The expression for φ has been chosen so that

$$\left.\frac{\partial\varphi}{\partial n}\right|_{z=-h} = 0$$

is satisfied; the condition of Poisson for the free surface can be expressed in the form,

$$4\pi^2/T^2 = gk \tanh (kh). \tag{4}$$

The boundary conditions must include the statement that the velocity normal to the solid walls is everywhere zero, i.e., $dF/dn = 0$ along the intersection of the plane of the free surface and the solid walls. The only boundary condition remaining is that for φ in the plane of the entrance.

To be rigorous, continuity of the two velocity fields immediately inside and immediately outside the entrance would be required, although this stringent requirement is not entirely essential from the physical point of view. Hence only the necessary condition that the velocity potential give the same normal velocity as defined by (1) is imposed. From (2), this condition becomes

$$dF/dn = A, \qquad (y = 0, \alpha \leq x \leq \beta), \tag{5}$$

where n is the inward-drawn normal. The problem is thus the definition of F in such a manner as to satisfy (3) and the conditions imposed.

It is essential to demonstrate that the boundary-value problem is properly posed and to determine whether a solution is possible. To this end, it is noted that the value of k is known from (4) if values are assigned to the depth h and the period T of the oncoming wave. The quantities $k_{n,m}$ and $\Phi_{n,m}$ ($n, m = 0, 1, \cdots, \infty$), which are, respectively, the eigen-values and eigen-functions of (3) for the free surface with the boundary condition $dF/dn = 0$ around the edge Γ, are essential to the development. It follows that

$$\nabla^2\Phi_{n,m} + k^2_{n,m}\Phi_{n,m} = 0, \tag{6}$$

and $d\Phi_{n,m}/dn_1 = 0$ along Γ. These values and functions are as follows:

$$k_{n,m}^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right),$$

$$\Phi_{0,0} = \frac{1}{(ab)^{1/2}}, \qquad \Phi_{n,m} = \frac{2}{(ab)^{1/2}} \cos\frac{n\pi x}{a} \cos\frac{m\pi y}{b}, \tag{7}$$

$$\Phi_{n,0} = \left(\frac{2}{ab}\right)^{1/2} \cos\frac{n\pi x}{a}, \qquad \Phi_{0,m} = \left(\frac{2}{ab}\right)^{1/2} \cos\frac{m\pi y}{b}.$$

If $d\sigma$ is an element of D,

$$\iint_D \Phi_i\Phi_j \, d\sigma = \delta_{ij} \tag{8}$$

in which the double subscript n, m has been replaced for convenience by i, and δ_{ij} is the Kronecker delta. The uniqueness of this result is established in classic reference works such as [3].

Were it not for the singularities at either edge of the entrance, the existence of a solution could be readily demonstrated. For a simple approximation to reality, however, V_n was assumed to be independent of y (and hence discontinuous), and for this case the classical results must be modified. To this end an arbitrary function $f(x, y)$ must be selected which is finite and continuous as are also its first and second derivatives throughout D except for the boundary Γ. It is then possible to write

$$F(x, y) = f(x, y) + u(x, y). \tag{9}$$

From (3), u is a solution of the non-homogeneous equation,

$$\nabla^2 u + k^2 u = -(\nabla^2 f + k^2 f) = f_1(x, y), \tag{10}$$

the second part of which is known, subject to the condition that $du/dn = 0$ on Γ. It is now possible to develop u in an absolutely and uniformly convergent series in Φ_i over the entire surface:

$$u(x, y) = \sum_i^\infty c_i\Phi_i(x, y), \tag{11}$$

where c_i are constants in a Fourier series. As u is an analytic function, the series (11) can be expressed in a doubly infinite series (a point which is to be verified *a posteriori*). Combining (10) and (11), one obtains

$$\sum_i^\infty (k^2 - k_i^2)c_i\Phi_i = f_1$$

and, with reference to (8),

$$c_i = \frac{\iint_D f_1\Phi_i \, d\sigma}{k^2 - k_i^2} \tag{12}$$

which is useful if the function f_1 is integrable in the domain D. Thus the solution is seen to lead to a unique result.

