ON SOME EFFECTS OF VELOCITY DISTRIBUTION IN ELECTRON STREAMS*

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Abstract. Based on the Liouville theorem an integral equation is obtained for the solution of electron beam problems having a velocity spread. Assuming a rectangular velocity distribution (which is justified later) the integral equation is solved by Laplace Transforms to obtain the solution of the problems of small-signal velocity modulation in a drifting electron stream, and a drifting electron stream initially possessing full shot noise in each velocity class.

It is shown that one obtains from the above integral equation the same results as those given by the Llewellyn equations for the case of a single valued velocity stream. The problem of finite but narrow velocity spread in the case of an accelerated electron stream is briefly considered.

1. Introduction. To investigate the consequences of treating the electron stream as a plasma one can use either one of the two following methods:

1) treat the electron stream as being composed of a number of beams whose velocities are discrete, or

2) use the distribution function treatment.

The first method has been used by many; quite recently Bohm and Gross [1] used the same to discuss many characteristics of the plasma. The second method has been used by Vlasov [2], Landau [3], and recently by Watkins [4] to discuss the behavior of a plasma, the effect of a velocity distribution on noise in electron streams, etc. We will use below the distribution function method.

The plasma is completely described by a distribution function of $f(\mathbf{r}, \mathbf{v}, t)$ such that $-f(\mathbf{r}, \mathbf{v}, t)/e$ gives the average number of electrons in a small range of position and velocity at a time t in the phase space whose coordinates are position $\mathbf{r}$ and velocity $\mathbf{v}$. The behavior of the distribution function is completely specified by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_r f + \frac{d\mathbf{v}}{dt}\cdot\nabla_v f = \left(\frac{\delta f}{\delta t}\right)_{\text{collisions}}, \tag{1}$$

where $d\mathbf{v}/dt$ is determined by the interparticle forces and any external impressed fields. The physical assumptions involved in the above are based on the following [5]. The forces acting on a particle in a plasma can be divided into two parts. One is the short-range force acting on a particle when a close collision is experienced by it during which there is a heavy momentum transfer, and this is taken care of by the term on the right hand side in (1). The other part is the long-range coulomb force, due to the other particles of the system. In the second type of collision, the momentum transfer is small. At any instant a single electron is engaged in a many body collision, and in (1) the effect of the same is taken into account as a smoothed out force. This latter force depends on the distribution function itself and gives rise to many of the plasma characteristics.

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That it is so has been shown by Pines and Bohm [5]. In the type of plasmas encountered in vacuum tubes the collision term can be neglected, since the collisions would have mainly a minor damping effect, although the same procedure cannot be followed in the case of a dense plasma. In (1) the value of $d\mathbf{v}/dt$ can be obtained by using Maxwell's equations. Also, in the above equation any currents due to the motion of positive ions is neglected. To take it into account one can introduce another distribution function $g(\mathbf{r}, \mathbf{v}, t)$ and write an equation similar to (1). If one assumes that positive ions are uniformly smeared out one need not consider $g(\mathbf{r}, \mathbf{v}, t)$.

Equation (1) and the corresponding Maxwell's equations form a set of coupled, non-linear integro-differential equations which are extremely difficult to solve.

2. Formulation of the problem for a diode type region for small signals. To simplify the mathematics we consider below only one dimensional electron streams.

Consider a parallel plane diode region across which there is impressed a d-c accelerating potential and a small a-c potential. The distribution function f satisfies the specialized Boltzmann equation neglecting collisions

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \eta E \frac{\partial f}{\partial v} = 0, \tag{2}$$

where x is the space coordinate normal to the diode planes, v is the x directed velocity, $\eta = e/m$ for an electron and E the electric field.

We can write

$$E = -\frac{\partial V_0}{\partial x} + E_1 e^{i\omega t}, \tag{3}$$

where V_0 is the d-c potential and E_1 is the amplitude of the a-c electric field.

