and when

$$W_2 + V_3 = -\psi_1$$
$$U_3 + W_1 = -\psi_2$$
$$V_1 + U_2 = -\psi_3$$

the matrix is anti-symmetric, and its elements represent the Morera stress-functions which when substituted into Eqs. (11) to (16) yield the following stress components:

$$\begin{aligned} Xx &= \frac{\partial^2 \psi_1}{\partial y\, \partial z}, \qquad Yy = \frac{\partial^2 \psi_2}{\partial z\, \partial x}, \qquad Zz = \frac{\partial^2 \psi_3}{\partial x\, \partial y}, \\ Xy &= Yx = -\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} - \frac{\partial \psi_3}{\partial z}\right), \\ Xz &= Zx = -\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right), \\ Yz &= Zy = -\frac{1}{2}\frac{\partial}{\partial z}\left(-\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right). \end{aligned} \tag{19}$$

Thus, the stress tensor expressed by Maxwell and Morera functions was derived by a direct method from diagonal and antisymmetric matrices.

A REMARK ON INTEGRAL INVARIANTS*

By H. D. BLOCK (*University of Minnesota*)

Let the $2n$ variables $q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n$ be related to the $2n$ variables $Q_1, Q_2, \cdots, Q_n, P_1, P_2, \cdots, P_n$ by a canonical transformation. Let σ be the unit square: $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, and let $q_i = f_i(u, v)$, $p_i = g_i(u, v)$, $(i = 1, 2, \cdots, n)$, where f_i and g_i have continuous derivatives on σ. This induces the relationships $Q_i = F_i(u, v)$, $P_i = G_i(u, v)$, $(i = 1, 2, \cdots, n)$. Let $s_i = \bigcup_{(u,v)\epsilon\sigma} (f_i(u, v), g_i(u, v))$ and $S_i = \bigcup_{(u,v)\epsilon\sigma} (F_i(u, v), G_i(u, v))$, i.e. the maps of σ on the (q_i, p_i) and (Q_i, P_i) planes respectively. Let

$$\sum_{i=1}^{n} \iint_{s_i} dq_i\, dp_i \qquad \text{and} \qquad \sum_{i=1}^{n} \iint_{S_i} dQ_i\, dP_i$$

be denoted respectively by

$$\iint_{s} \sum dq_i\, dp_i \qquad \text{and} \qquad \iint_{S} \sum dQ_i\, dP_i\,.$$

It is widely[1] believed that under the conditions stated

$$\iint_{s} \sum dq_i\, dp_i = \iint_{S} \sum dQ_i\, dP_i\,.$$

*Received Oct. 28, 1953.

[1]Cf. e.g. Goldstein *Classical Mechanics*, Addison Wesley, 1950, pp. 247-250. Also Corben and Stehle *Classical Mechanics*, Wiley, 1950, pp. 292-3.

It is the purpose of this paper, (1) to show by a simple example that this is not true, and, (2) to discuss the sources of the mistake and point out correct formulations.

1. **Example.** Let $Q_1 = q_1$, $Q_2 = p_2$, $P_1 = p_1 - 2p_2$, $p_2 = -2q_1 - q_2$. It is easily verified that this is a canonical transformation. Let $q_1 = q_2 = u$ and $p_1 = p_2 = v$, so that $Q_1 = u$, $P_1 = -v$, $Q_2 = v$, $P_2 = -3u$. Then s_1 is the square $0 \leqq q_1 \leqq 1$, $0 \leqq p_1 \leqq 1$; s_2 is the square $0 \leqq q_2 \leqq 1$, $0 \leqq p_2 \leqq 1$; S_1 is the square: $0 \leqq Q_1 \leqq 1$, $-1 \leqq P_1 \leqq 0$; and S_2 is the rectangle $0 \leqq Q_2 \leqq 1$, $-3 \leqq P_2 \leqq 0$. Then

$$\iint_s \sum dq_i \, dp_i = 1 + 1 = 2,$$

while

$$\iint_S \sum dQ_i \, dP_i = 1 + 3 = 4.$$

2. **Sources of the mistake.** "Proofs" of the incorrect statement may be made in several ways, two of which we now examine. First[2] one can show correctly that

$$\sum_{i=1}^{n} \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} \right) = \sum_{i=1}^{n} \left(\frac{\partial Q_i}{\partial u} \frac{\partial P_i}{\partial v} - \frac{\partial P_i}{\partial u} \frac{\partial Q_i}{\partial v} \right).$$

