THE TRANSFER FUNCTION OF NETWORKS WITHOUT MUTUAL REACTANCE*

By

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1. **Introduction.** In this paper we develop a complete theory of the transfer function of general two terminal-pair networks containing resistance, capacitance and self-inductance, but no mutual coupling or ideal transformers. These results constitute an extension of those obtained in a previous paper [5], for networks containing two kinds of elements only. In part, our present techniques depend upon the ideas and methods employed in [5] to which reference will be made in the course of the proofs.

Bott and Duffin [2] have characterized two-terminal networks without mutual reactance. The present research is a first step toward solving the corresponding problem for two terminal-pair networks. We do not require the results of [2] in this paper.

Recently several papers [8, 9, 10] have appeared which deal with the transfer function of RLC networks. These do not develop a complete theory but are concerned primarily with the synthesis of a transfer function up to a constant multiplier. Even with this restriction on the multiplicative constant, the synthesis procedures for grounded networks described in these papers are all of limited applicability for realizing the general transfer function of these networks. Furthermore none of the papers gives the properties of the transfer function which are peculiar to general grounded networks and distinguish it from the transfer function of general two terminal-pair networks.

On the other hand the present paper besides characterizing the transfer functions of both grounded and general two terminal-pair networks, yields a synthesis realizing any multiplicative constant up to the theoretical maximum which is allowable.

The transfer function $A(p)$ is defined as the ratio of steady state output voltage to input voltage in the domain of the complex frequency variable p. It is a real rational function which we may write in the form

$$A(p) = K \frac{N}{D} = K \frac{p^n + a_1 p^{n-1} + \cdots + a_n}{p^m + b_1 p^{m-1} + \cdots + b_m}, \tag{1.1}$$

where K is a constant and N and D are polynomials which have no common factors.

We first consider the general grounded two terminal-pair networks (3 external terminals) abbreviated 3 T.N. The conditions on $A(p)$ in this case are given in Theorem 1 and are here described in an equivalent form as follows. The poles of $A(p)$ are in the left-half plane or on its boundary except that $p = 0$ and $p = \infty$ are excluded. A pole on the imaginary axis must be simple and have a pure imaginary residue. The zeros of $A(p)$ cannot be positive real but are otherwise arbitrary. The range of K is an interval $0 < K \leq K_0$ where K_0 is the minimum value** of the function $D(p)/N(p)$ for $0 \leq p \leq \infty$. Conversely when a function $A(p)$ satisfies these conditions a 3 T.N. may be synthesized whose transfer function is the given $A(p)$. In the sequel, this synthesis is performed assuming an open circuit termination, but the technique may be modified to take account of any finite resistive load without any essential change in the above results.

For the general two terminal-pair network (4 external terminals) abbreviated 4

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**K_0 is a realizable value of K if and only if the minimum value is assumed only at $p = 0$ or $p = \infty$ or both.

T.N. the results are given in Theorem 2. They differ from those stated above for a 3 T.N. in that there is no restriction on the zeros of $A(p)$, and the range of K is now $-K_0 \leq K \leq K_0$ where here K_0 is the minimum value* of $| D(p)/N(p) |$ for $0 \leq p \leq \infty$.

In the synthesis procedure, it is convenient to distinguish two cases. These are: Case I. The poles of $A(p)$ all lie in the interior of the left-hand plane. Case II. $A(p)$ has at least one pole on the pure imaginary axis. The synthesis in Case I is relatively simple to handle while that in Case II is considerably more complicated.

A synthesis procedure in Case II for a 3 T.N., even up to a constant multiplier, has not been considered heretofore in the literature. For a 4 T.N., a method due to Kahal [8] has appeared which claims to realize the transfer functions of Case II up to a constant factor as a symmetric lattice. However, both his proof and conclusions are erroneous. As a counter-example, we note that the realizable** transfer function $A_1(p) = K(p^2 - 0.5\,p + 0.5)/(p^2 + 1)(p + 1)$ which satisfies his conditions may not be synthesized as a symmetric lattice† for any $K \neq 0$. The foregoing example is an illustration of a theorem†† on lattice realization, the proof of which will appear elsewhere. On the other hand, Weinberg [9] overlooks Case II completely, and erroneously states [9, pp. 37, 53] that every transfer function may be realized as a symmetric lattice.

2. The grounded two terminal-pair network.

Theorem 1: *Necessary and sufficient conditions that a real rational function $A(p)$ given by* (1.1) *be the transfer function of a* 3 *T.N. are*: (i) *The zeros of D are anywhere in the left-half plane or on the imaginary axis with the origin excluded.*

(ii) *At a pure imaginary zero of D, $A(p)$ has a simple pole with pure imaginary residue.*

(iii) *The zeros of N can not be positive real but are otherwise arbitrary.*

(iv) $m \geq n$.

(v) *The number K satisfies the inequalities $0 < K < K_0$ where K_0 is the least of the three quantities K_d, b_m/a_n, 1 if $m = n$ and of the first two quantities if $m > n$. If $K_0 \neq K_d$ then K may equal K_0. Here K_d is the least positive value of κ (if it exists) for which the equation $D - \kappa N = 0$ has a positive multiple root.*

Proof: (a) *Necessity*

In the case of either a 3 T.N. or 4 T.N. we may write the transfer function (1.1) in the equivalent form

$$A(p) = Y_{12}/Y_{22} \tag{2.1}$$

where Y_{12} and Y_{22} are the short circuit transfer and driving point admittances respectively of the network, [6, pp. 134–136]. The zeros of D in (1.1) are either the poles of Y_{12} or the zeros of Y_{22} which as is well known must lie in the left-half plane or on its boundary, [1, Chap. VII]. This establishes (i) except for the exclusion of $p = 0$. This last physically evident condition will drop out of later considerations.

*K_0 is a realizable value of K if and only if the minimum value is assumed only at $p = 0$ or $p = \infty$ or both.

**The transfer function $A_1(p)$ with $K = 2$ is realized by the parallel connection of the networks of Figure 5.

†This follows from the fact that the fraction $(1 - A_1)/(1 + A_1)$ has zeros and poles in the right-half plane for any $K \neq 0$, whereas for a symmetric lattice this fraction should be the quotient of the two constituent impedances of the lattice.

††The theorem is: Let $A(p)$ be a 4 T. N. realizable transfer function (i.e. one satisfying the conditions of Theorem 2) belonging to Case II. Define $X_n = \mathbf{Re}\,\{i^n \cdot d(A^{-1})/dp \cdot d^n(A^{-1})/dp^n\}$, $n = 2, 3, \cdots$. Then $A(p)$ can be synthesized up to a multiplicative constant by means of a symmetric lattice if and only if at each pure imaginary pole of A the first non-zero value of X_n occurs for n even and is negative.