Let

$$f(x, v, t) = f_0(x, v) + f_1(x, v)e^{i\omega t}. \tag{4}$$

It is assumed that $f_1 \ll f_0$ and $E_1 \ll E_0$. One can then split Eq. (2) into two parts as shown below, one containing the d-c terms only, and the other a-c terms as well.

$$v \frac{\partial f_0}{\partial x} + \eta \frac{\partial V_0}{\partial x} \frac{\partial f_0}{\partial v} = 0, \tag{5}$$

$$j\omega f_1 + v \frac{\partial f_1}{\partial x} + \eta \frac{\partial V_0}{\partial x} \frac{\partial f_0}{\partial v} = \eta E_1 \frac{\partial f_0}{\partial v} \tag{6}$$

dropping the products of a-c terms.

It is evident from the above that f_0 is a function of u only, where

$$u^2 = v^2 - 2\eta V_0 . \tag{7}$$

The d-c potential V_0 is measured with respect to the cathode in the case of temperature limited flow and potential minimum in case of space-charge limited flow. Similarly the velocity u is measured at the cathode or the potential minimum depending upon the fact that the flow is either temperature limited or space-charge limited.

Changing then the variables from x and v to x and u we have,

$$v\,dv = u\,du + \eta \frac{\partial V_0}{\partial x} dx, \qquad dx = dx,$$

$$\frac{\partial}{\partial v} = \frac{v}{u} \frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \eta \frac{1}{u} \frac{\partial V_0}{\partial x} \frac{\partial}{\partial u}. \tag{8}$$

We also have from Maxwell's equations

$$\frac{\partial E_1}{\partial x} = -\frac{1}{\epsilon_0}\int_{-\infty}^{\infty} f_1\, dv, \tag{9}$$

where ϵ_0 is the dielectric constant of free space. After changing the variables according to the above scheme, substitution of the same into (6) yields

$$j\frac{\omega}{v} f_1 + \frac{\partial f_1}{\partial x} = \eta E_1 u^{-1}\frac{\partial f_0}{\partial u}. \tag{10}$$

Integrating the above we obtain

$$\begin{aligned} f_1(x, u) = {} & f_1(0, u) \exp\left[-j\omega\int_0^x [v(x', u)]^{-1}\, dx'\right] \\ & + \int_0^x \eta u^{-1}\frac{\partial f_0}{\partial u} E_1(x') \exp\left[-j\omega\int_{x'}^x [v(\xi, x)]^{-1}\, d\xi\right] dx'. \end{aligned} \tag{11}$$

Hence, $f_1(x, u)$ can be determined from the above integral equation (11), knowing $v(x, u)$, $f_1(0, u)$, $f_0(u)$ and $E_1(x)$. The only boundary condition needed is the value of $f_1(0, u)$. We can also obtain an integral equation for the current as follows:

$$i_1(x) = \int_0^\infty v f_1(x, v)\, dv = \int_0^\infty u f_1(x, u)\, du \tag{12}$$

if the diode is open-circuited for a-c* the total a-c current density is zero, i.e.,

$$\frac{\partial}{\partial x}[i_1 - j\omega\epsilon_0 E_1] = 0,$$

yielding

$$E_1 = \frac{i_1}{j\omega\epsilon_0}. \tag{13}$$

Hence, we obtain after substitution the following integral equation satisfied by the current**

$$\begin{aligned} i_1(x) = {} & \int_0^\infty u f_1(0, u) \exp\left[-j\omega\int_0^x [v(x', u)]^{-1}\, dx'\right] du \\ & + \frac{\eta}{j\omega\epsilon_0}\int_0^x i_1(x')\, dx' \int_0^\infty \frac{\partial f_0}{\partial u} \exp\left[-j\omega\int_{x'}^x [v(\xi, u)]^{-1}\, d\xi\right] du. \end{aligned} \tag{14}$$

This is a non-homogeneous integral equation of the Volterra type whose exact solution in a general case can be found only by numerical integration.

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*An examination of the Llewellyn-Peterson equations makes it evident that there are a large number of problems where the condition of total a-c current is equal to zero gives the correct result for the a-c quantities in the electron stream.

**It has been found by the author through a private communication that the integral equations (11) and (14) have also been obtained independently by L. R. Walker of the Bell Telephone Laboratories, and in a somewhat different form by D. A. Watkins presently with the Electronics Research Laboratory of Stanford University.

