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THE DISTRIBUTION OF RANDOM DETERMINANTS*

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1. Introduction. The random determinants considered here are those whose elements are independent random variables having a common distribution, which is symmetrical about zero and (for convenience only) has unit variance.

They arise naturally in considering the problem of solving large systems of linear equations on an automatic computing machine, and have a bearing on the relation of the machine precision and the order of the largest system which should be tried.

With no further restriction on the common distribution than has been given above, Fortet [1] has determined the second moment to be

$$E(D_n^2) = n! \tag{1.1}$$

with D_n the nth order determinant in question and E, as usual, the expectation. In §2, the fourth moment is determined for the same general conditions. Also, the sixth moment for which no general formula has been found is given for $n = 1, 2, 3, 4$, and the simpler cases for two kinds of discrete distribution are displayed.

The remainder of the paper is devoted to the normal (Laplacian or Gaussian) distribution. Here the moments are unexpectedly simple; indeed

$$E(D_n^{2k}) = n!\,\frac{(n+2)!}{2!}\cdots\frac{(n+2k-2)!}{(2k-2)!}. \tag{1.2}$$

As was first brought to our notice by a reader of an earlier draft, and as has since been stated by Forsythe and Tukey [2] to whom thanks are due for the opportunity of seeing a draft of their paper, this is a consequence of results of S. S. Wilks [4] on the generalized variance. Nevertheless we give our original derivation, which takes only slightly more space, because it leads more easily to the expression of the probability density as an integral of Mellin-Barnes type, from which both series and asymptotic expansions are obtained, and because it seems more natural in this setting.

By symmetry, the odd moments of D_n are all zero and the probability density, $p_n(D_n)$, is an even function. The first four probability density functions are, with $x = D_n$

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and for $x \geq 0$,

$$p_1(x) = (2\pi)^{-1/2} \exp(-x^2/2),$$
$$p_2(x) = 2^{-1}e^{-x},$$
$$p_3(x) = (2\pi)^{-1/2} \int_0^\infty r \exp\left(-\frac{r^2}{2} - \frac{x}{r}\right) dr, \tag{1.3}$$
$$p_4(x) = \frac{x}{2} K_2(2x^{1/2}).$$

K_2 is a Bessel function of the second kind for imaginary argument. The result for $p_2(x)$ is well known.

2. Moments—general distribution. If a_{ij} is the element of D_n in row i, column j, D_n may be written as

$$D_n = \sum \pm a_{1j_1}a_{2j_2} \cdots a_{nj_n} \tag{2.1}$$

with $(j_1, j_2, \cdots, j_n)$ a permutation of $(1, 2, \cdots n)$, the sum extending over all possible $(n!)$ permutations, and the sign according to the parity (evenness or oddness) of the permutation.

Then

$$E(D_n^2) = E[\Sigma \pm a_{1j_1} \cdots a_{nj_n}]^2, \tag{2.2}$$

and it is apparent, because $E(a_{ij}) = 0$, that the only contributions are from terms like $a_{11}^2 \cdots a_{nn}^2$ so that

$$E(D_n^2) = n!$$

since by convention $E(a_{ij}^2) = 1$.

For the fourth moment by the same reasoning, contributions are only from terms in D_n^4 in which elements appear either two or four times. In the latter case, the contribution of the element to a typical product in D_n^4 is $m_4 = E(a_{ij}^4)$. The corresponding combinatorial problem is that of determining the sum of the weights of all $4 \times n$ rectangles, each row of which is a permutation of numbers 1 to n (the numbers $j_1, j_2, \cdots j_n$ of (2.1)). The weight of a rectangle is the product of the weights of its n columns. The weight of a column is m_4 if the four numbers in it are all alike. The weight is unity if the four numbers are two pairs of like numbers, the numbers in one pair being different from those in the other. The weight is zero for any other combination of four numbers.

Putting the first row in natural order $(1, 2, \cdots n)$ we may write

$$E(D_n^4) = n!y_n . \tag{2.3}$$

Here, y_n is a function of m_4 as well as n, but this need not be indicated explicitly. y_n is the total weight of all of the rectangles which have the first row in natural order.