The case where D has a zero at $p = i\omega_0$ can only arise if Y_{22} has a simple zero and Y_{12} has neither a zero nor a pole at $p = i\omega_0$. The remaining possibilities, as is known, may be excluded by use of the general residue condition

$$r_{11}r_{22} - r_{12}^2 \geq 0 \tag{2.2}$$

for $\mathbf{Re}(p) \geq 0$, [6, pp 216–218]. Here r_{11}, r_{22}, r_{12}, $\mathbf{Re}(p)$ represent the real parts of the short circuit driving point and transfer admittances Y_{11}, Y_{22}, Y_{12} and of p respectively. It follows that at such a zero $p = i\omega_0$ of D, $A(p)$ has a simple pole. If $Y_{22} = a(p - i\omega_0) + \cdots$, $a > 0$ then the residue of $A(p)$ at $p = i\omega_0$ is evidently $Y_{12}(i\omega_0)/a$. Since $r_{22}(i\omega_0) = 0$, $r_{11}(i\omega_0) \neq \infty$, (2.2) implies that $r_{12}(i\omega_0) = 0$. Hence $Y_{12}(i\omega_0)/a$ is pure imaginary. This proves (ii).

Condition (iv) is well known and follows from (2.2) and its consequences. To show the necessity of the remaining conditions, the 3 T.N. in Fig. 1 is considered upon a nodal basis.

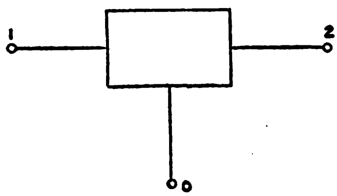

FIG. 1

Here the ground terminal is taken as node 0, the other input and output terminals as nodes 1 and 2 respectively. The remaining nodes are identified so that each branch is an R, L and C in parallel. Hence the admittance y_{ij} $(i \neq j)$ of the branch between nodes i and j is of the form $ap + b + c/p$, $a \geq 0$, $b \geq 0$, $c \geq 0$. Of course $y_{ij} = y_{ji}$, and as usual we write

$$y_{ii} = \sum_{\substack{j=0\\ j\neq i}} y_{ij} \qquad (i = 1, 2, \cdots, t),$$

where $t + 1$ is the total number of nodes. Now let $y'_{ij} = py_{ij}$ $(i, j = 0, 1, \cdots, t, i$ and j not both zero). Then each y'_{ij} is a quadratic polynomial in p with non-negative coefficients, and $y'_{ii} = \sum_{j=0, j\neq i}^{t} y'_{ij}$ $(i = 1, 2, \cdots, t)$. Using the nodal equations of the network, it follows* that we may express the transfer function $A(p)$ in the form

$$A(p) = \Delta_1/\Delta_2,$$

where

$$\Delta_1 = \begin{vmatrix} y'_{21} & y'_{23} & \cdots & y'_{2t} \\ -y'_{31} & y'_{33} & \cdots & -y'_{3t} \\ \cdot & \cdot & \cdot & \cdot \\ -y'_{t1} & -y'_{t3} & \cdots & y'_{tt} \end{vmatrix} = c_0p^s + c_1p^{s-1} + \cdots + c_s, \tag{2.3}$$

$$\Delta_2 = \begin{vmatrix} y'_{22} & -y'_{23} & \cdots & -y'_{2t} \\ -y'_{32} & y'_{33} & \cdots & -y'_{3t} \\ \cdot & \cdot & \cdot & \cdot \\ -y'_{t2} & -y'_{t3} & \cdots & y'_{tt} \end{vmatrix} = d_0p^s + d_1p^{s-1} + \cdots + d_s, \; d_0 \neq 0. \tag{2.4}$$

*Cf. [5, §2(a)] for details.

Here Δ_1 and Δ_2 may have common factors and by (iv) the degree of Δ_1 may actually be less than s. In [5, Appendix A], it was shown that $\Delta_2 = \Delta_1 + \Delta_1'$ where Δ_1' is the determinant obtained from Δ_1 by replacing the elements of its first column by y'_{20} , $-y'_{30}$, $\cdots$, $-y'_{t0}$ respectively. It was further shown that a determinant of the form Δ_1 (or Δ_1') is a polynomial in the y'_{ij} $(i \neq j)$ with positive coefficients. These two results prove that

$$0 \leq c_j \leq d_j , \qquad (j = 0, 1, \cdots , s). \tag{2.5}$$

As an immediate consequence of (2.5), it follows that $p = 0$ cannot be a pole of $A(p)$. For $d_s = d_{s-1} = \cdots = d_{s-k} = 0, d_{s-k-1} \neq 0$ in (2.4) now imply $c_s = c_{s-1} = \cdots = c_{s-k} = 0$ in (2.3). This completes the proof of (i). Incidentally, it also follows directly from (2.5) that $0 \leq A(p) \leq 1$ *for* p *in the range* $0 \leq p \leq \infty$. (The equality sign may hold only if $p = 0$ or $p = \infty$).

Having established (2.5), the remainder of the proof for the necessity of (v) now follows word for word as in [5, §2(a)] starting with Eq. (2.6) there.

(b) *Sufficiency.*

It is useful at this point to introduce a few notations which will simplify the exposition. Capital Latin letters (except for A, Y, Z and K) unless otherwise identified will always denote polynomials in p with real coefficients. If $R = \sum_{i=0}^{n} \alpha_i p^i$ and $S = \sum_{i=0}^{n} \beta_i p^i$ then $R \ll S$ denotes that $\alpha_i \leq \beta_i$ $(i = 0, 1, \cdots , n)$. If P is a given polynomial, P_e and P_0 will denote the polynomials consisting of the even powers and the odd powers of P respectively. More generally, without reference to a given polynomial, the subscript e and 0 will indicate an even or odd polynomial respectively.

In terms of this notation, the following result is implied by the sufficiency proof of [5, §2(b)]:

(A) *Let* S_e *have only simple pure imaginary zeros and let* $0 \ll R_e \ll S_e$. *Then an* $LC - 3$ *T.N. exists whose transfer function is* $A(p^2) = R_e/S_e$ *and whose* $Y_{22} = S_e/T_0$ *with* T_0 *any odd polynomial relatively prime to* S_e *such that* S_e/T_0 *is an LC admittance,* T_0 *being one degree lower* than* S_e .

For replacing p^2 by p in $A(p^2)$ gives what we called an R-function in [5, §2(b)]. In the RC-network corresponding to this R-function, as given by our synthesis procedure** taking $p^{1/2}\ S_e(p^{1/2})/T_0(p^{1/2})$ as the Y_{22} , replace each resistance R by an inductance L with $L = R$ and the LC-network required for (A) results.