Definition of auxiliary function. The relations (9), (11) and (12) resolve completely the problem as posed for the non-resonant oscillation in Chap. I of [1] for which $k^2 \neq k_i^2$, with the reservation that $f_1(x, y)$ must be defined and the differentiation of (11) term by term must be justified. For this problem, the development from (11) and (12), although absolutely convergent throughout the domain, does not have all of the properties of regularity. The validity of the formal solution is studied in a subsequent paragraph.

The definition of the function f_1 is the next step in the analysis. For a rectangular harbor, a complex analytic function λ is defined:

$$\lambda(z) = \frac{A}{\pi}[(z - \alpha)\ln(z - \alpha) - (z - \beta)\ln(z - \beta)] = \psi_1(x, y) + i\psi_2(x, y), \tag{13}$$

where $z = x + iy$. The logarithms are evaluated in such a way as to be real if z is real and greater than β. It is evident that $\lambda(z)$ is analytic throughout the domain except for the points $z = \alpha$ and $z = \beta$. If the polar notation

$$\rho_1 = |z - \alpha|, \qquad \rho_2 = |z - \beta|,$$
$$\theta_1 = \arg(z - \alpha), \qquad \theta_2 = \arg(z - \beta),$$

is used,

$$\frac{d\lambda}{dz} = \frac{\partial\psi_2}{\partial y} - i\frac{\partial\psi_1}{\partial y} = \frac{A}{\pi}\ln\frac{\rho_1}{\rho_2} + \frac{iA}{\pi}(\theta_1 - \theta_2)$$

and hence

$$\frac{\partial\psi_1}{\partial y} = -\frac{A}{\pi}(\theta_1 - \theta_2).$$

For the portion of r defined by $y = 0$, the harmonic function $\psi_1(x, y)$ satisfies the following conditions:

$$\partial\psi_1/\partial n = 0, \qquad (0 \le x \le \alpha,\ \beta \le x \le a)$$
$$\partial\psi_1/\partial n = A, \qquad (\alpha \le x \le \beta)$$

and is regular for the remainder of Γ.

Another regular function $\psi(x, y)$ is now sought such that $d\psi/dn = 0$ on the boundary $y = 0$ and $d\psi/dn = d\psi_1/dn$ on the other three sides. One can take

$$f = \psi_1 - \psi, \tag{14}$$

where, from (13),

$$\psi_1 = \frac{A}{\pi}[(x - \alpha)\ln\rho_1 - (x - \beta)\ln\rho_2 + y(\theta_2 - \theta_1)]. \tag{15}$$

It is evidently possible to form an infinite number of functions ψ which satisfy the required conditions; the one chosen herein has certain advantages for numerical calculations. For convenience the analytic functions

$$p(x) = \left.\frac{\partial\psi_1}{\partial y}\right|_{y=b} = -\frac{d\psi_1}{dn} \qquad 0 \le x \le a,$$

$$\left.\begin{aligned} q(x) &= \left.\frac{\partial\psi_1}{\partial x}\right|_{x=0} = \frac{\partial\psi_1}{dn} \\ r(x) &= \left.\frac{\partial\psi_1}{\partial x}\right|_{x=a} = -\frac{d\psi_1}{dn} \end{aligned}\right\} \qquad (0 \le y \le b)$$

are defined. Now, with reference to (14), it can be verified that the function,

$$\psi(x, y) = p(x)\frac{y^2}{2b} + q(y)\left(x - \frac{x^2}{2a}\right) + r(y)\frac{x^2}{2a} - \left\{p'(0)x + [p'(a) - p'(0)]\frac{x^2}{2a}\right\}\frac{y^2}{2b} \qquad (16)$$

can be chosen. As the order of differentiation is immaterial, $q'(0) = r'(0) = 0$, $q'(b) = p'(0)$, $r'(b) = p'(a)$.

From (14), (15) and (16), f can be determined, and from (10),

$$f_1(x, y) = \nabla^2\psi + k^2\psi - k^2\psi_1 . \qquad (17)$$

The quantity $\nabla^2\psi + k^2\psi$ is analytic and regular throughout D and on Γ. The function ψ_1 is harmonic and regular except for $z = \alpha$ and $z = \beta$, at which points it is continuous but presents singularities of the type

$$\rho_1 \cos\theta_1 \ln\rho_1 - y\theta_1 , \qquad \rho_2 \cos\theta_2 \ln\rho_2 - y\theta_2 .$$

In the vicinity of $z = \alpha$, for example,

$$\frac{\partial\psi_1}{\partial x} = \frac{A}{\pi}\ln\rho_1 + H, \qquad \frac{\partial\psi_1}{\partial y} = -\frac{A}{\pi}\theta_1 ,$$

in which H is an analytic function.