Of several ways we have found to determine y_n, the following seems to be the simplest. Consider only the rectangles of non-zero weight and classify them according to the condition of column n. Either four n's are in column n, or just two are there (the other two in some other column). In the first case, the contribution to y_n is $m_4 y_{n-1}$. In the second case there are three ways of choosing the row for the second n, $n - 1$ ways for choosing the column for the remaining two n's and the contribution may be written

$3(n-1)z_n$, z_n being the total weight of the rectangles which satisfy any one of these choices. Hence

$$y_n = m_4 y_{n-1} + 3(n-1)z_n . \tag{2.4}$$

Take for z_n the rectangles whose $(n-1)$th and nth columns are $n-1$, $n-1$, n, n and n, n, x, x respectively, where x may be any of the numbers 1 to $n-1$. If x is $n-1$ the contribution is y_{n-2}. For each of the remaining $n-2$ choices of x, it is z_{n-1}; for element n may be removed and the last two columns coalesced to the single column $n-1$, $n-1$, x, x thus repeating the form given above. Hence

$$z_n = y_{n-2} + (n-2)z_{n-1} . \tag{2.5}$$

Elimination of z_n from (2.4) and (2.5) shows that

$$y_n = (m_4 + n - 1)y_{n-1} + (n-1)(3-m_4)y_{n-2} . \tag{2.6}$$

With the boundary conditions, $y_1 = m_4$, $y_2 = m_4^2 + 3$, this completely determines y_n. The next two values are:

$$y_3 = m_4^3 + 9m_4 + 6, \qquad y_4 = m_4^4 + 18m_4^2 + 24m_4 + 45.$$

Introducing the exponential generating function

$$Y(t) = \sum_0^\infty y_n t^n/n!, \tag{2.7}$$

it is found by straightforward calculation that, if $y_0 = 1$,

$$Y(t) = (1-t)^{-3} \exp t(m_4 - 3). \tag{2.8}$$

It is interesting to notice that for $m_4 = 0$, this is the exponential generating function for the number of $4 \times n$ rectangles in which each column has exactly two pairs and is also the cube of the corresponding generating function for "rencontres" or sub-factorial numbers. There is also a purely combinatorial interest in the instance $m_4 = 1$. Equation (2.8) leads to the explicit expression for the fourth moment

$$E(D_n^4) = \frac{(n!)^2}{2} \sum_{k=0}^{n} \frac{(n-k+1)(n-k+2)}{k!} (m_4 - 3)^k.$$

For the sixth moment, we have been able to find only the following special values:

$$E(D_1^6) = m_6 ,$$

$$E(D_2^6) = 2!(m_6^2 + 15m_4^2),$$

$$E(D_3^6) = 3!(m_6^3 + 45m_6m_4^2 + 30m_4^3 + 270m_4^2 - 90),$$

$$E(D_4^6) = 4![m_6(m_6^3 + 90m_6m_4^2 + 120m_4^3 + 1080m_4^2 - 360) + 45(17m_4^4 + 56m_4^3 + 144m_4^2 - 72m_4 + 96)].$$

These results are verified by the particular distributions given below in Tables I and II. In the first, the elements of the determinant take the values -1 and $+1$, each with probability 1/2, so both the variance and m_4 are unity. In the second, the elements take values $-\sqrt{2}$, 0, and $\sqrt{2}$ with probabilities 1/4, 1/2, and 1/4 respectively, and hence have variance unity but $m_4 = 2$. The tables give values of $k_n p_n(x)$ where $p_n(x)$

TABLE I

Density Functions for Determinants of Elements ± 1

n	k_n	$k_n p_n(x)$			
		$x = 0$	$x = 2^{n-1}$	$x = 2^n$	$x = 3.2^{n-1}$
1	2	0	1		
2	2^2	2	1		
3	2^4	10	3		
4	2^9	338	84	3	
5	2^{14}	10744	2505	300	15

TABLE II

Density Functions for Determinants of Elements 0, $\pm\sqrt{2}$

n	k_n	$k_n p_n(x)$				
		$x = 0$	$x = 2^{n/2}$	$x = 2.2^{n/2}$	$x = 3.2^{n/2}$	$x = 4.2^{n/2}$
1	2^2	2	1			
2	2^6	38	12	1		
3	2^{11}	1214	327	78	9	3

is the probability that a determinant of order n has the value x, and k_n is a power of two chosen for convenience of tabulation. Both distributions of course are discrete, the values of x being multiples of 2^{n-1} and $2^{n/2}$, respectively, in the two cases, and symmetrical: $p(-x) = p(x)$. In both cases, as n increases, the density tends to peak at the origin.