Now suppose that $A(p)$ as given by (1.1) satisfies the conditions of Theorem 1. We shall construct a 3 T.N. whose transfer function is $A(p)$. The first step in the synthesis procedure is to write $A(p)$ in a special form analogous to the special form (R-function) used for RC networks. The argument of [5, Appendix B] may be used here to show the existence of a Hurwitz polynomial† U (actually the zeros of U may be taken to be negative real and distinct except that in the special case where N and D are both even functions of p we may use a non-Hurwitz polynomial U having only pure imaginary zeros distinct from the zeros of D and from each other) such that in

$$A(p) = KUN/UD = G/H$$

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*This restriction on T_0 is easily removed but we do not stop to do this here.

**Of the synthesis procedures mentioned in [5], the last alternative method in the footnote on p. 120 which was later given in detail in [4] usually yields a simpler network.

†In this paper we shall use the term Hurwitz polynomial in the strict sense, i.e., a real polynomial having a positive leading coefficient whose zeros are in the interior of the left-half plane.

we have

$$0 \ll G \ll H. \tag{2.6}$$

For an examination of the proof in [5, Appendix B] shows that in addition to (iii), (iv) and (v) of Theorem 1, use was made only of the fact that the zeros of D were not positive real or zero, and this is guaranteed by (i) of Theorem 1. This common factor technique achieves positive coefficients in the transfer function as is stated in (2.6). Recently Weinberg [J. Appl. Phys. **24**, 1526 (1953)], citing references to Bode, has stated that the use of common factors for similar purposes is well established and almost common knowledge. This is hardly justified, for examination of these and other references reveals at best a superficial similiarity to our method. Furthermore, in view of his statement it is surprising that in none of the literature on transfer functions prior to [5] including his own thesis [9] has our or a similar method been employed.

(b_1) *Synthesis*: *Case I*.

We now assume that H is a Hurwitz polynomial. Consider $H = H_0 + H_e$, $H_0 = pH'_e$. As is well known [7, p 400] H_e and H'_e are relatively prime polynomials having simple pure imaginary zeros and such that H_0/H_e is an LC-admittance. In view of (2.6) we have $G = G_e + G_0$, $G_0 = pG'_e$ with $0 \ll G_0 \ll H_0$, $0 \ll G'_e \ll H'_e$. Hence by (A) the functions

$$A_1 = G'_e/H'_e, \qquad A_2 = G_e/H_e$$

are $LC - 3$ T.N. transfer functions.

We may write

$$\frac{H_e}{pH'_e} = \alpha p + \frac{\beta}{p} + \frac{F_0}{H'_e}, \qquad \frac{pH'_e}{H_e} = \alpha' p + \frac{F'_0}{H_e},$$

where $\alpha \geq 0$, $\alpha' \geq 0$, $\beta > 0$ and F_0/H'_e and F'_0/H_e are LC impedances, F_0 and F'_0 being of lower degree than H'_e and H_e respectively. Of course one of α and α' is always zero.

Now according to (A) realize the transfer functions A_1 and A_2 by two $LC - 3$ T.N. Γ_1 and Γ_2 whose Y_{22}'s are $Y_{22}^{(1)} = H'_e/F_0$, $Y_{22}^{(2)} = H_e/F'_0$ respectively. Modify the networks Γ_1 and Γ_2, without changing their transfer functions, as shown in Fig. 2 to form Γ'_1 and Γ'_2 taking

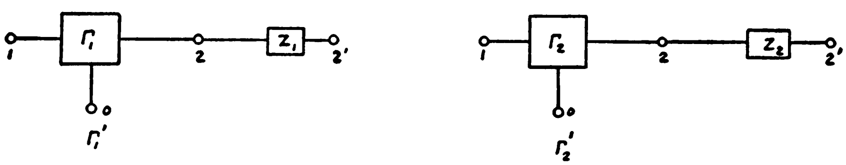

FIG. 2

$$Z_1 = 1 + \alpha p + \frac{\beta}{p}, \qquad Z_2 = 1 + \alpha' p.$$

The Y_{22}'s of the networks Γ'_i $(i = 1, 2)$ are evidently $1/(Z_i + 1/Y_{22}^{(i)})$, $(i = 1, 2)$ or $H_0/(H_0 + H_e)$ and $H_e/(H_0 + H_e)$ respectively.* In view of (2.1) the corresponding

*The method used here is analogous to the last alternative procedure mentioned in [5], footnote on page 120 and given in [4]. Similarly, techniques based on the other procedures given in [5] may also be employed here.

Y_{12}'s are G_0/H and G_e/H respectively. Now connect the networks Γ_1' and Γ_2' in parallel to form a new 3 T.N. Γ whose $Y_{12} = G/H$ and whose $Y_{22} = 1$ so that the transfer function $A(p)$ of Γ is G/H. This completes the synthesis in Case I.

(b$_2$) *Synthesis*: *Case II*.

We now assume that H has some pure imaginary zeros. If all the zeros of H are pure imaginary, we may apply (A) to obtain an LC-network realization. Otherwise write $H = P_e J$ where

$$P_e = \prod_{k=1}^{s} (p^2 + \omega_k^2) \tag{2.7}$$

and J is a Hurwitz polynomial. Of course the ω_k are distinct and different from zero and the residue of $A(p)$ at $p = \pm i\omega_k$ is pure imaginary.

The synthesis method here consists of a reduction procedure by which the realization of $A(p)$ is made to depend upon the realization of successively simpler transfer functions until we finally reach a stage in which we may use the result (A) to effect an actual realization. In order to apply this method of reduction, it is necessary that the transfer function be written in a particular form M/Q (described in Lemma 1) and that a positive real function Q/S having special properties (also given in Lemma 1) be associated with $A(p)$. In Lemma 2 it is shown how to transform the transfer function into the required form as well as how to construct the function Q/S. The essential ideas of the reduction algorithm are now described. The justification for the various steps is given in Appendix I.

Starting with the positive real function Q/S, Q and S of the same degree, we effect the decomposition

$$\frac{Q}{S} = \frac{pQ'}{S} + \frac{Q''}{S},$$

where pQ'/S and Q''/S are again positive real functions, with Q' and Q'' one degree lower than Q. Corresponding to the partition $Q = pQ' + Q''$ we may form the partition $M = pM' + M''$ so that $A_1 = M'/Q'$ and $A_2 = M''/Q''$ are of the same special form as M/Q but of lower degree.

