LATERAL VIBRATIONS OF TWISTED RODS*

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1. Introduction. Figure 1 shows part of a straight rod which is twisted, like a

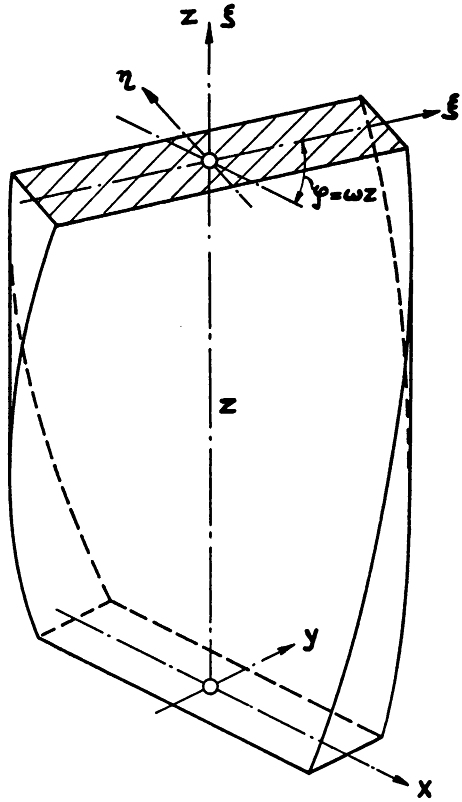

FIG. 1. Twisted rod.

propeller blade, in the unloaded state. The bending vibrations of such a rod have been dealt with by H. Reissner,[1] who derived the differential equations under most general assumptions, without, however, to the best of our knowledge, proceeding to their solution. Subsequently, a number of authors have treated particular cases by approximate or experimental methods.[2] In view of the possibilities offered by modern computers, the problem actually is ripe for rigorous treatment. It is solved in this paper for an isotropic,

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[1]H. Reissner, *Ing. Arch.* **4**, 557 (1933).

[2]E. Maier, *Ing. Arch.* **11**, 73 (1940),
E. R. Love, J. P. O. Silberstein, J. R. M. Radok, Report ACA 36 (1947),
J. Geiger, *Schweiz. Bauzeitung* **68**, 17 (1950),
D. D. Rosard, *J. Appl. Mech.* **20**, 241 (1953).

homogeneous rod built in at one end, under the assumptions that the mass per unit length, μ, the twist per unit length, ω, and the principal flexural rigidities, α, β, be constant (one of them being taken infinitely large).

The eigenvalue problem, which is easily established by a method recently used in a stability investigation,[3] is of 8th order. The eigenvalue equation is obtained by equating an 8-row determinant to zero, the elements of which contain the eigenvalue in the form of an exponent. For small and large values of the total twist, ϕ, the natural frequencies of the rod can be computed by development. In the intermediate range, they have been evaluated by means of the sequence controlled computer of the Institute for Applied Mathematics of the Eidg. Technische Hochschule.[4]

2. Reference frames. Figure 2 shows the deflection curve of an originally vertical

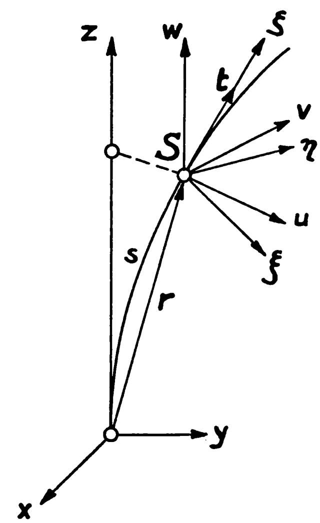

Fig. 2. Coordinate systems.

rod, S denoting the center of gravity of a generic section given by the arc s. The curve can be referred to three different coordinate frames, viz.,

(i) the fixed frame x, y, z, defined by the principal axes of an end section and the axis of the straight rod;

(ii) the principal frame ξ, η, ζ, defined by the principal axes of the section at s, together with the tangent of the deflection curve;

(iii) the raised frame u, v, w with vertical axis w, obtained from ξ, η, ζ by rotation about the nodal line, i.e. the horizontal line in the plane ξ, η passing through S.