3. The average value of an arbitrary function of D_n, elements normally distributed. Let $F(z)$ be an arbitrarily assigned function of a real variable z such that the various averages defined below exist. Later on we shall take $F(z)$ to be one or the other of

$$F(z) = |z|^{s-1}, \qquad R(s) > 1, \tag{3.1}$$

$$F(z) = 1 \quad \text{for} \quad x < z < x + \Delta x \quad \text{and 0 for other values of } z, \tag{3.2}$$

where the vertical bars denote "absolute value of", and x, $x + \Delta x$ are two fixed real numbers. Let $\varphi_n(u)$ denote the average value (expectation) of $F(uD_n)$, so that

$$\varphi_n(u) = \text{av.}_1\,[\text{av.}_2\, F(uD_n)], \tag{3.3}$$

where av.$_1$ denotes an average taken over the elements $a_{11}, \cdots, a_{1n}$ of the first row in D_n, and av.$_2$ an average taken over the remaining elements $a_{21}, \cdots, a_{nn}$ with $a_{11}, \cdots, a_{1n}$ held constant.

This two-stage averaging is for the purpose of building a recurrence, as will appear.

Consider $a_{11}, \cdots, a_{1n}$ to be the components of a vector in n-dimensional Euclidean space and let $t_1^{(1)} \cdots, t_n^{(1)}$ be the components of a unit vector having the same direction. Form a determinant T by taking the components $t_j^{(1)}$, $j = 1, 2, \cdots n$, to be the elements of the first column, $t_j^{(2)}$ the elements of the second column, and so on to $t_j^{(n)}$ where the only restriction on the elements $t_j^{(s)}$, $s = 2, 3, \cdots n$ is that they be components of a unit vector perpendicular to the $n - 1$ remaining unit vectors and that their signs be chosen so that the determinant of T be unity. Such a determinant is always possible.

The properties of T and the rule for multiplying determinants show that

$$D_n = D_n T = \begin{vmatrix} b_{11} & \cdot & b_{1n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \cdot & b_{nn} \end{vmatrix} = b_{11} B, \tag{3.4}$$

where

$$b_{11} = (a_{11}^2 + \cdots + a_{1n}^2)^{1/2}, \qquad b_{1k} = 0, \qquad k = 2, 3, \cdots, n \tag{3.5}$$

and

$$B = \begin{vmatrix} b_{22} & \cdot & b_{2n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \cdot & b_{nn} \end{vmatrix}.$$

The inner average shown in (3.3) is equal to the average of $F(ub_{11}B)$ taken over $a_{21}, \cdots a_{nn}$ with the n elements $a_{11}, \cdots a_{1n}$ held fixed. Since b_{11} and the $t_j^{(s)}$ $(s, j = 1, \cdots, n)$ depend only on $a_{11}, \cdots, a_{1n}$, they also are fixed. From (3.4) it is seen that the elements of B are linear sums, with fixed coefficients, of terms chosen from $a_{21}, \cdots a_{nn}$. Since the a_{ij}'s are normally distributed about zero, so are the b_{rs}'s for the purposes of the present averaging process (and for $r, s = 2, \cdots, n$). The orthogonal and normal properties of the $t_j^{(s)}$ lead to

$$\text{av.}_2\, b_{rs} b_{lm} = \delta_{rl} \delta_{sm}, \tag{3.6}$$

where $\delta_{rl} = 1$ if $r = l$ and 0 if $r \neq l$. Consequently, in averaging over $a_{21}, \cdots a_{nn}$, the elements of B behave like $(n - 1)^2$ independent random variables distributed normally about zero with unit standard deviation. Thus B is the same sort of random variable as D_n except that its order is $(n - 1)$ instead of n. From the definition of $\varphi_n(u)$ as an average we get

$$\text{av.}_3\, F(ub_{11}B) = \varphi_{n-1}(ub_{11}), \tag{3.7}$$

where av.$_3$ denotes an average taken over $b_{22}, \cdots, b_{nn}$.