We have

$$\frac{S}{pQ'} = \frac{a}{p} + \frac{S'}{Q'}, \qquad \frac{S}{Q''} = bp + \frac{S''}{Q''}, \qquad a > 0, \qquad b > 0,$$

where Q'/S' and Q''/S'' are positive real functions of the same special form as Q/S. Suppose that the transfer functions A_1 and A_2 are realized by 3 T.N. Γ_1 and Γ_2 whose short circuit driving-point admittances are $Y_{22}^{(1)} = Q'/S'$ and $Y_{22}^{(2)} = Q''/S''$ respectively. Then connecting the impedances $Z_1 = a/p$ and $Z_2 = bp$ to Γ_1 and Γ_2 respectively, as in Fig. 2, we get 3 T.N. Γ_1', Γ_2' whose Y_{22}'s are $\mathcal{Y}_{22}^{(1)} = pQ'/S$, $\mathcal{Y}_{22}^{(2)} = Q''/S$ respectively and whose transfer functions are A_1 and A_2 respectively. Thus the Y_{12}'s of Γ_1' and Γ_2' are $\mathcal{Y}_{12}^{(1)} = pM'/S$, $\mathcal{Y}_{12}^{(2)} = M''/S$ respectively. The parallel connection of Γ_1' and Γ_2' gives a network Γ whose transfer function is

$$\frac{(pM'/S) + (M''/S)}{(pQ'/S) + (Q''/S)} = \frac{M}{Q}$$

and whose $Y_{22} = Q/S$. We have thus made the realization of the pair $A(p)$, Q/S depend upon the realization of the simpler pairs A_1, Q'/S'; A_2, Q''/S.

As shown in the remarks following Lemma 1, the reduction process may be continued until we reach transfer functions belonging to either one of the two following categories which we designate as subcases (α) and (β) respectively and whose synthesis we now describe.

(α) Here $A = M/Q$ with Q Hurwitz and of degree $2s$. Also $0 \ll M \ll Q$, $Q = P_e + Q_0$, $M_e = \alpha P_e$, $0 \leq \alpha \leq 1$. The required Y_{22} is Q/S with $S = P_e$.

If $s \geq 2$, we may expand the LC-impedance $P_e/Q_0 = P_e/pQ_e'$ as

$$\frac{P_e}{pQ_e'} = \beta p + \frac{\gamma}{p} + \frac{F_0}{Q_e'}$$

where $\beta > 0$, $\gamma > 0$ and F_0/Q_e' is an LC impedance with F_0 of lower degree than Q_e'. Now apply (A) to realize the LC transfer function $(M_0/p)/Q_e'$ by means of a network Γ_1 whose $Y_{22} = Q_e'/F_0$. Form the network Γ_1' as in Fig. 2 taking $Z_1 = \beta p + \gamma/p$ so that the Y_{22} of Γ_1' is Q_0/P_e. Consider the L-network Γ_2 whose series arm is $Y_a = \alpha$ and whose shunt arm is $Y_b = 1 - \alpha$. The parallel connection of Γ_1' and Γ_2 gives a network whose transfer function is M/Q and whose $Y_{22} = Q/S$.

If $s = 1$ i.e. $A(p) = [\delta p + \alpha(p^2 + \omega_1^2)]/[\epsilon p + p^2 + \omega_1^2]$ where $0 \leq \delta \leq \epsilon$ and the required $Y_{22} = 1 + \epsilon p/(p^2 + \omega_1^2)$, form the parallel combination of the previous network Γ_2 and an L-network Γ_1 whose series and shunt admittances are $y_{1a} = \delta p/(p^2 + \omega_1^2)$ and $y_{1b} = (\epsilon - \delta)p/(p^2 + \omega_1^2)$ respectively.

(β) Here $A = M/Q$ with $0 \ll M = M_e \ll Q = P_e$. The required Y_{22} is Q/S where $S = P_e + S_0$, S Hurwitz and of degree $2s$.

Realize the LC transfer function M_e/P_e using (A) to get a network Γ whose $Y_{22} = P_e/S_0$. Then form Γ_1' as in Fig. 2 taking $Z_1 = 1$. This is the required network.

The synthesis method of Case II may easily be modified to provide an alternate synthesis for Case I also.

A second synthesis method for Case II which is also inductive in nature (but for which we give no details) may be described as follows. The transfer function A is written as $A = rA_1 + SA_2$ where $r > 0$, $s > 0$, $r + s \leq 1$ and A_1 and A_2 are 3 T.N. transfer functions. It is easy to show that A is realizable if A_1 and A_2 are. We may choose the function A_1 so that it has the same pure imaginary poles as A but its numerator consists of even (or odd) powers only. The synthesis of A_1 is then accomplished easily by a modification of the procedure of Case I. As for A_2, it may be taken to have one less pole than A on the imaginary axis and the same process is now applied to it until all the pure imaginary poles are eliminated, when the results of Case I apply. This synthesis results in a network of about the same complexity as the one previously given.

3. The two-terminal pair network

Theorem 2. *Necessary and sufficient conditions that a real rational function* $A(p)$ *given by* (1.1) *be the transfer function of a* 4 *T.N. are* (i), (ii), (iv) *of Theorem* 1 *and* (v'): *The number* K *satisfies the inequalities* $-K_0 < K < K_0$ *where* K_0 *is the least of the three quantities* $|K_d|$, $|b_m/a_n|$, 1 *if* $m = n$, *and of the first two quantities if* $m > n$. *If* $K_0 \neq |K_d|$ *then* K *may actually equal* $\pm K_0$. *Here* K_d *is that real value of* κ *of smallest absolute value* (*if it exists*) *for which the equation* $D - \kappa N = 0$ *has a positive multiple root.*

Proof: (a) *Necessity*

By the remarks in Section 2 (a) only the proof of (v') remains. Consider the 4 T.N. on a nodal basis taking the input terminals as nodes 0 and 1, the output terminals as nodes 2 and 3, and choosing the remaining nodes as in the 3 T.N. case. Then in the notation of Section 2 we have*

$$A(p) = (\Delta_1 - \Delta_3)/\Delta_2$$

with Δ_1 and Δ_2 as in (2.3), (2.4) and

$$\Delta_3 = \begin{vmatrix} y'_{31} & y'_{32} & y'_{34} & \cdots & y'_{3t} \\ -y'_{21} & y'_{22} & -y'_{24} & \cdots & -y'_{2t} \\ -y'_{41} & -y'_{42} & y'_{44} & \cdots & -y'_{4t} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -y'_{t1} & -y'_{t2} & -y'_{t4} & \cdots & y_{tt} \end{vmatrix} = c'_0 p^s + c'_1 p^{s-1} + \cdots + c'_s .$$

Since Δ_3 is of the same form as Δ_1, we have $0 \leq c'_j \leq d_j$ $(j = 0, 1, \cdots, s)$ in addition to (2.5). As a consequence of these equations, we note that $0 \leq |A(p)| \leq 1$ for p in the range $0 \leq p \leq \infty$, with the possibility that $|A(p)| = 1$ existing only when $p = 0$ or $p = \infty$. The remainder of the proof now follows word for word as in [5, §4(a)] following equation (4.2) there.

(b) *Sufficiency.*

Let $A(p)$ satisfy the conditions of Theorem 2. We will realize $A(p)$ by showing it can be written as $A = A_1 - A_2$ with A_1 and A_2 transfer functions of 3 T.N. If Γ_1 and Γ_2 are 3 T.N. corresponding to A_1 and A_2 respectively, then the 4 T.N. Γ of Fig. 3 corresponds to A.