Let **t** denote the unit vector along the tangent of the deflection curve, while **a** is an arbitrary vector the first two components of which are small. Provided that the inclination of the deflection curve is small, the components of **a** in the last two reference frames are connected by the relations[5]

$$a_u = a_\xi + t_u a_\zeta , \qquad a_v = a_\eta + t_v a_\zeta , \qquad a_w = a_\zeta . \tag{2.1}$$

[3]H. Ziegler, *ZAMP* 2, 268 (1951).

[4]The authors are indebted to the Director of this Institute, Prof. Dr. E. Stiefel, who put the computer at their disposal, and to Prof. Dr. H. Pallmann, President of the Schweiz. Schulrat, for granting them very favourable conditions. Also, they are obliged to Dr. H. Rutishauser for valuable advice with regard to the establishment of the computing program.

[5]These follow at once from Fig. 2; see also H. Ziegler, loc. cit., (4.1).

If, for the present, s is interpreted as a measure of time, the principal frame slides along the deflection curve with unit speed. Its state of motion consists of a translation of velocity $\mathbf{t}$ and of a rotation of angular velocity $\boldsymbol{\kappa} + \omega\mathbf{t}$, provided that $\omega = d\varphi/ds$ denotes the angle of twist per unit length of the unloaded rod, while the components of the deformation vector $\boldsymbol{\kappa}(\kappa_\xi, \kappa_\eta, \kappa_\zeta)$ represent in turn the curvatures of the deflection curve [measured in the principal planes η, ζ and ξ, ζ] and the elastic twist per unit length. In the raised system the components of $\boldsymbol{\kappa}$ are, according to (2.1),

$$\kappa_u = \kappa_\xi + t_u\kappa_\zeta , \qquad \kappa_v = \kappa_\eta + t_v\kappa_\zeta , \qquad \kappa_w = \kappa_\zeta , \tag{2.2}$$

hence, $\boldsymbol{\kappa} + \omega\mathbf{t}$ has the components

$$\kappa_\xi + t_u(\kappa_\zeta + \omega), \qquad \kappa_\eta + t_v(\kappa_\zeta + \omega), \qquad \kappa_\zeta + \omega.$$

The first two components describe the motion of the principal frame with respect to the raised one, whereas the last component represents the rotation of the raised frame.

3. Differential equations of motion. Let $\mathbf{k}$ denote the external force and $\mathbf{m}$ the external moment, both taken per unit length. If $\mathbf{K}$ and $\mathbf{M}$ represent the internal forces in section s, the conditions of equilibrium for an element of unit length are

$$\frac{d\mathbf{K}}{ds} + \mathbf{k} = 0, \qquad \frac{d\mathbf{M}}{ds} + \mathbf{t} \times \mathbf{K} + \mathbf{m} = 0. \tag{3.1}$$ [6]

Since the angular velocity of the raised system is $\mathbf{u}(0, 0, \kappa_w + \omega)$, the equilibrium conditions, for an observer taking part in this motion, are, instead of (3.1),

$$\frac{d\mathbf{K}}{ds} + \mathbf{u} \times \mathbf{K} + \mathbf{k} = 0, \qquad \frac{d\mathbf{M}}{ds} + \mathbf{u} \times \mathbf{M} + \mathbf{t} \times \mathbf{K} + \mathbf{m} = 0. \tag{3.2}$$

Resolving them and denoting derivatives with respect to s by primes, we obtain

$$\begin{aligned} K_u' - (\kappa_w + \omega)K_v + k_u &= 0, \\ K_v' + (\kappa_w + \omega)K_u + k_v &= 0, \\ K_w' \qquad\qquad\qquad + k_w &= 0; \end{aligned} \tag{3.3}$$

$$\begin{aligned} M_u' - (\kappa_w + \omega)M_v + t_vK_w - t_wK_v + m_u &= 0, \\ M_v' + (\kappa_w + \omega)M_u + t_wK_u - t_uK_w + m_v &= 0, \\ M_w' \qquad\qquad\qquad + t_uK_v - t_vK_u + m_w &= 0. \end{aligned} \tag{3.4}$$

Denoting by α, β, γ the flexural rigidities with respect to the principal axes and the torsional rigidity, we have further, provided that the strains remain small,

$$M_\xi = \alpha\kappa_\xi , \qquad M_\eta = \beta\kappa_\eta , \qquad M_\zeta = \gamma\kappa_\zeta . \tag{3.5}$$

When the rod executes free vibrations, $\mathbf{k}$ is the inertia force per unit length and $\mathbf{m}$ its moment. For small vibrations, k_u, k_v and m_w can be treated as small quantities of the first order. Excluding longitudinal vibrations and the higher modes of flexural vibrations, we are justified in replacing s by z and in suppressing the quantities k_w, m_u, m_v, since they are small of higher order. So is K_w, while K_u, K_v, M_u, M_v, M_w, t_u, t_v, κ_u, κ_v, κ_w are of the first order and $t_w = 1$.