3. Drifting electron stream: *Simplified integral equation.* For a drifting electron stream one can write the following simplified forms of (11) and (14):

$$f_1(x, u) = f_1(0, u) \exp\left(-\frac{j\omega x}{u}\right) + \int_0^x \eta u^{-1} \frac{\partial f_0}{\partial u} E_1(x') \exp\left[-\frac{j\omega(x - x')}{u}\right] dx', \tag{15}$$

$$i_1(x) = \int_0^\infty u f_1(0, u) \exp\left(-\frac{j\omega x}{u}\right) du + \frac{\eta}{j\omega\epsilon_0} \int_0^x i_1(x')\, dx' \int_0^\infty \frac{\partial f_0}{\partial u} \exp\left[-\frac{j\omega(x - x')}{u}\right] du \tag{16}$$

which are to be interpreted as follows. The coordinate x is measured from the entrance plane of the drift region, and u is the velocity at the entrance plane.

Drifting electron stream with small-signal velocity modulation. We will use below the integral equation (16) for the solution of a problem treated by Watkins [4] in a different manner.

Instead of using a Maxwellian distribution of velocities as done elsewhere [4] we use below a rectangular type distribution for $f_0(u)$; the justification for this is given in Appendix I making use of the Tchebycheff inequality.

For the computation of $f_1(0, u)$ the model chosen is as follows. The stream is allowed to pass through two parallel grids across which there appears a voltage $V_1 e^{j\omega t}$. It is also assumed that no transit time effects occur and the grids are perfectly permeable.

To the left of the gap, we have

$$f_0(u) = \frac{\rho_0}{w} [S(u - u_s) - S(u - u_s - w)], \tag{17}$$

where ρ_0 is the d-c charge density at the entrance plane, w the width of the distribution function, $S(u - u_s)$ equals unity for $u \geq u_s$ and zero for $u < u_s$, u_s being the smallest velocity of an electron in the drift region defined by its d-c potential. At the right of the gap, we have

$$f_0 + f_1 e^{j\omega t} = \frac{\rho_0}{w} \{s[u - (u_s^2 + 2\eta V_1 e^{j\omega t})^{1/2}] - S[u - (u_s^2 + w^2 + 2\eta V_1 e^{j\omega t})^{1/2}]\}. \tag{18}$$

By expanding the step function in a Taylor series and retaining only first order terms, we obtain

$$f_1(0, u) \cong -\eta \frac{V_1}{u} \frac{\rho_0}{w} \delta(u - u_s) + \frac{\eta V_1 \rho_0}{(u_s + w)w} \delta(u - u_s - w), \tag{19}$$

where δ denotes a Dirac delta function having the properties $\delta(x - x_1) = 1$ for $x = x_1$, and $\delta(x - x_1) = 0$ for $x \neq x_1$.

Now we have to solve the integral equation

$$i_1(x) = \int_0^\infty u f_1(0, u) \exp\left(-\frac{j\omega x}{u}\right) du + \frac{\eta}{j\omega\epsilon_0} \int_0^x i_1(x')\, dx' \int_0^\infty \frac{\partial f_0}{\partial u} \exp\left[-\frac{j\omega(x - x')}{u}\right] du. \tag{16}$$

To solve the above by Laplace Transforms, introduce the Laplace transform $i_1(s)$ defined by

$$i_1(s) = \int_0^\infty i_1(x)e^{-sx}\,dx, \qquad \text{real part of } s > 0. \tag{20}$$

Taking now the transform of (16) and using the rule for the transform of a convolution we have

$$i_1(s) = \left[\int_0^\infty \frac{uf_1(0,u)}{s + j(\omega/u)}\,du\right]\left[1 - \frac{\eta}{j\omega\epsilon_0}\int_0^\infty \frac{\partial f_0}{\partial u}\frac{1}{s + j(\omega/u)}\,du\right]^{-1}, \tag{21}$$

$$i_1(x) = \frac{1}{2\pi j}\int_{\beta - j\infty}^{\beta + j\infty} i_1(s)e^{sx}\,ds. \tag{22}$$

Here, β is so chosen that all the singularities of $i_1(x)$ lie in the region where the real part of $s < \beta$.