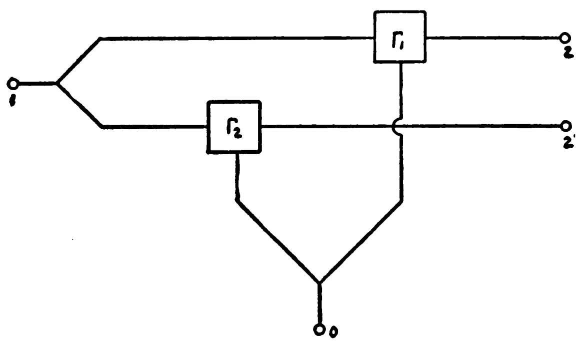

FIG. 3

By the argument of [5, §4(b)] together with the remarks in Section 2(b) following (2.6), we show that $A(p)$ may be written in the form $A(p) = G/H$ where $H \pm G \gg 0$.

(b$_1$) *Synthesis: Case I.*

If H is Hurwitz and G_1, $-G_2$ consist respectively of the terms of G having positive or negative coefficients, then $A = (G_1/H) - (G_2/H)$ where each of the fractions is a 3 T.N. transfer function in the realizable form for Case I.

(b$_2$) *Synthesis: Case II.*

If $H = P_e J$, $P_e = \Pi(p^2 + \omega_k^2)$, J Hurwitz then apply the method** of Appendix II

*For details cf. [5, §4(a)].

**The only modification in Appendix II consists of considering $J_e \pm G_e^*$ instead of G_e^* and $J_e - G_e^*$ in the argument following (II.1).

to get A in the form $A(p) = M/P_e Q^*$ with Q^* Hurwitz, Q_e^* divisible by P_e and with $M_e = P_e M_e^*$, $P_e Q_0^* \pm M_0 \gg 0$, $Q_e^* \pm M_e^* \gg 0$. Denoting the positive terms of M_e^* by M_e' and the negative terms by $-M_e''$ and using M_0', M_0'' similarly with respect to M_0 we have $A = A_1 - A_2$ where $A_1 = (M_0' + P_e M_e')/P_e Q^*$ and $A_2 = (M_0'' + P_e M_e'')/P_e Q_e^*$ are 3 T.N. transfer functions of the form considered in §2(b_2).

We note if one desires the synthesis of $A(p)$ up to a constant multiplier only, a procedure based directly upon Case I and having the simplicity inherent in LC synthesis techniques is possible. A brief description follows: Write $A(p) = KN/D$ where $D = P_e D'$, $P_e = \Pi(p^2 + \omega_k^2)$ D' Hurwitz. Decompose A so that $A = (N_1/P_e) + (N_2/D')$ where N_1 must be an even polynomial. For sufficiently small positive constants α, β the transfer function $A_1 = \alpha N_1/P_e$ may be realized by an LC network Γ_1, and $A_2 = \beta N_2/D'$ may be realized by a Case I RLC network. Then the function $[\beta A_1/(\alpha + \beta)] + [\alpha A_2/(\alpha + \beta)] = \alpha \beta A/(\alpha + \beta)$ is realized by first connecting a suitable RLC impedance Z_1 and a suitable LC impedance Z_2 to Γ_1 and Γ_2 respectively (similar to the connection shown in Fig. 2) and then putting the modified networks in parallel.

4. Examples. (i) The following example illustrates the theory and synthesis procedure for Case I. Consider the synthesis of the 3 T.N. transfer function $A(p) = G/H = (3p^2 + p + 2)/(p^3 + 3p^2 + 2p + 3)$. Calculation shows that $K_0 = K_d = 4.468$. We shall synthesize the given $A(p)$ where $K = 3 < K_0$. Here H is Hurwitz and $0 \ll G \ll H$, so that $A(p)$ is in the realizable form of Case I.

We consider the LC $-$ 3 T.N. transfer functions

$$A_1(p^2) = \frac{(G_0/p)}{(H_0/p)} = \frac{1}{p^2 + 2}, \qquad A_2(p^2) = \frac{G_e}{H_e} = \frac{3p^2 + 2}{3p^2 + 3}.$$

Since

$$\frac{H_e}{pH_e'} = \frac{3}{2} \cdot \frac{1}{p} + \frac{3}{2} \cdot \frac{p}{p^2 + 2}$$

and

$$\frac{pH_e'}{H_e} = \frac{p}{3} + \frac{1}{3} \cdot \frac{p}{p^2 + 1},$$

we must realize $A_1(p^2)$ and $A_2(p^2)$ with Y_{22}'s respectively $Y_{22}^{(1)} = 2(p^2 + 2)/3p$ and $Y_{22}^{(2)} = 3(p^2 + 1)/p$. To do this, first realize $A_1(p) = 1/(p + 2)$ and $A_2(p) = (3p + 2)/(3p + 3)$ as RC-networks whose Y_{22}'s are respectively $2(p + 2)/3$ and $3(p + 1)$. This results in two L-networks Γ_1 and Γ_2 whose series and shunt arm admittances are $y_{1a} = 2/3$, $y_{1b} = 2(p + 1)/3$; $y_{2a} = 3p + 2$, $y_{2b} = 1$ respectively. In Γ_1 and Γ_2 replace each resistor R by an inductance L with $R = L$. We get L-networks Γ_1^*, Γ_2^* with series and shunt arm admittances $y_{1a}^* = 2/3p$, $y_{1b}^* = 2(p^2 + 1)/3p$; $y_{2a}^* = 3p + 2/p$, $y_{2b}^* = 1/p$. Networks Γ_1^* and Γ_2^* realize $A_1(p^2)$ and $A_2(p^2)$ with the required Y_{22}'s.

Now connect $Z_1 = 1 + 3/2p$ and $Z_2 = 1 + p/3$ to Γ_1^* and Γ_2^* respectively as in Fig. 2 to form Γ_1' and Γ_2' whose parallel combination Γ given in Fig. 4 is the required 3 T.N. (ii) The transfer function $A(p) = K(p^2 - 0.5p + 0.5)/(p^2 + 1)(p + 1)$, mentioned in the introduction as unrealizable by means of a symmetric lattice, belongs to Case II. Here K_0 is found to be 2 and we synthesize $A(p)$ with $K = K_0$ as a 3 T.N. In order to apply the procedure for Case II, we work with $A(p)$ in the form $A(p) = M/Q = (2p^3 + p^2 + 1)/(p^2 + 1)(p + 1)^2$ together with $Q/S = (p^2 + 1)(p + 1)^2/(p^4 + 5p^3 + 5p^2 +$

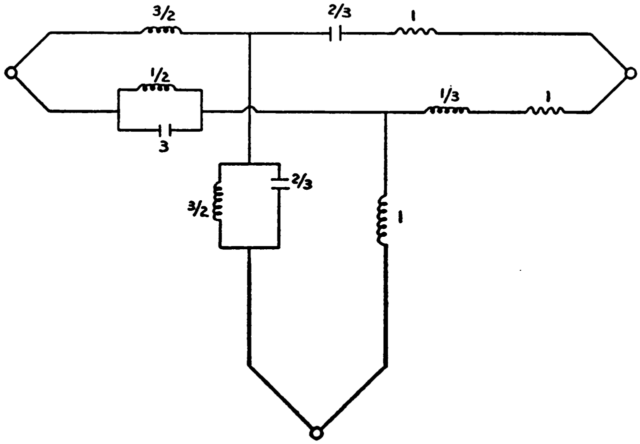

FIG. 4

$5p + 2$) which satisfy the conditions of Lemma 1. The synthesis leads us to the network formed by the parallel connection of those of Fig. 5 as a realization of $A(p)$. For lack of space, all details must be omitted.