From this and (3.3) it follows that

$$\varphi_n(u) = \text{av.}_1\, \varphi_{n-1}(ub_{11}). \tag{3.8}$$

At this stage b_{11} is no longer regarded as fixed but as a random variable by virtue of its definition (3.5) in terms of the random variables $a_{11}, \cdots, a_{1n}$. Indeed, writing v for b_{11}^2, the probability density of v is

$$\frac{[v/2]^{n/2-1}}{2\Gamma(n/2)} \exp(-v/2), \tag{3.9}$$

and therefore (3.8) becomes

$$\varphi_n(u) = \frac{2^{-n/2}}{\Gamma(n/2)} \int_0^\infty \varphi_{n-1}(uv^{1/2}) v^{n/2-1} e^{-v/2}\, dv \tag{3.10}$$

which, in principle, may be used to compute $\varphi_2(u), \cdots, \varphi_n(u)$ in succession starting with $\varphi_1(u)$.

We now use (3.10) to compute the average value of $| D_n |^{s-1}$ where $R(s) > 1$ and the vertical bars denote absolute value. This calls for the form (3.1) for $F(z)$. Since $D_1 = a_{11}$ and a_{11} is distributed normally about zero with unit variance, equation (3.3) gives

$$\varphi_1(u) = \text{av.}_1 \, | ua_{11} |^{s-1} = \pi^{-1/2} 2^{(s-1)/2} \Gamma(s/2) u^{s-1}, \tag{3.11}$$

where we take $0 < u < \infty$. Equation (3.11) gives the 0, 2, 4, $\cdots$ moments of the normal distribution upon setting $u = 1$ and $s = 1, 3, 5, \cdots$.

The integrals encountered when (3.10) is used to compute $\varphi_n(u)$, starting from (3.11), are of the gamma function type. The general result is

$$E[| D_n |^{s-1}] = \varphi_n(1) = 2^{n(s-1)/2} \frac{\Gamma(s/2)\Gamma([s+1]/2) \cdots \Gamma([s+n-1]/2)}{\Gamma(1/2)\Gamma(2/2) \cdots \Gamma(n/2)}. \tag{3.12}$$

This equation may also be obtained by transforming the $(s - 1)/2$th moment of Wilks' generalized variance.

As in (3.11), the various moments for the probability distribution of D_n may be obtained from (3.12) *ny* setting $s = 1, 3, 5, \cdots$ Equation (1.2) is obtained by taking $s = 2k + 1$, adjusting the fraction in (3.12) to have $2k$ gamma functions in both numerator and denominator, and using the duplication formula for the gamma function.

Let $p_n(D_n)$ denote the probability density of D_n and let $F(z)$ have the form shown in (3.2). The average value of $F(uD_n)$ is then the probability that uD_n lies between x and $x + \Delta x$. When $\Delta x \to 0$ this and equation (3.3) give

$$\varphi_n(u) = p_n(x/u) \, \Delta x/u. \tag{3.13}$$

A recurrence relation for $p_n(D_n)$ is obtained when (3.13) is placed in the recurrence relation (3.10) for $\varphi_n(u)$:

$$p_n(x) = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty p_{n-1}(xv^{-1/2}) v^{(n-3)/2} e^{-v/2} \, dv. \tag{3.14}$$

Starting with $p_1(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, the higher $p_n(x)$'s may be computed in succession. The integrations steadily become more difficult and the work has not been carried beyond the case $n = 4$ stated in the introduction.