We also have,

$$\frac{\partial f_0}{\partial u} = \frac{\rho_0}{w}\{\delta(u - u_s) - \delta(u - u_s - w)\}. \tag{23}$$

Substituting (19) and (23) into (21) and intergrating, we obtain

$$i_1(s) = \frac{j\omega V_1\rho_0\eta}{u_s^2(1 + w/u_s)}\left[\left(s + \frac{j\omega}{u_s}\right)\left(s + \frac{j\omega}{u_s + w}\right) + \frac{\omega_P^2}{u_s^2(1 + w/u_s)}\right], \tag{24}$$

where $\omega_P^2 = \eta\rho_0/\epsilon_0$. One then has to find the poles of $i_1(s)$. These are

$$s_{1,2} = -\frac{j}{2}\left[\frac{\omega}{u_s} + \frac{\omega}{u_s(1 + w/u_s)}\right] \pm \frac{j}{2}\left\{\left[\frac{\omega}{u_s} - \frac{\omega}{u_s(1 + w/u_s)}\right]^2 + \frac{4\omega_P^2}{u_s^2(1 + w/u_s)}\right\}^{1/2}. \tag{25}$$

Then, $i_1(x) = Ae^{s_1x} + Be^{s_2x}$, i.e.,

$$i_1(x) = j\omega V_1\rho_0\eta[e^{s_1x} - e^{s_2x}]\left\{j\left[u_s^2\left(1 + \frac{w}{u_s}\right)\right]\right.$$
$$\left.\cdot\left[\left(\frac{\omega}{u_s} - \frac{\omega}{u_s(1 + w/u_s)}\right)^2 + \frac{4\omega_P^2}{u_s^2(1 + w/u_s)}\right]^{1/2}\right\}^{-1}. \tag{26}$$

Neglecting terms of the order greater than one in w/u_s, (which is applicable to a narrow distribution), we obtain after some manipulation,

$$i_1(x) \cong \frac{j\eta V_1}{u_s}\rho_0\left(1 - \frac{w}{u_s}\right)\left(\frac{\omega}{\omega_P}\right)\left\{\sin\left[\left(1 - \frac{w}{u_s}\right)\frac{\omega_P x}{u_s}\right]\right\}\exp\left[-j\omega\left(1 - \frac{w}{u_s}\right)\frac{x}{u_s}\right]. \tag{27}$$

Notice that as we let w go to zero, we obtain a result identical with that from the analyses of Hahn [6] and Ramo [7].

For the sake of comparison the result of Watkins [4] is quoted below in (28).

$$i_1(x) \cong -\frac{j\eta V_1}{u_s}\frac{I_0}{u_s}\left(1 - \frac{3}{4\sigma}\right)\frac{\omega}{\omega_P}\left\{\sin\left[\left(1 - \frac{3}{4\sigma}\right)\frac{\omega_P x}{u_s}\right]\right\}\exp\left[-j\omega\left(1 - \frac{1}{2\sigma}\right)\frac{x}{u_s}\right], \tag{28}$$

where, $-I_0/u_s = \rho_0$ of the present analysis, see Eq. (27), and $\sigma = eV_0/kT_c$, k being the Boltzmann constant and T_c the cathode temperature.

Equation (28) has been obtained using a Maxwellian distribution and a series method for solution. One can see from Eqs. (27), (28) and Appendix I that the current amplitude obtained with a rectangular distribution is a little less than the one assuming a Maxwellian velocity distribution.

Drifting electron stream initially possessing full shot noise. We next consider another simple example to determine the effect of the thermal velocity spread on noise. Consider a drifting electron stream initially possessing full shot noise in each velocity class, or electrons having a velocity in a small interval around some particular velocity. For this computation one needs to find the convection current produced by shot noise in each velocity class, and assuming that the noises due to different velocity classes add in a mean square manner we obtain the total mean square noise current. This problem has been treated by Watkins [4] and by Pierce [8] in different manners from the following.