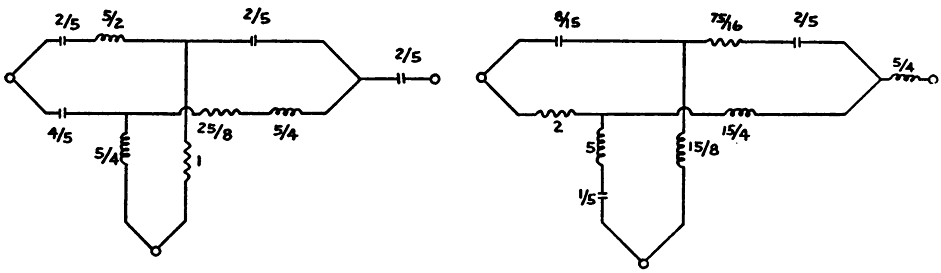

FIG. 5

Appendix I

Lemma 1. *Let M/Q and Q/S be a real rational function and a positive real rational function respectively satisfying the following conditions*:

(i) *The zeros of Q are in the left half-plane including the boundary.*

(ii) $0 \ll M \ll Q$.

(iii) *Q and S are each of the same degree $r + 2s$, $r > 0$, $s > 0$.*

(iv) *Q and S have no common pure imaginary zeros including $p = 0$.*

(v)

$$\operatorname*{Re}_{p=i\omega}\left[\frac{Q}{S}\right] = \frac{P_e^2(i\omega)(\beta_0\omega^{2r} + \beta_1\omega^{2r-2} + \cdots + \beta_r)}{|\,S(i\omega)\,|^2}, \qquad \beta_j > 0\ (j = 0, 1, \cdots, r)^*.$$

(vi) *Either* (*a*) $M_e = P_eM_e^*$, $Q_e = P_eQ_e^*$, $S_0 = P_eS_0^*$, $0 \ll M_e^* \ll Q_e^*$; *or*

(*b*) $M_0 = P_eM_0^*$, $Q_0 = P_eQ_0^*$, $S_e = P_eS_e^*$, $0 \ll M_0^* \ll Q_0^*$.

Then polynomials S', S'' and a decomposition

$$M = pM' + M'', \qquad Q = pQ' + Q''$$

*The polynomial P_e is defined by (2.7).

exist such that each pair of functions M'/Q', Q'/S' *and* M''/Q'', Q''/S'' *satisfy all the preceding conditions with* r *replaced by* $r - 1$ *and* $S = aQ' + pS' = bpQ'' + S''$ *with* $a > 0, b > 0$.

Proof: It suffices to consider the case in which vi(a) holds, the other case being treated in a similar manner. Our first goal is the decomposition of Q/S. We note that the zeros of S are also in the left half-plane with simple zeros on the boundary because of (i), (iv) and the fact that Q/S is a positive real function. Since if

$$\operatorname*{Re}_{p=i\omega}\left[\frac{B(p)}{C(p)}\right] = \frac{\theta(\omega^2)}{|\,C(i\omega)\,|^2}$$

then

$$\operatorname*{Re}_{p=i\omega}\left[\frac{C(p)}{B(p)}\right] = \frac{\theta(\omega^2)}{|\,B(i\omega)\,|^2},$$

it follows from (v) that

$$\operatorname*{Re}_{p=i\omega}\left[\frac{S}{Q}\right] = \frac{P_e^2(i\omega)(\beta_0\omega^{2r} + \beta_1\omega^{2r-2} + \cdots + \beta_r)}{|\,Q(i\omega)\,|^2}.$$

Hence any pure imaginary zeros of S must be zeros of P_e. Thus we may write $S = P'_e T$ where $P'_e = \Pi'(p^2 + \omega_k^2)$ of degree $2s' \leq 2s$, contains all the pure imaginary zeros of S and, of course T is Hurwitz of degree $2(s - s') + r$.

It follows that Q/S may be expanded to yield

$$\frac{Q}{S} = \sum{}' \frac{\alpha_k p}{p^2 + \omega_k^2} + \frac{R}{T} = \frac{L_0}{P'_e} + \frac{R}{T} \tag{I.1}$$

where $\sum'$ ranges over the zeros of P'_e and $\alpha_k > 0$. Also from (v)

$$\operatorname*{Re}_{p=i\omega}\left[\frac{R}{T}\right] = [P_e(i\omega)/P'_e(i\omega)]^2 \cdot \frac{(\beta_0\omega^{2r} + \beta_1\omega^{2r-2} + \cdots + \beta_r)}{|\,T(i\omega)\,|^2}.$$

Let β'_i and β''_i $(i = 1, 2, \cdots, r - 1)$ be any positive numbers such that $\beta_i = \beta'_i + \beta''_i$. Let $\beta'_0 = \beta_0$, $\beta''_r = \beta_r$. Form the positive real rational functions pR'/T and R''/T whose numerators are of degree $2(s - s') + r$ at most and where*

$$\operatorname*{Re}_{p=i\omega}\left[\frac{pR'}{T}\right] = [P_e(i\omega)/P'_e(i\omega)]^2 \cdot \frac{(\beta'_0\omega^{2r} + \beta'_1\omega^{2r-2} + \cdots + \beta'_{r-1}\omega^2)}{|\,T(i\omega)\,|^2}, \tag{I.2}$$

$$\operatorname*{Re}_{p=i\omega}\left[\frac{R''}{T}\right] = [P_e(i\omega)/P'_e(i\omega)]^2 \cdot \frac{(\beta''_1\omega^{2r-2} + \beta''_2\omega^{2r-4} + \cdots + \beta''_r)}{|\,T(i\omega)\,|^2}. \tag{I.3}$$

Then both R' and R'' are of degree $2(s - s') + r - 1$. We have $(p\,R'/T) + (R''/T) = R/T$ since the real parts of each member are equal for $p = i\omega$ and in this case the function is uniquely determined by its real part.