4. An integral for the probability density of D_n. Since $p_n(D_n)$ is an even function of D_n, (3.12) gives for $R(s) > 1$

$$\frac{1}{2} E[| D_n |^{s-1}] = \int_0^\infty | D_n |^{s-1} p_n(D_n) \, dD_n \,. \tag{4.1}$$

The expression for $p_n(x)$ obtained heuristically from (4.1) by Mellin's inversion formula is

$$p_n(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} 2^{-1+n(s-1)/2} \frac{\Gamma(s/2)\Gamma([s+1]/2) \cdots \Gamma([s+n-1]/2)}{\Gamma(1/2)\Gamma(2/2) \cdots \Gamma(n/2)} \, ds, \tag{4.2}$$

where $x > 0$ and $c > 0$. The integrand in (4.2) is an analytic function of s when $R(s) > 0$ and the integral converges absolutely by virtue of the asymptotic behavior of the gamma functions. This integral is closely related to those given by S. Kullback [3] for the distribution of the generalized variance.

That (4.2) actually gives $p_n(x)$ may be verified by induction from the recurrence relation (3.14) for $p_n(x)$ (the order of integration may be inverted because both integrals converge absolutely).

A series for $p_n(x)$ may be obtained by closing the path of integration in (4.2) on the left by a large semicircle. This has actually been done by Kullback in treating the analogous case for the generalized variance. The complication introduced by the multiple poles is illustrated by the following series for $n = 3$.

$$p_3(x) = (2\pi)^{-1/2}\left(1 + \sum_{k=1}^{\infty} \frac{(-x^2/2)^k}{k!(2k-1)!}\left[\log(2^{-1/2}x) - \frac{1}{2}\Psi(k) - \Psi(2k-1)\right]\right) - \frac{x}{2}\sum_{k=0}^{\infty}\frac{(-x^2/2)^k}{(3/2)_k(2k)!}. \tag{4.3}$$

Here $(\alpha)_0 = 1$, $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$, and $\Psi(x) = \Gamma'(x+1)/\Gamma(x+1)$. When k is an integer

$$\Psi(k) = -C + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}, \qquad \Psi(0) = -C,$$

$$\Psi(k-1/2) = -C - 2\log 2 + 2 + \frac{2}{3} + \frac{2}{5} + \cdots + \frac{2}{2k-1}, \tag{4.4}$$

$$\Psi(-1/2) = -C - 2\log 2,$$

where $C = .577\cdots$.

The first few terms of the power series for $n \geq 4$ are

$$p_n(x) = \frac{2^{-n/2}}{\Gamma(n/2)} - \frac{x}{(n-2)!2} + \frac{x^2 2^{-1-n/2}}{(n-2)!\Gamma[(n-2)/2]} \cdot\left[-\log(2^{-n}x^2) + (-3C - 2\log 2 + 3) + \sum_{j=4}^{n}\Psi\left(\frac{j-5}{2}\right)\right] + \cdots. \tag{4.5}$$

It must be remembered that all this assumes $x > 0$ and that $p_n(x)$ is an even function of x.

Equation (4.5) shows how $p_n(x)$ behaves for small values of x. An asymptotic expression holding for large values of x may be obtained from (4.2) by the method of steepest descent. If we assume for the moment that the appropriate saddle-point $s = s_0$ is far from $s = 0$, so that the gamma functions may be replaced by their asymptotic expressions, we find that $(s_0)_n \approx x^2$. The proper root to use turns out to be $s_0 \approx x^{2/n} - (n-1)/2$. When we take $c = s_0$ in (4.2), the method of steepest descent gives the following asymptotic expression:

$$p_n(x) \sim \frac{\pi^{(n-1)/2}2^{-(n^2-3n+4)/4}x^{(n-1)(n-2)/2n}}{n^{1/2}\Gamma(1/2)\Gamma(2/2)\cdots\Gamma(n/2)}\exp(-nx^{2/n}/2). \tag{4.6}$$

As a check on (4.6), note that it reduces to the exact forms when $n = 1$ and 2. As a further check, we note that it satisfies the differential equation (which may be obtained by differentiating (4.2) n times)

$$\frac{d^n}{dx^n}p_n(x) - (-1)^n x^{2-n}p_n(x) = 0, \tag{4.7}$$

in the sense of being the leading term of an asymptotic series solution. In fact (4.7) may be used to extend (4.6). For example, if $n = 3$, we can assume

$$p_3(x) \sim Ax^{1/3} \exp[-3x^{2/3}/2][1 + a_1x^{-2/3} + a_2x^{-4/3} + \cdots]$$

and find from (4.7) that $a_1 = 5/18$ and $a_2 = -35/648$.

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