Consider then the ith velocity class, or electrons having a velocity in a small interval around u_i.

$$f_1(0, u_i) = [2eu_i\rho_0(u_i)\ \Delta f]^{1/2}u^{-1}\ \delta(u - u_i), \tag{29}$$

i.e., we assume that the input is pure shot noise. The condition on the above expression is $u_s \leq u_i < u_s + w$, assuming again a rectangular type distribution. We have to solve the equation

$$i_1(s) = \left[\int_0^\infty \frac{uf_1(0, u)}{s + j(\omega/u)}\, du\right]\left[1 - \frac{\eta}{j\omega\epsilon_0}\int_0^\infty \frac{\partial f_0}{\partial u}\frac{1}{s + j(\omega/u)}\, du\right]^{-1} \tag{21}$$

$$= \left\{\int_0^\infty \frac{u[2eu_i\rho_0(u_i)\ \Delta f]^{1/2}}{s + j(\omega/u)}\, u^{-1}\ \delta(u - u_i)\, du\right\}.$$

$$\cdot\left\{1 - \frac{\eta}{j\omega\epsilon_0}\int_0^\infty \frac{\rho_0\{\delta(u - u_s) - \delta(u - u_s - w)\}}{w[s + j(\omega/u)]}\, du\right\}^{-1} \tag{30}$$

$$= \frac{(2eu_i\rho_0(u_i)\ \Delta f)^{1/2}}{s + j(\omega/u_i)}\left\{1 + \frac{\eta}{j\omega\epsilon_0}\frac{j\omega\rho_0}{u_s(u_s + w)}\left[\left(s + \frac{j\omega}{u_s}\right)\left(s + \frac{j\omega}{u_s + w}\right)\right]^{-1}\right\}^{-1}.$$

Define now,

$$[2eu_i\rho_0(u_i)\ \Delta f]^{1/2} = i_i\ . \tag{31}$$

Then we have

$$\frac{i_1(s)}{i_i} = \left\{\left(s + \frac{j\omega}{u_i}\right)\left[1 + \left\{\frac{\omega_P^2}{u_s^2(1 + w/u_s)}\right\}\left\{\left(s + \frac{j\omega}{u_s}\right)\left(s + \frac{j\omega}{u_s + w}\right)\right\}^{-1}\right]\right\}^{-1} \tag{32}$$

$$= \left[(s - s_1)(s - s_2) - \left(\frac{\omega_P}{u_s}\right)^2\left(1 - \frac{w}{2u_s}\right)\right][(s - s_1)(s - s_2)(s - s_3)]^{-1}. \tag{33}$$

At this point to simplify the computations we neglect terms of order greater than one in w/u_s and write

$$s_{1,2} \cong -j\frac{\omega}{u_s}\left(1 - \frac{w}{u_s}\right) \pm j\frac{\omega_P}{u_s}\left(1 - \frac{w}{u_s}\right)$$

$$\cong -j\frac{\omega}{u_s} \pm j\frac{\omega_P}{u_s},$$

$$s_3 = -j\frac{\omega}{u_i}. \tag{34}$$

Define now

$$(u_i - u_s)/u_i = \epsilon . \tag{35}$$

Then the inverse Laplace transformation is taken and after making the indicated approximations, one obtains

$$i_1(x) \cong -i_i[\exp - j\omega x/u_s]\{\cos(\omega_P x/u_s) + j\epsilon(\omega/\omega_P)\sin(\omega_P x/u_s)\}. \tag{36}$$

The minus sign appears in the above because of the present notation regarding ρ_0 and ω_p^2. From (36), we find

$$|i_1(x)|^2 \cong i_i^2\{\cos^2(\omega_P x/u_s) + (\omega/\omega_P)^2\epsilon^2\sin^2(\omega_P x/u_s)\}. \tag{37}$$

Expression (37) is the noise current at x due to an injected noise current in the ith velocity class at $x = 0$. We have

$$i_i^2 = 2eI_i\,\Delta f, \tag{38}$$

where $I_i = \rho_0(u_i)$. u_i is the current carried by the ith velocity class.

We have assumed at the start that the noise due to electrons of different velocity classes are independent, i.e., they add in a mean square manner. As is done also by Pierce [8] we can write

$$|i|^2 \cong 2eI_0\,\Delta f\{\cos^2(\omega_P x/u_s) + \langle\epsilon^2\rangle(\omega/\omega_P)^2\sin^2(\omega_P x/u_s)\} \tag{39}$$

where, $I_0 = \sum_i I_i$ and

$$\langle\epsilon^2\rangle = (1/I_0)\sum_i \epsilon_i^2 I_i = (1/I_0u_s^2)\sum_i I_i(u_i - u_s)^2, \tag{40}$$

i.e., $\langle\epsilon^2\rangle$ is of the order of the mean square velocity distribution divided by the mean square velocity. Notice that the above result agrees with that of Watkins [4].