In the course of the experiments in the Neyrpic Hydraulics Laboratory, vertical vortices were observed to form periodically at either end of the entrance (Chap. V of [1]). These are the physical counterpart of the mathematical singularities already discussed.

The development of a Fourier series, which is convergent in the domain D, can now be written:

$$f_1(x, y) = \sum_0^\infty\left(\sum_0^\infty \gamma_{n,m}\cos\frac{m\pi y}{b}\right)\cos\frac{n\pi x}{a} , \qquad (18)$$

where

$$\gamma_{n,m} = \iint_D f_1\Phi_{n,m}\, d\sigma. \qquad (19)$$

It can be shown that

$$|\gamma_{n,m}| \le \frac{C}{nm} ,$$

in which C is a constant. From (7) and (12)

$$|C_{n,m}| \le \frac{C_1}{nm(n^2 + m^2)} ,$$

in which C_1 is a finite number, so that the convergence of (11) is absolute and uniform. The development (11) is differentiable term by term, once in x and once in y; the resulting series are still absolutely and uniformly convergent in D and Γ, but the convergence of the quantities $\partial^2u/\partial x^2$ and $\partial^2u/\partial y^2$ is not known to be absolute.

Resonant motion. From an inspection of (12), the method of the preceding paragraph is not valid for the case designated as resonant, for which $k^2 = k_i^2$. It is evident that there can be added to the solution for $u(x, y)$ an unknown quantity of the form $B\Phi_{n,m}(x, y)$ in which B is an arbitrary constant.

If ds is an element of Γ, from (6) and (10) and from Green's theorem, which is applicable because of the assumed regularity of u,

$$\iint_D (\Phi_{n,m}\nabla^2 u - u\nabla^2\Phi_{n,m})\, d\sigma = \int_\Gamma \left(\Phi_{n,m}\frac{du}{dn} - u\frac{d\Phi_{n,m}}{dn}\right) ds = \iint_D f_1\Phi_{n,m}\, d\sigma = 0.$$

The existence of the function u to which Green's theorem is applicable is postulated and demonstrated subsequently.

The necessary condition for the existence of a solution, distinct from $B\Phi_{n,m}$ in the resonant case, is the orthogonality of f_1 and $\Phi_{n,m}$. This condition is not in general fulfilled if A is not zero. From the physical point of view, this would mean that the crest of the clapotis must, in general, coincide with the plane of the entrance (so that $V_n = 0$) if $k = k_{n,m}$. Thus only the resonant solution of the type $B\Phi_{n,m}(x, y)$ is possible, and the movement is similar to that which would occur for the same harbor shape and wave period and with the entrance closed. This result corresponds to that observed in the laboratory. Some important though isolated exceptions do, however, occur.

The partial derivatives of ψ_1 are infinite at the singular points but integrable. It can be shown that Green's theorem is applicable to ψ_1 and hence to f_1 throughout the domain [3]. From (6),

$$f_1 = -\nabla^2 f - k_{n,m}^2 f$$

and, hence,

$$\begin{aligned}\iint_D f_1\Phi_{n,m}\, d\sigma &= -\iint_D (\Phi_{n,m}\nabla^2 f - f\nabla^2\Phi_{n,m})\, d\sigma \\ &= \int_\Gamma \left(\Phi_{n,m}\frac{df}{dn} - f\frac{d\Phi_{n,m}}{dn}\right) ds = A\int_\alpha^\beta \Phi_{n,m}(x, 0)\, dx \\ &= \frac{2A}{\pi n}\left(\frac{a}{b}\right)^{1/2}\left(\sin\frac{n\pi\beta}{a} - \sin\frac{\pi n\alpha}{a}\right).\end{aligned}$$

Finally, the relationship,

$$A\left(\sin\frac{\pi n\beta}{a} - \sin\frac{\pi n\alpha}{a}\right) = 0$$

is obtained.