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[6]See, for instance, H. Ziegler, loc. cit., (2.1).

If terms of the first order alone are retained, the third equation (3.3) becomes trivial. The third equation (3.4) reduces to $M'_w + m_w = 0$, thus, torsional vibrations can be treated separately. Excluding them, we have $m_w = 0$, $M_w = 0$. From (2.1) follows $m_\zeta = 0$, $M_\xi = M_u$, $M_\eta = M_v$, $M_\zeta = 0$ and, owing to the third equation (3.5),

$$\kappa_\xi = \kappa_u , \qquad \kappa_\eta = \kappa_v , \qquad \kappa_\zeta = \kappa_w = 0. \tag{3.6}$$

Thus, the four remaining equations (3.3), (3.4) reduce to

$$\begin{aligned} K'_u - \omega K_v + k_u &= 0, \\ K'_v + \omega K_u + k_v &= 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} M'_u - \omega M_v - K_v &= 0, \\ M'_v + \omega M_u + K_u &= 0, \end{aligned} \tag{3.8}$$

while the first two relations (3.5) may be replaced by

$$M_u = \alpha \kappa_u , \qquad M_v = \beta \kappa_v . \tag{3.9}$$

According to Section 2, the angular velocity $\mathbf{u}^*$ of the principal frame with respect to the raised one has the components

$$\kappa_\xi + t_u(\kappa_\zeta + \omega), \qquad \kappa_\eta + t_v(\kappa_\zeta + \omega), \qquad 0.$$

Owing to (3.6), it can be written $\mathbf{u}^*(\kappa_u + \omega t_u , \kappa_v + \omega t_v , 0)$. For an observer moving with the raised frame

$$\frac{d\mathbf{t}}{ds} = \mathbf{u}^* \times \mathbf{t};$$

hence

$$\kappa_u = -t'_v - \omega t_u , \qquad \kappa_v = t'_u - \omega t_v . \tag{3.10}$$

If, finally, $\mathbf{r}$ denotes the radius vector of S (Fig. 2),

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} + \mathbf{u} \times \mathbf{r};$$

thus

$$t_u = r'_u - \omega r_v , \qquad t_v = r'_v + \omega r_u . \tag{3.11}$$

Eliminating κ_u, κ_v and t_u, t_v from (3.9) to (3.11), we obtain the bending moments

$$\begin{aligned} M_u &= -\alpha(r''_v + 2\omega r'_u - \omega^2 r_v), \\ M_v &= \beta(r''_u - 2\omega r'_v - \omega^2 r_u) \end{aligned} \tag{3.12}$$

and, by substitution in (3.8), the shear forces

$$\begin{aligned} K_u &= -\beta r'''_u + (\alpha + 2\beta)\omega r''_v + (2\alpha + \beta)\omega^2 r'_u - \alpha\omega^3 r_v , \\ K_v &= -\alpha r'''_v - (2\alpha + \beta)\omega r''_u + (\alpha + 2\beta)\omega^2 r'_v + \beta\omega^3 r_u . \end{aligned} \tag{3.13}$$

Introducing them in (3.7), we have

$$\begin{aligned} &\beta r^{\text{iv}}_u - 2(\alpha + \beta)\omega r'''_v - 2(2\alpha + \beta)\omega^2 r''_u + 2(\alpha + \beta)\omega^3 r'_v + \beta\omega^4 r_u - k_u = 0, \\ &\alpha r^{\text{iv}}_v + 2(\alpha + \beta)\omega r'''_u - 2(\alpha + 2\beta)\omega^2 r''_v - 2(\alpha + \beta)\omega^3 r'_u + \alpha\omega^4 r_v - k_v = 0, \end{aligned} \tag{3.14}$$

i.e. a system of 8th order for r_u, r_v.