The minima of the above expression (39) occur at

$$\omega_P x/u_s = \pi/2 + n\pi \tag{41}$$

and the minimum value is

$$\langle i^2\rangle_{\min} \cong \langle\epsilon^2\rangle(\omega/\omega_P)^2 2eI_0\,\Delta f. \tag{42}$$

We next note that Pierce [8] has shown that one obtains the same result when one treats the above problem using the Llewellyn-Peterson equations. The assumption in the above procedure is that there is no correlation between velocity and current fluctuations.

The author [9] has shown recently that the above assumption is valid provided one considers that

a) The fluctuation quantity associated with an electron is independent of the rest of the fluctuations,

b) the fluctuation quantities are identically distributed, and

c) the process is stationary in time, and observations are made after a steady state is reached.*

4. Solution of integral equation in an accelerating region—single velocity stream. It is desired to obtain the Llewellyn form of equations for an accelerating region. To

*For further details see Ref. 9.

do this one has to assume that all electrons at a given plane have the same velocity at any instant. We have

$$i_1(x) = \int_0^\infty u f_1(0, u) \exp\left[-j\omega \int_0^x (v(x', u))^{-1} dx'\right] du$$
$$+ \frac{\eta}{j\omega\epsilon_0} \int_0^x i_1(x')\, dx' \int_0^\infty \frac{\partial f_0}{\partial u} \exp\left[-j\omega \int_{x'}^x (v(\xi, u))^{-1} d\xi\right] du. \qquad (14)$$

With

$$\tau = \int_0^x [v(x', u)]^{-1} dx',$$

we can write (14) as

$$i_1(x) = \int_0^\infty u f_1(0, u) \exp(-j\omega\tau)\, du + \int_0^x i_1(x')\, dx' \int_0^\infty \frac{\eta}{j\omega\epsilon_0} \frac{\partial f_0}{\partial u} \exp[-j\omega(\tau - \tau')]\, du,$$

where it is understood that $\tau = \tau(x, u)$ and, $\tau' = \tau'(x', u)$. By a proper choice of $f_1(0, u)$ one can consider either current or velocity modulation or both simultaneously.

Input current modulation alone. The above integral equation can be written in the form

$$i_1(x) = i_1(0) \exp(-j\omega\tau) + \int_0^x i_1(x') K(x, x')\, dx', \qquad (43)$$

where

$$K(x, x') = \int_0^\infty \left(\frac{\eta}{j\omega\epsilon_0}\right) \frac{\partial f_0}{\partial u} \exp[-j\omega(\tau - \tau')]\, du \qquad (43a)$$

is called the kernel. The solution of (43) can be written as

$$i_1(x) = i_1(0) \exp(-j\omega\tau) + \int_0^x i_1(0) L(x, x') \exp[-j\omega\tau']\, dx', \qquad (44)$$

where $L(x, x')$ is called the resolving kernel; it satisfies the following integral equation

$$L(x, x') = K(x, x') + \int_{x'}^x K(x'', x) L(x', x'')\, dx'' \qquad (45)$$

as follows from the Fredholm theory [10] of integral equations. Thus, to solve the integral equation (14), one has to solve (45) to obtain the resolving kernel $L(x, x')$.

Now, for the Llewellyn approximation we write

$$f_0(u) = u^{-1} I_0\, \delta(u - u_i), \qquad (46)$$

where I_0 is the d-c current and u_i is the entering velocity, $\delta(u - u_i)$ is a Dirac delta function.

We then can evaluate $K(x, x')$ as follows:

$$K(x, x') = \frac{\eta}{j\omega\epsilon_0} \int_0^\infty \frac{\partial f_0}{\partial u} \exp[-j\omega(\tau - \tau')]\, du$$
$$= -\frac{\eta}{j\omega\epsilon_0} \int_0^\infty f_0 \left\{\frac{\partial}{\partial u} \exp[-j\omega(\tau - \tau')]\right\} du.$$

Substituting then the value of f_0 we have

$$K(x, x') = -\left[\frac{\eta I_0}{j\omega\epsilon_0 u_i}\right] \frac{\partial}{\partial u_i} \{\exp\,[-j\omega\tau(x, u_i) + j\omega\tau'(x', u_i)]\} \tag{47}$$

since the delta function vanishes at the two limits.