Now split each α_k in (I.1) so that $\alpha_k = \alpha'_k + \alpha''_k$, $\alpha'_k > 0$, $\alpha''_k > 0$ and write $L'_0/P'_e = \sum' (\alpha'_k p)/(p^2 + \omega_k^2)$, $L''_0/P'_e = \sum' (\alpha''_k p)/(p^2 + \omega_k^2)$. Then

$$\frac{Q}{S} = \frac{pQ'}{S} + \frac{Q''}{S} \tag{I.4}$$

*For the construction of a positive real function with given denominator and given real part on the imaginary axis see [3, pp. 97-98].

where

$$pQ' = L_0'T + pR'P_e' , \qquad Q'' = L_0''T + P_e'R''. \tag{I.5}$$

Q' and Q'' are each of degree $2s + r - 1$. The desired decomposition of Q/S is given by (I.4) and (I.5).

Before proceeding to obtain a corresponding split of M, we must first show that P_e divides both Q_0' and Q_e''. As a preliminary to this we need the result

$$T_0 = P_e U_0 . \tag{I.6}$$

We have

$$\underset{p=i\omega}{\mathbf{Re}} \left[\frac{Q}{S}\right] = \frac{Q_e(i\omega)S_e(-i\omega) + Q_0(i\omega)S_0(-i\omega)}{| S(i\omega) |^2},$$

$$= \frac{P_e(i\omega)[Q_e^*(i\omega)S_e(-i\omega) + Q_0(i\omega)S_0^*(-i\omega)]}{| S(i\omega) |^2},$$

where we have used vi(a) to get the last form. In view of (v) it follows that $P_e(i\omega)$ (and a fortiori P_e' $(i\omega)$) divides $[Q_e^*$ $(i\omega)$ S_e $(-i\omega)$ $+$ Q_0 $(i\omega)$ S_0^* $(i\omega)]$. Now P_e' $(i\omega)$ divides S_e $(-i\omega)$; but because of (iv) P_e' $(i\omega)$ is relatively prime to Q_0 $(i\omega)$. Hence S_0^* $(i\omega)$ is divisible by P_e' $(i\omega)$. Thus $T_0 = S_0/P_e' = P_e$ (S_0^*/P_e') which is (I.6) with $U_0 = S_0^*/P_e'$.

Coming back to Q' and Q'', we show now that

$$Q_e'' = P_e C_e . \tag{I.7}$$

In view of (I.3) and using an argument similar to that given above in the proof of (I.6), we find that $[R_e''$ $(i\omega)$ T_e $(-$ $i\omega)$ $+$ R_0'' $(i\omega)$ T_0 $(-$ $i\omega)]$ is divisible by $[P_e$ $(i\omega)/P_e'$ $(i\omega)]^2$. Since T is Hurwitz and by virtue of (I.6) we have P_e dividing T_0 but relatively prime to T_e . Hence P_e/P_e' divides R_e''. Using this result in the second equation of (I.5) and again noting (I.6), we find that (I.7) holds. A similar argument using (I.2) and the first equation of (I.5) shows that

$$Q_0' = P_e D_0' . \tag{I.8}$$

We are now ready to decompose M. Equating the odd and even parts of the numerators in (I.4) and deleting the common factor P_e , we get

$$Q_0 = pQ_e' + Q_0'' , \qquad Q_e^* = pD_0' + C_e . \tag{I.9}$$

Since $0 \ll M_0 \ll Q_0$ and $0 \ll M_e^* \ll Q_e^*$ by (ii) and (via), we may obtain* polynomials N_0', N_0'', B_e', B_e'' such that $N_0' + N_0'' = M_0$, $B_e' + B_e'' = M_e^*$ and $0 \ll N_0' \ll pQ_e'$, $0 \ll N_0'' \ll Q_0''$, $0 \ll B_e' \ll pD_0'$, $0 \ll B_e'' \ll C_e$. Now define $M' = (N_0' + P_e B_e')/p$, $M'' = N_0'' + P_e B_e''$. Then $M = pM' + M''$. Since B_e' is divisible by p^2, M' is actually a polynomial. Also P_e divides both M_0' and M_e''.

Finally we obtain polynomials S' and S''. In pQ'/S, Q''/S, take out the zero at $p = 0$ and $p = \infty$ respectively in the usual way. Then we may write

$$\frac{S}{pQ'} = \frac{a}{p} + \frac{S'}{Q'}, \qquad \frac{S}{Q''} = bp + \frac{S''}{Q''}, \qquad a > 0, \qquad b > 0.$$

---

*We are here using the following result whose proof is straightforward. If R and S are any polynomials with real coefficients and $0 \ll R \ll S$, $0 \ll S_1$, $0 \ll S_2$, $S_1 + S_2 = S$ then there exist R_1 and R_2 such that $0 \ll R_1 \ll S_1$, $0 \ll R_2 \ll S_2$, $R_1 + R_2 = R$.

Here S'/Q' and S''/Q'' are positive real functions. The relations

$$S = aQ' + pS' = bpQ'' + S'' \tag{I.10}$$

hold, and together with (I.7), (I.8) and vi(a) imply that P_e divides both S'_e and S''_0. Also we have **Re** $(p = i\omega)$ $[S'/Q']$ = **Re** $(p = i\omega)$ $[S/pQ']$; and since by (I.5) and (I.2)

$$\underset{p=i\omega}{\mathbf{Re}}\left[\frac{pQ'}{S}\right] = \underset{p=i\omega}{\mathbf{Re}}\left[\frac{pR'}{T}\right] = P_e^2(i\omega)\cdot\frac{(\beta'_0\omega^{2r} + \beta'_1\omega^{2r-2} + \cdots + \beta'_{r-1}\omega^2)}{|\,S(i\omega)\,|^2},$$

it follows that

$$\underset{p=i\omega}{\mathbf{Re}}\left[\frac{Q'}{S'}\right] = P_e^2(i\omega)\cdot\frac{(\beta'_0\omega^{2r-2} + \beta'_1\omega^{2r-4} + \cdots + \beta'_{r-1})}{|\,S'(i\omega)\,|^2}.$$

Similarly, we get

$$\underset{p=i\omega}{\mathbf{Re}}\left[\frac{Q''}{S''}\right] = P_e^2(i\omega)\cdot\frac{(\beta''_1\omega^{2r-2} + \beta''_2\omega^{2r-4} + \cdots + \beta''_r)}{|\,S''(i\omega)\,|^2}.$$

By the last two equations S' and S'' are both of degree $2s + r - 1$. We have now obtained the required pairs of functions M'/Q', Q'/S' and M''/Q'', Q''/S''. The preceding proof contains a verification that they satisfy all the conditions of Lemma 1 except (iv), (with r replaced by $r - 1$). In particular the first pair satisfies vi(b) and the second pair vi(a). To establish (iv) for Q' and S' note that by (I.10) a common factor $p^2 + \alpha^2$ of Q' and S' must divide S and hence P'_e. But then by the first equation of (I.5) it must divide L'_0 which is impossible. A similar argument proves (iv) for Q'' and S''. This completes the proof of Lemma 1.