Since the rod is loaded only by its inertia forces,

$$k_u = -\mu \ddot{r}_u \, , \qquad k_v = -\mu \ddot{r}_v \, . \tag{3.15}$$

Substituting (3.15) in (3.14), we obtain the partial differential equations of the flexural vibrations. Taking

$$r_u = U \cos \kappa t, \qquad r_v = V \cos \kappa t \tag{3.16}$$

they yield, for the eigenfunctions, the differential equations

$$\begin{aligned} &\beta U^{iv} - 2(\alpha + \beta)\omega V''' - 2(2\alpha + \beta)\omega^2 U'' + 2(\alpha + \beta)\omega^3 V' + (\beta\omega^4 - \mu\kappa^2)U = 0, \\ &\alpha V^{iv} + 2(\alpha + \beta)\omega U''' - 2(\alpha + 2\beta)\omega^2 V'' - 2(\alpha + \beta)\omega^3 U' + (\alpha\omega^4 - \mu\kappa^2)V = 0. \end{aligned} \tag{3.17}$$

If the coordinate z is replaced by the angle of rotation of the corresponding section,

$$\varphi = \omega z, \tag{3.18}$$

we have finally, denoting derivatives with respect to φ by primes,

$$\begin{aligned} &\beta U^{iv} - 2(\alpha + \beta) V''' - 2(2\alpha + \beta) U'' + 2(\alpha + \beta) V' + \beta\left(1 - \frac{\mu\kappa^2}{\beta\omega^4}\right) U = 0, \\ &\alpha V^{iv} + 2(\alpha + \beta) U''' - 2(\alpha + 2\beta) V'' - 2(\alpha + \beta) U' + \alpha\left(1 - \frac{\mu\kappa^2}{\alpha\omega^4}\right) V = 0. \end{aligned} \tag{3.19}$$

4. Boundary conditions. The eigenfunctions are determined by the differential equations (3.19) in connection with the boundary conditions. In the case of a rod built in at the lower end the displacements r_u , r_v of section $\varphi = 0$ are zero; thus, by (3.16),

$$U = V = 0. \qquad (\varphi = 0). \tag{4.1}$$

Besides, for $\varphi = 0$ the tangent of the deflection curve is vertical; hence, by (3.11), (3.16) and (4.1),

$$U' = V' = 0, \qquad (\varphi = 0) \tag{4.2}$$

primes indicating derivatives with respect to φ.

If l denotes the length of the rod and $\phi = \omega l$ its total twist, the bending moments (3.12) and the shear forces (3.13) vanish for $\varphi = \phi$; hence,

$$\beta(U'' - 2V' - U) = \alpha(V'' + 2U' - V) = 0, \qquad (\varphi = \phi) \tag{4.3}$$

and

$$\beta(U''' - 2V'' - U') = \alpha(V''' + 2U'' - V') = 0. \qquad (\varphi = \phi). \tag{4.4}$$

The eigenvalue problem defined by (3.19) and (4.1) to (4.4) is somewhat similar to the one of the corresponding buckling problem[7]; yet, since its order is twice as high, it is more complicated.

5. Solution. The fundamental solutions of (3.19) are of the type

$$U = A \exp(\lambda\varphi), \qquad V = B \exp(\lambda\varphi). \tag{5.1}$$

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[7]H. Ziegler, *Schweiz. Bauzeit.* **66**, 463 (1948).

Substituting them in (3.19), we obtain

$$\begin{gathered}\left[\beta\lambda^4 - 2(2\alpha + \beta)\lambda^2 + \beta\left(1 - \frac{\mu\kappa^2}{\beta\omega^4}\right)\right]A - 2(\alpha + \beta)\lambda(\lambda^2 - 1)B = 0,\\ 2(\alpha + \beta)\lambda(\lambda^2 - 1)A + \left[\alpha\lambda^4 - 2(\alpha + 2\beta)\lambda^2 + \alpha\left(1 - \frac{\mu\kappa^2}{\alpha\omega^4}\right)\right]B = 0\end{gathered} \tag{5.2}$$

and the characteristic equation

$$\lambda^8 + 4\lambda^6 + \left[6 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\frac{\mu\kappa^2}{\omega^4}\right]\lambda^4 + 2\left[2 + 3\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\frac{\mu\kappa^2}{\omega^4}\right]\lambda^2 + \left(1 - \frac{\mu\kappa^2}{\alpha\omega^4}\right)\left(1 - \frac{\mu\kappa^2}{\beta\omega^4}\right) = 0. \tag{5.3}$$