Next, one has to find the resolving kernel $L(x, x')$. At this stage we note that if

$$\phi(x) = \psi(x) + \int_0^x \phi(x')K(x, x')\,dx', \tag{48}$$

then,

$$\phi(x) = \psi(x) + \int_0^x \psi(x')L(x, x')\,dx', \tag{49}$$

$L(x, x')$ being independent of the form of $\phi(x)$ and $\psi(x)$. We can therefore use the resolving kernel obtained by Knipp [11]:

$$L(x, x') = -\{\eta I_0[\tau(x, u_i) - \tau'(x', u_i)]\exp\,[-j\omega(\tau - \tau')]\}[\epsilon_0 v(x, u_i)v(x', u_i)]^{-1}. \tag{50}$$

Going back now to the integral equation (43) we have

$$\begin{aligned} i_1(x) &= i_1(0)\exp\,(-j\omega\tau) \\ &\quad - \int_0^x i_1(0)\{\exp\,(-j\omega\tau')\}\eta I_0(\tau - \tau')\{\exp\,[-j\omega(\tau - \tau')]\}[\epsilon_0 v(x, u_i)v(x', u_i)]^{-1}\,dx' \\ &= i_1(0)\{\exp\,(-j\omega\tau)\}\left[1 - \frac{\eta I_0}{\epsilon_0 v(x, u_i)}\int_0^x (\tau - \tau')[v(x', u_i)]^{-1}\,dx'\right]. \end{aligned} \tag{51}$$

We note that

$$\frac{dx'}{v(x', u_i)} = d\tau'. \tag{52}$$

Integrating (51) we obtain

$$i_1(x) = i_1(0)[1 - (\eta I_0/2\epsilon_0)(\tau^2/v(x, u_i))]\exp\,(-j\omega\tau). \tag{53}$$

Equation (53) gives precisely the same result as the Llewellyn [12] equations for initial current modulation alone. Equation (53) can be written in the form

$$i_1(x) = E^* i_1(0)\exp\,(-j\omega\tau) \tag{54}$$

as in the notation of Llewellyn.

Input velocity modulation alone. We have again for the Llewellyn approximation

$$f_0(u) = u^{-1}I_0\,\delta(u - u_i).$$

We assume that the velocity gap is rather narrow. The above equation represents the conditions to the left of the gap; to the right of the gap we have

$$f_0 + f_1 e^{j\omega t} = u^{-1}I_0\,\delta(u - u_i - v_1(0)e^{j\omega t}), \tag{55}$$

where $v_1(0)$ is the amplitude of the velocity modulation. To obtain the Llewellyn form of equations, we now have to consider that the entrance plane of the electron stream is permeable, and $v_1(0) \ll u_i$ (the a-c velocity modulation is small in comparison with the entrance d-c velocity).

Expanding now the delta function in (55) and retaining only the first order terms, we obtain

$$f_1(0, u_i) \cong -v_1(0)u^{-1}I_0 \ \delta'(u - u_i), \tag{56}$$

where $\delta'(u - u_i)$ is the derivative of the delta function with respect to u.

We use again the same integral equation (43) and the resolving kernel $L(x, x')$ given by (50). Making use of (56), (43) and (50) as before we obtain

$$i_1(x) = v_1(0)[v(x, u_i)]^{-1}j\omega\tau I_0 \exp(-j\omega\tau) \ . \tag{57}$$

Equation (57) can also be written as

$$i_1(x) = v_1(0)\eta^{-1}j\omega\epsilon_0 u_i L(x, 0) \ . \tag{58}$$

Equation (57) gives precisely the same result as the Llewellyn [12] equations for initial velocity modulation alone, i.e.,

$$i_1(x) = F^* v_1(0) \exp(-j\omega\tau) \tag{59}$$

as in the notation of Llewellyn. E^* and F^* can be expressed in the forms given in equations (53) and (57) after eliminating the space-charge factor ζ from the Llewellyn coefficients by the use of the equation

$$\eta \frac{I_0}{\epsilon_0} = [u_i + v_0(x)]2\zeta[\tau(x, u_i)]^{-2}; \tag{60}$$

$v_0^2(x) = 2\eta(V_0(x)$, and $V_0(x)$ is the d-c potential at x.