If the dimensions of the harbor do not satisfy one of the conditions

$$\frac{\beta - \alpha}{a} = \frac{2N}{n}, \qquad \frac{\beta + \alpha}{n} = \frac{2N + 1}{n}, \tag{20}$$

where N is any positive integer, f_1 and $\Phi_{n,m}$ can only be orthogonal if $A = 0$, as already stated. If, on the contrary, one of the conditions in (18) is met, (11) and (12) will give the following solution to (10):

$$u(x, y) = \sum_i^{\infty}{}' c_i\Phi_i + B\Phi_{n,m}, \tag{21}$$

where the symbol $\sum'$ indicates that the (n, m) term is excluded from the summation.

Calculation of the surface displacements. The function f_1 from (10) was introduced only for consideration of the convergence of (11); the singularities are thus isolated, and the expressions

$$F_{p,q}(x, y) = f(x, y) + \sum_{n,m}^{p,q} c_{n,m}\Phi_{n,m}(x, y)$$

satisfy for any values of p and q the appropriate boundary conditions for (5). From the practical point of view, the values $\partial F_{p,q}/\partial x$ and $\partial F_{p,q}/\partial y$ are acceptable approximations to the values $\partial F/\partial x$ and $\partial F/\partial y$ for small values of p and q throughout D. The foregoing method is thus recommended for the calculation of the velocities from (2). In laboratory experiments, however, the point of primary interest is the amplitude of the oscillations of the free surface. The relationship from which these are determined is well known:

$$z = -\frac{1}{g}\frac{\partial \phi}{\partial t}\bigg|_{z=0} .$$

From (2), the calculation of z requires only the definition of $F(x, y)$, and not that of its derivatives. Development of F in series is hence desirable so as to avoid the difficult calculation of the coefficients $\gamma_{n,m}$. As F is a regular function, the procedure can be developed as follows:

$$F = \sum_{n,m}^{\infty} d_{n,m}\Phi_{n,m} , \tag{22}$$

in which

$$d_{n,m} = \iint_D F\Phi_{n,m}\, d\sigma. \tag{23}$$

The development of F in series converges absolutely, but, as can be shown, it may not be differentiable term by term, and the partial summations in (22) would then not satisfy the boundary conditions. From (3) and (23),

$$k^2 d_{n,m} = -\iint_D \Phi_{n,m}\nabla^2 F\, d\sigma.$$

Also, Green's theorem is applicable, so that

$$k^2 d_{n,m} = \iint_D F\nabla^2\Phi_{n,m}\, d\sigma + \int_\Gamma \left(\Phi_{n,m}\frac{dF}{dn} - F\frac{d\Phi_{n,m}}{dn}\right) ds$$

$$= k^2_{n,m} d_{n,m} + \frac{2A}{\pi n}\left(\frac{a}{b}\right)^{1/2}\left(\sin\frac{\pi n\beta}{a} - \sin\frac{\pi n\alpha}{a}\right)$$

or, if $k^2 \neq k^2_{n,m}$,

$$d_{n,m} = \frac{2A}{\pi n(k^2 - k^2_{n,m})}\left(\frac{a}{b}\right)^{1/2}\left(\sin\frac{\pi n\beta}{a} - \sin\frac{\pi n\alpha}{a}\right). \tag{24}$$

Equations (22), (23) and (24) give the expression sought for F. The Fourier series thus obtained is absolutely and uniformly convergent in D and Γ, but it is not differentiable term by term. To repeat, the boundary conditions are not satisfied for dF/dn calculated by differentiating the development (22).

Conclusion. From the equations derived, computations can be made of the local amplitudes of both the velocity and the surface displacement. The double-series solution obtained for the velocity potential from (9) is absolutely convergent. A more direct solution in a double series is provided for the surface displacement. Two classes of solutions designated as resonant and non-resonant have been clearly delineated.

For the non-resonant case, the general solution is given by (9) and (11–16). For the resonant motions, the simple solution from the appropriate eigen-function is obtained unless one of the conditions in (18) is satisfied. In the latter case (19) is used in place of (11). The more direct method for the calculation of the surface displacements is comprised of (22–24). In addition, the basic relationships (1–3) are used in each instance.

Although the singularities and the lack of continuity for the tangential component of velocity at the entrance require further study for the establishment of the analytic validity of the solution, the physical validity and usefulness of this type of result were demonstrated in [1].

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