By (5.2), any one of its eight roots λ_k yields a ratio

$$\frac{B_k}{A_k} = \frac{\lambda_k^4 - 2[1 + 2(\alpha/\beta)]\lambda_k^2 + 1 - (\mu\kappa^2/\beta\omega^4)}{2[1 + (\alpha/\beta)]\lambda_k(\lambda_k^2 - 1)} \tag{5.4}$$

and therefore one of eight fundamental solutions. The general solution is

$$U = \sum_{k=1}^{8} A_k \exp(\lambda_k\varphi), \qquad V = \sum_{k=1}^{8} B_k \exp(\lambda_k\varphi), \tag{5.5}$$

the coefficients A_k, B_k being determined by (5.4) and by the boundary conditions (4.1) to (4.4), i.e. by

$$\begin{aligned}
&\sum_1^8 A_k = 0, && \beta\sum_1^8 [(\lambda_k^2 - 1)A_k - 2\lambda_k B_k]\exp(\lambda_k\phi) = 0,\\
&\sum_1^8 \lambda_k A_k = 0, && \beta\sum_1^8 [(\lambda_k^2 - 1)A_k - 2\lambda_k B_k]\lambda_k\exp(\lambda_k\phi) = 0,\\
&\sum_1^8 B_k = 0, && \alpha\sum_1^8 [(\lambda_k^2 - 1)B_k + 2\lambda_k A_k]\exp(\lambda_k\phi) = 0,\\
&\sum_1^8 \lambda_k B_k = 0, && \alpha\sum_1^8 [(\lambda_k^2 - 1)B_k + 2\lambda_k A_k]\lambda_k\exp(\lambda_k\phi) = 0.
\end{aligned} \tag{5.6}$$

In order to simplify the problem, we restrict ourselves, by the limiting process $\beta \to \infty$, to the most interesting case of a perfectly narrow cross section. If this process is carefully applied, (5.3) reduces to

$$(\lambda^2 + 1)^4 - (\lambda^4 - 6\lambda^2 + 1)\frac{\mu\kappa^2}{\alpha\omega^4} = 0; \tag{5.7}$$

(5.4) yields

$$B_k = \frac{\lambda_k^2 - 1}{2\lambda_k} A_k, \tag{5.8}$$

and the system (5.6) becomes

$$
\begin{aligned}
&\sum_1^8 A_k = 0, && \sum_1^8 \frac{(\lambda_k^2+1)^2 + (\mu\kappa^2/\alpha\omega^4)}{\lambda_k^2 - 1} \exp(\lambda_k\phi) A_k = 0,\\
&\sum_1^8 \lambda_k A_k = 0, && \sum_1^8 \frac{(\lambda_k^2+1)^2 + (\mu\kappa^2/\alpha\omega^4)}{\lambda_k^2 - 1} \lambda_k \exp(\lambda_k\phi) A_k = 0,\\
&\sum_1^8 \frac{\lambda_k^2 - 1}{\lambda_k} A_k = 0, && \sum_1^8 \frac{(\lambda_k^2+1)^2}{\lambda_k} \exp(\lambda_k\phi) A_k = 0,\\
&\sum_1^8 (\lambda_k^2 - 1) A_k = 0, && \sum_1^8 (\lambda_k^2+1)^2 \exp(\lambda_k\phi) A_k = 0.
\end{aligned}
\tag{5.9}
$$