5. The problem of an accelerated stream with a narrow velocity distribution. The successful solution of the simple problems considered in the preceding sections justifies the hope that it will be possible to solve more general problems. One can for instance make use of the integral equation approach for the case of a finite but narrow distribution of velocities in an accelerating region as follows. Consider a rectangular velocity distribution,

$$f_0(u) = \frac{I_0}{uw}[S(u - u_i) - S(u - u_i - w)], \tag{61}$$

where w is the width of the rectangular distribution, $S(u - u_i)$ and $S(u - u_i - w)$ are step functions, i.e.,

$$\begin{aligned} S(u - u_i) &= 1 \quad \text{for} \quad u \geq u_i \ , \\ &= 0 \quad \text{for} \quad u < u_i \ . \end{aligned}$$

For the case $w \ll u_i$, one then obtains the following expression for $K(x, x')$ from (43a):

$$K(x, x') \cong -\frac{\eta I_0}{j\omega\epsilon_0 u_i}\left(1 - \frac{w}{u_i}\right)\frac{\partial}{\partial u_i}\{\exp[-j\omega\tau(x, u_i) + j\omega\tau'(x', u_i)]\} + \Delta, \tag{62}$$

where Δ, a correction term in (62) can be written approximately as

$$\Delta \cong \frac{\eta I_0}{j\omega\epsilon_0 u_i^2}\left(1 - \frac{w}{u_i}\right)\exp[-j\omega\tau(x, u_i) + j\omega\tau'(x', u_i)]. \tag{63}$$

If we neglect the correction term for the time being, one can see from (62) that the amplitude of $K(x, x')$ is lower in this case than in the case of the Llewellyn approximation.

Finally, there is the general problem considering a Maxwellian distribution of velocities instead of the rectangular distribution mentioned above. It appears that this problem might be solved by numerical methods employing the above approach.

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Appendix

Let ξ be a random variable, m its mean value, σ its root mean square derivation and n a number. Then the following called the Tchebycheff inequality* applies to any kind of distribution of ξ,

$$P(|\,\xi - m\,| \geq n\sigma) \leq n^{-2}. \tag{A.1}$$

The inequality states that the quantity of mass** in the distribution situated outside the interval $m - n\sigma < \xi < m + n\sigma$ is at most equal to n^{-2}, and thus gives a good idea of the sense in which σ may be used as a measure of dispersion or concentration.

If one assumes that electrons are emitted from the cathode according to a Maxwellian distribution of velocities, then the spread in velocity is due to this distribution of velocities. Consequently, one can use the above inequality to terminate the distribution at some point instead of considering electrons with all possible velocities. The Maxwellian distribution of velocities can be written as

$$n(v)\,dv = \left(\frac{n_0}{kT_c}\right)\left[\exp\left(\frac{-mv^2}{2kT_c}\right)\right]mv\,dv, \tag{A.2}$$

where $n(v)\,dv$ is the number of electrons in the velocity range v and $v + dv$, n_0 is the total number of electrons, m the mass of an electron, k the Boltzmann constant and T_c the cathode temperature.

We then have

$$\int_0^\infty n(v)\,dv = n_0$$

$$\langle v^2\rangle = 2kT_c/m, \qquad \langle v\rangle^2 = \pi kT_c/m$$

$$\sigma_v^2 = \langle v^2\rangle - \langle v\rangle^2 = (4 - \pi)kT_c/2m.$$

Choosing a value of $n = 10$, we have $n\sigma_v = 10[(4 - \pi)kT_c/2m]^{1/2}$ we conclude then that the number of electrons having a velocity not considered in the above approximation is at most one percent. Actually this is a very good approximation which can be seen from the solutions of problems in section III of this paper.

In practical beam-type tubes, electrons are accelerated to a high potential of a few hundred to a few thousand volts before they enter the drift region. Consequently a little thought would show that one can neglect a small pip of energy $\sim$0.1 volts superposed on an energy of the order of 1000 V. Hence, we can use a rectangular distribution whose width is $w = n\sigma_v = 10\sigma_v$.

*See for example S. S. Wilks, *Mathematical statistics*, Princeton University Press, 1943.

**The probability distribution is here interpreted as a distribution of mass.

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