The reduction procedure of Lemma 1 may be continued for each pair of functions until we reach pairs for which $r = 0$. Let then $A = M/Q$ and Q/S satisfy the conditions of Lemma 1. with $r = 0$. We now investigate what this implies as regards the form of M, Q and S. First suppose condition vi(a) holds. We may take $Q = P_e + Q_0$. Since S is of degree $2s$ and P_e divides S_0 we must have $S_0 \equiv 0$. But then using (v) we find that $P_e(i\omega)\,S_e(-i\omega)$ is divisible by $P_e^2(i\omega)$. Thus $S = S_e = dP_e$, $d > 0$ and without loss of generality we may take $d = 1$. In view of (i), (iv) and (v), Q is Hurwitz. Also by vi(a) $M_e = \alpha P_e$, $0 \le \alpha \le 1$.

Next suppose condition vi(b) holds. Then $Q_0 \equiv 0$, and by (ii), $M_0 \equiv 0$. We may take $S = P_e + S_0$, and as in the preceding case show that $Q = Q_e = P_e$, and that S is Hurwitz.

Appendix II

Lemma 2. *Let* $A(p) = G/H$ *where** $0 \ll G \ll H$, $H = P_eJ$, J *Hurwitz, and suppose* $A(p)$ *satisfies the residue condition* (ii) *of Theorem* 1. *Then functions* M/Q *and* Q/S *can be constructed having the properties stated in Lemma* 1 *and such that* $A(p) = M/Q$.

Proof:** If P_e does not divide J_e then we begin by constructing a Hurwitz polynomial F such that $[FJ]_e$ is divisible by P_e. Let $J_e = P'_e J^*_e$ where $P'_e = \Pi'(p^2 + \omega_k^2)$ contains the common zeros of P_e and J_e.

---

*The polynomial P_e is defined by (2.7).

**Our construction will yield a pair of functions which satisfies vi(a) of Lemma 1. By a parallel procedure we can get a pair which satisfies vi(b).

Let $i\omega_a$ be a zero not included in P'_e and suppose $J(i\omega_a) = \alpha + i\beta$, $\alpha \neq 0$. Form the LC-impedance $\eta F'_0/F'_e$ with η constant and F'_0 and F'_e relatively prime polynomials subject to the following further conditions: (i) P'_e divides F'_0; (*iia*). If $\beta = 0$, $p^2 + \omega_a^2$ divides F'_e, $\eta = 1$; (iib). If $\beta \neq 0$ choose the remaining zeros of F'_0 and the zeros of F'_e such that sgn $[iF'_e(i\omega_a)/F'_0(i\omega_a)] =$ sgn (β/α). Take $\eta = i\alpha F'_e(i\omega_a/\beta F'_0(i\omega_a)$.

It follows that $\eta > 0$ and $F' = F'_e + \eta F'_0$ is Hurwitz. One readily verifies that $[F'J]_e$ is divisible by $(p^2 + \omega_a^2) P'_e$. The process is continued with $F'J$ to determine an F'', etc. until all of the ω_k are exhausted. Then F may be taken as the product* $F' F'' \cdots$.

Now consider $A(p) = GF/P_eJF$. Since P_e divides $[JF]_e$ it follows from the residue condition (ii) of Theorem 1, that P_e divides $[GF]_e$. We may therefore suppose that function $A(p)$ of Lemma 2 has already been prepared so that $G_e = P_e G_e^*$, $J_e = P_e J_e^*$.

Of course

$$0 \ll G_0 \ll P_e J_0 , \qquad 0 \ll G_e = P_e G_e^* \ll P_e J_e . \tag{II.1}$$

If also $0 \ll G_e^* \ll J_e$, then we have one of the special forms of the transfer function required for Lemma 1. In the contrary case, we will bring this about by means of a suitable common factor. Consider the polypomials G_e^* and $J_e - G_e^* = (P_eJ_e - G_e)/P_e$. In view of (II.1), both of these even polynomials have positive leading coefficients and have no positive real roots. Hence by [5, Appendix B] (with p replaced by p^2) there exists a polynomial $E_e = \Pi(p^2 + \delta_i^2)$ with the δ_i real, distinct and different from the $\pm\, \omega_k$ and zero, such that

$$0 \ll E_e G_e^* \ll E_e J_e . \tag{II.2}$$

We now choose as our common factor the Hurwitz polynomial

$$B = P_e P'_0 + E_e E'_e$$

where P'_0 and E'_e are any polynomials making $P_eP'_0/E_eE'_e$ an LC impedance with $P_eP'_0$ relatively prime to $E_eE'_e$.

Now define

$$M = GB, \qquad Q^* = JB, \qquad Q = P_e Q^*,$$

so that $A(p) = M/Q$. Then we have

$$M_e = P_e[G_0P'_e + G_e^*E_eE'_e], \qquad Q_e^* = P_eP'_0J_0 + P_eJ_e^*E_eE'_e .$$

Hence P_e divides Q_e^* and by (II.1) and (II.2), $0 \ll M_e/P_e = M_e^* \ll Q_e^*$. Thus M/Q is in the form required by Lemma 1.

We will now construct the function Q/S mentioned in Lemma 2. We note that Q^* is Hurwitz and let its degree be r. Choose any positive numbers δ'_k $(k = 1, 2, \cdots, s)$ and β_k $(k = 1, 2, \cdots, r)$, and form the positive real rational function**

$$\frac{S}{Q} = \sum_{k=1}^{s} \frac{\delta'_k p}{p^2 + \omega_k^2} + \frac{V}{Q^*} \tag{II.3}$$

where

$$\operatorname*{Re}_{p=i\omega} \left[\frac{V}{Q^*}\right] = \frac{\beta_0\omega^{2r} + \beta_1\omega^{2r-2} + \cdots + \beta_r}{|\, Q^*(i\omega)\,|^2} = \frac{\theta(\omega)}{|\, Q^*(i\omega)\,|^2} . \tag{II.4}$$

---

*A more complicated algorithm which handles all of the ω's simultaneously and results in a polynomial F of much lower degree can also be given.

**Cf. footnote regarding Eq. (I.2).

Then S and Q are both of degree $r + 2s$. It follows from (II.3) that S and Q have no common pure imaginary zeros. Also since P_e divides Q_e^* and

$$S = P_e Q^* \sum \frac{\delta_k' p}{p^2 + \omega_k^2} + P_e$$

we have that $S_0 = P_e S_0^*$. From (II.3) and (II.4) we get

$$\operatorname*{Re}_{p=i\omega} \left[\frac{Q}{S} \right] = \frac{P_e^2(i\omega)\theta(\omega)}{| S(i\omega) |^2} .$$

Thus Q/S has all the properties required by Lemma 1.

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*Since the manuscript was submitted a number of further papers by Weinberg have appeared in the Proc. I.R.E. and the J. Appl. Phys. These are based on sections of [9].