Equating the determinant of (5.9) to zero, we observe that (5.7) is solved by pairs of roots, $\lambda_1 = -\lambda_5, \cdots, \lambda_4 = -\lambda_8$. We further make use of (5.7) and of the fact that any row may be supplemented by a linear combination of other rows. Thus, we obtain the eigenvalue equation

$$
\begin{vmatrix}
1 & 0 & \cdots \\
0 & \lambda_1 & \cdots \\
0 & \dfrac{1}{\lambda_1} & \cdots \\
\lambda_1^2 & 0 & \cdots \\
(1+\lambda_1^2)^2 \cosh \lambda_1\phi & (1+\lambda_1^2)^2 \sinh \lambda_1\phi & \cdots \\
\dfrac{(1+\lambda_1^2)^2}{\lambda_1} \sinh \lambda_1\phi & \dfrac{(1+\lambda_1^2)^2}{\lambda_1} \cosh \lambda_1\phi & \cdots \\
\dfrac{3-\lambda_1^2}{1+\lambda_1^2} \cosh \lambda_1\phi & \dfrac{3-\lambda_1^2}{1+\lambda_1^2} \sinh \lambda_1\phi & \cdots \\
\dfrac{1-3\lambda_1^2}{\lambda_1(1+\lambda_1^2)} \sinh \lambda_1\phi & \dfrac{1-3\lambda_1^2}{\lambda_1(1+\lambda_1^2)} \cosh \lambda_1\phi & \cdots
\end{vmatrix} = 0,
\tag{5.10}
$$

the three remaining double columns being obtained from the first one by substituting $\lambda_2, \lambda_3, \lambda_4$ for λ_1.

In order to evaluate (5.7) and (5.10), we introduce, in place of the natural frequency κ, the quantity

$$
m^4 = \frac{\mu\kappa^2 l^4}{\alpha},
\tag{5.11}
$$

denoted by $k^4 l^4$ by some authors. According to S. Timoshenko[8], the eigenvalue equation, in the absence of twist, takes the simple form

$$
1 + \cos m^* \cosh m^* = 0,
\tag{5.12}
$$

[8]S. Timoshenko, *Vibration Problems in Engineering*, Van Nostrand, New York 1935, p. 234.

yielding the eigenvalues

$$m_1^* = 1.875, \; m_2^* = 4.694, \; m_3^* = 7.855, \; m_4^* = 10.996, \cdots \tag{5.13}$$

Owing to (5.10) and the relation $\phi = \omega l$, the constant in (5.7) is

$$\frac{\mu\kappa^2}{\alpha\omega^4} = \frac{m^4}{\phi^4} = p^4; \tag{5.14}$$

hence, (5.7) can be written

$$(1 + \lambda^2)^4 - (1 - 6\lambda^2 + \lambda^4)p^4 = 0. \tag{5.15}$$

In order to obtain the natural frequencies

$$\kappa_i = \frac{m_i^2}{l^2}\left(\frac{\alpha}{\mu}\right)^{1/2} \tag{5.16}$$

in the presence of twist, it is best to start from a given value of p and to calculate the corresponding roots $\lambda_1, \cdots, \lambda_4$ of (5.15). Introducing them in (5.10) and solving for ϕ, we obtain a sequence of values of the total twist, every one of them, by (5.14), yielding a value

$$m = p\phi. \tag{5.17}$$

By repetition of this process (cumbersome since the real or complex character of the roots λ_k depends on the magnitude of p), the functions $m_1(\phi)$, $m_2(\phi)$, $\cdots$ can be evaluated.

6. Computer work. The process described above has been carried through by means of the sequence controlled computer mentioned in **1**. For any of 12 values of p (chosen in the interval $0.06 \leqq p^4 \leqq 26{,}000$) 9 to 18 values of the determinant (5.10) were computed. The roots λ_k of (5.15) were introduced right from the start; the angular functions of ϕ near the probable zeros were taken from Tables and transferred to punched strips for use by the computer. The determinant was evaluated by the method of Gauss and Banachiewicz[9]. Controlled by a cyclic program, the computer worked nearly without supervision, printing the results from which the zeros were obtained by interpolation.

Tables 1 to 4 give the values calculated for $m_1(\phi), \cdots, m_4(\phi)$, together with their

TABLE 1

ϕ	$m_1(\phi)$	$m_1^2(\phi)$
0	1,875	3,5156
0,1477	$1{,}875_5$	$3{,}51_8$
0,6003	$1{,}879_0$	$3{,}53_1$
1,6209	$1{,}903_0$	$3{,}62_1$
2,4328	$1{,}934_7$	$3{,}74_3$
2,9273	$1{,}957_6$	$3{,}83_2$
4,0638	$2{,}011_2$	$4{,}04_5$

[9]See R. Zurmühl, *Matrizen*, Springer, Berlin, Göttingen, Heidelberg 1950, p. 249.