Now, for each i in $(1, 2, \cdots, n)$,

$$\iint_{s_i} dq_i \, dp_i = \int_0^1 \int_0^1 \left| \frac{\partial(q_i, p_i)}{\partial(u, v)} \right| du \, dv,$$

and

$$\iint_{S_i} dQ_i \, dP_i = \int_0^1 \int_0^1 \left| \frac{\partial(Q_i, P_i)}{\partial(u, v)} \right| du \, dv.$$

The error is now introduced by neglecting the absolute value signs in the integrand. Summing on i completes the "proof". [Since vertical lines are used to denote both determinants and absolute values and since the printing of both symbols: $|| \quad ||$ is widely used with a different meaning, it would be highly desirable if the formula for transforming of integrals were written, say,

$$\iint_A dx \, dy = \iint_{A^*} \left| \frac{\partial(x, y)}{\partial(u, v)} \right|_+ du \, dv^3$$

in order to avoid such mistakes.]

A correct formula would therefore be

$$\sum_{i=1}^{n} \operatorname{sgn} \frac{\partial(q_i, p_i)}{\partial(u, v)} \iint_{s_i} dq_i \, dp_i = \sum_{i=1}^{n} \operatorname{sgn} \frac{\partial(Q_i, P_i)}{\partial(u, v)} \iint_{S_i} dQ_i \, dP_i .$$

A second incorrect "proof" proceeds as follows[4] Suppose that Λ is a real interval,

[2]Cf. Goldstein, loc. cit.

[3]This notation is used e.g. in Mood, *Introduction to the Theory of Statistics*, McGraw-Hill, 1950.

[4]Cf. Corben and Stehle, loc. cit.

say $0 \leqq \lambda \leqq 1$. Let $q_i = \varphi_i(\lambda)$, $p_i = \psi_i(\lambda)$, $(i = 1, 2, \cdots, n)$, where φ_i, ψ_i have continuous derivatives on Λ and (φ_i, ψ_i) traces out a simple closed curve c_i as λ traverses Λ. This induces the relationships $Q_i = \Phi_i(\lambda)$, $P_i = \Psi_i(\lambda)$. Because the transformation is canonical, $(\Phi_i(\lambda), \Psi_i(\lambda))$ also traces out a simple closed curve C_i in the (Q_i, P_i) plane as λ traverses Λ, and

$$\sum_{i=1}^{n}\left[\oint_{C_i} P_i\,dQ_i - \oint_{c_i} p_i\,dq_i\right] = \sum_{i=1}^{n}\left[\int_0^1 \Psi_i(\lambda)\,d\Phi_i(\lambda) - \int_0^1 \psi_i(\lambda)\,d\varphi_i(\lambda)\right]$$

$$= \int_0^1 dH(q(\lambda), p(\lambda)) = 0.$$

Thus, correctly,

$$\sum_{i=1}^{n} \oint_{C_i} P_i\,dQ_i = \sum_{i=1}^{n} \oint_{c_i} p_i\,dq_i\,.$$

Now the mistake is made of using Stokes' theorem $\oint_C x\,dy = \iint_A dx\,dy$ to complete the "proof". But for this formula to hold, C must be positively oriented. Now in the case at hand C_i and c_i have the orientations which are given to them by the mapping from Λ, and these are not necessarily all the same (examine the Example). We could use Stokes' theorem in the form

$$\iint_{s_i} dq_i\,dp_i = (-1)^{\sigma_i} \oint_{c_i} p_i\,dq_i \qquad \text{and} \qquad \iint_{S_i} dQ_i\,dP_i = (-1)^{\rho_i} \oint_{C_i} P_i\,dQ_i\,,$$

where σ_i and ρ_i are 0 or 1 according as c_i or C_i respectively is traversed negatively or positively as λ traverses Λ. Thus a correct formulation is

$$\sum_{i=1}^{n} (-1)^{\sigma_i} \iint_{s_i} dq_i\,dp_i = \sum_{i=1}^{n} (-1)^{\rho_i} \iint_{S_i} dQ_i\,dP_i\,.$$

This subject, as well as other topics in mathematical physics, can be presented in a more unified and elegant fashion through the use of Cartan's exterior differential calculus. This the writer hopes to do in a later paper.

ON A GENERALIZATION OF SYNGE'S CRITERION FOR SUFFICIENT STABILITY OF PLANE PARALLEL FLOWS*

By S. I. PAI (*University of Maryland*)

In Ref. [1], Synge derived sufficient conditions for the stability of plane Couette flow and plane Poiseuille flow of incompressible fluid. In the present note, the conditions are generalized so that the problem of velocity profiles with point of inflection and with either finite or infinite boundary condition or both may be included.

*Received August 10, 1953. This note was written when the author visited the Aeronautical Engineering Department of the Pennsylvania State College during August of 1953, supported by ONR Contract Nonr 656(01).