TABLE 2

ϕ	$m_2(\phi)$	$m_2^2(\phi)$
0	4,694	22,03
0,3599	$4,57_0$	$20,8_9$
0,6775	$4,31_8$	$18,6_4$
0,9566	$4,06_2$	$16,5_0$
1,2230	$3,82_8$	$14,6_5$
1,5869	$3,54_8$	$12,5_9$
2,1245	$3,22_3$	$10,3_9$
2,5730	$3,02_1$	$9,1_2$
2,7629	$2,95_0$	$8,7_0$
3,0656	$2,85_3$	$8,1_4$
3,4585	$2,75_0$	$7,5_6$
3,9630	$2,65_0$	$7,0_2$
5,0854	$2,51_7$	$6,3_4$

TABLE 3

ϕ	$m_3(\phi)$	$m_3^2(\phi)$
0	7,855	61,70
0,6019	$7,64_2$	$58,4_1$
1,1400	$7,26_6$	$52,7_9$
1,6225	$6,88_9$	$47,4_6$
2,0898	$6,54_1$	$42,7_9$
2,7348	$6,11_5$	$37,4_0$
3,7083	$5,62_7$	$31,6_6$
4,5604	$5,35_4$	$28,6_7$
4,9419	$5,27_7$	$27,8_5$
5,5909	$5,20_3$	$27,0_7$
6,5430	$5,20_4$	$27,0_8$
7,9359	$5,30_7$	$28,1_7$
10,857	$5,37_3$	$28,8_7$

TABLE 4

ϕ	$m_4(\phi)$	$m_4^2(\phi)$
0	10,996	120,91
1,598	$10,18_5$	$103,7_3$
2,265	$9,61_7$	$92,4_9$
2,890	$9,04_6$	$81,8_3$
3,714	$8,30_5$	$68,9_7$
4,873	$7,39_4$	$54,6_7$
5,805	$6,81_6$	$46,4_6$
6,1975	$6,61_8$	$43,7_9$
6,8314	$6,35_7$	$40,4_2$
7,7011	$6,12_4$	$37,5_0$
8,981	$6,00_6$	$36,0_7$
11,888	$5,88_4$	$34,6_2$

squares which, according to (5.16), are proportional to the 4 lowest natural frequencies. Fig. 3 shows the corresponding $m(\phi)$-curves, points taken from Tables 1 to 4 being marked by small circles.

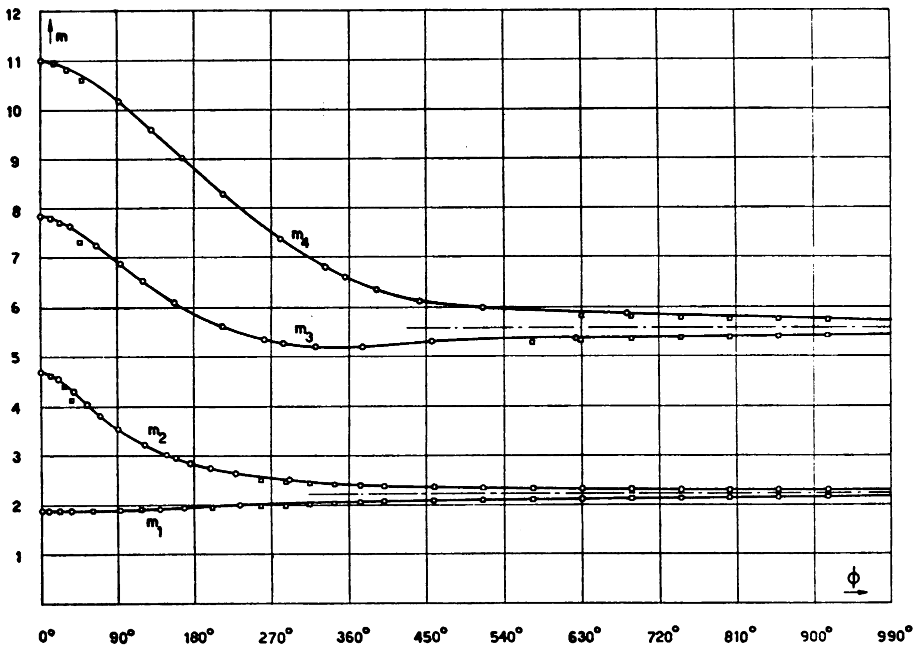

FIG. 3. Frequency curves.
Circular Frequencies $\kappa_i = (m_i^2/l^2)(\alpha/\mu)^{1/2}$
l: length of the rod α: flexural rigidity μ: mass per unit length ϕ: total twist

For higher modes, the numerical work is complicated by the appearance of differences of large numbers. Yet the computer (designed for 6 significant figures) would yield more than the 4 curves of Fig. 3.

7. Expansions. For large values of p, i.e. for small values of the total twist ϕ, the roots of (5.15) can be developed in the series

$$\lambda_1 = i\left(p + \frac{5}{2p} \cdots\right), \qquad \lambda_2 = p - \frac{5}{2p} \cdots , \qquad \lambda_{3,4} = \sqrt{2} \mp 1 \cdots . \tag{7.1}$$

The corresponding expansion of (5.10) is

$$(1 + \cos m \cosh m) + \frac{1}{p^2}\left[9 \sin m \sinh m - \frac{13}{2} m(\cos m \sinh m + \sin m \cosh m)\right] \cdots = 0. \tag{7.2}$$

Developing also

$$m = m^* + \frac{1}{p^2} m^{**} \cdots , \tag{7.3}$$

we obtain

$$m^{**} = \begin{cases} \left(9 - 13 \dfrac{m^*}{2} \cot \dfrac{m^*}{2}\right) \cot \dfrac{m^*}{2}, \\ -\left(9 + 13 \dfrac{m^*}{2} \tan \dfrac{m^*}{2}\right) \tan \dfrac{m^*}{2}, \end{cases} \tag{7.4}$$

according as m^* is an odd or an even root of (5.12).

Evaluating (7.4) and (7.3) for a number of values of p (chosen sufficiently large) and making use of (5.17), we obtain approximations of the frequency curves $m_i(\varphi)$ for small values of ϕ (marked, in Fig. 3, by small squares). For $\phi = 0$, the functions m_i are stationary; with increasing ϕ, however, m_1 increases slightly while $m_2, m_3, \cdots$ decrease.

For small values of p, the roots of (5.15) are

$$\lambda_{1,2} = i(1 \mp q \cdots), \qquad \lambda_{3,4} = \mp i + q \cdots . \qquad \left(q = \frac{p}{(2)^{1/4}} = \frac{p}{1.189}\right). \tag{7.5}$$

The corresponding expansion of (5.10) is

$$(1 + \cos n \cosh n)^2 - \frac{q^2}{4} (\cos n \sinh n - \sin n \cosh n)^2 = 0, \tag{7.6}$$

where

$$n = \frac{m}{\sqrt[4]{2}} = q\phi. \tag{7.7}$$

Developing

$$n = n^* + n^{**}q \cdots , \tag{7.8}$$

we find that n^* satisfies (5.12) and that $n^{**} = \pm 1/2$. Thus, (7.7) and (7.8) yield

$$n = \frac{2\phi}{2\phi \mp 1} n^* \cdots . \tag{7.9}$$

It follows from (7.9) that, for $\phi \to \infty$, two frequencies at a time tend to the same limit. The corresponding values of m, according to (7.7), are

$$M_i = \sqrt[4]{2}\, n_i^* \tag{7.10}$$

or, evaluated by means of (5.13),

$$M_1 = 2.230, \qquad M_2 = 5.582, \qquad M_3 = 9.341, \cdots . \tag{7.11}$$

According to (5.16), these limits correspond to the natural frequencies of an untwisted rod of flexural rigidity 2α.

In Fig. 3 the expansions for $\phi \to \infty$ (like those for $\phi \to 0$) are marked by small squares. For the fundamental mode, the expansions for $\phi \to 0$ and $\phi \to \infty$ approach each other closely. Evidently, they would suffice for many practical purposes. As to the higher modes, however, there remains a gap of increasing width between the two expansions (filled by the computing process described in 6). Particularly the expansion for $\phi \to 0$ is reliable only for very small values of the total twist.