

LOWER AND UPPER BOUNDS TO THE ULTIMATE LOADS OF BUCKLED REDUNDANT TRUSSES*

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Synopsis. In the buckled state, statically indeterminate, rigid-jointed trusses support loads which are generally in excess of those corresponding to initial instability. As buckling proceeds, the loads usually approach limiting values, called "ultimate loads". Two theorems are derived establishing lower and upper bounds to the ultimate loads. Elastic behavior is assumed throughout.

Introduction. If the rigid joints of a truss are subjected to a set of external loads, whose values are fixed except for a common multiplier λ_0 , the equations of equilibrium of the i -th joint are of the type¹

$$\sum_j S_{ij} \begin{Bmatrix} \cos \phi_{ij} \\ \sin \phi_{ij} \end{Bmatrix} + \lambda_0 \begin{Bmatrix} X_i \\ Y_i \end{Bmatrix} = 0 \quad (i = 1, 2, \dots, n) \quad (1a)$$

$$\sum_j M_{ij} = 0 \quad (i = 1, 2, \dots, n) \quad (1b)$$

where S_{ij} and M_{ij} are, respectively, the axial force and bending moment in bar (i, j) at joint i , ϕ_{ij} is the inclination of bar (i, j) , X_i and Y_i are external force components, and where the summation extends over all the bars adjoining i . If now all bars are composed of an elastic material and the elementary theory of beams in bending under axial forces is assumed to hold, the joint moments M_{ij} can be expressed as linear functions of the joint rotations θ_i [2]. Substitution of these expressions into Eqs. (1b) leads to a set of linear, homogeneous equations in the joint rotations of the type

$$\sum_{i=1}^n a_{ij} \theta_i = 0, \quad (i = 1, 2, \dots, n) \quad (2)$$

where the a_{ij} form a symmetric matrix whose components are transcendental functions of the physical characteristics of the bars, and of the axial forces. The equilibrium of the structure is stable only if the matrix is positive definite; a necessary condition of neutral equilibrium is given by

$$f(\lambda_0) \equiv |a_{ij}| = 0. \quad (3)$$

The lowest positive real root of Eq. (3) is the desired "critical" multiplier λ_0 at which buckling occurs.

By definition, the bar forces S_{ij} are uniquely determined by Eqs. (1a) for given λ_0 in a statically determinate truss. If, on the other hand, the truss be of m -th degree of redundancy relative to its axial force distribution, the most general expression for the bar forces is of the form

$$S_{ij} = \sum_{\xi=0}^m \lambda_{\xi} S_{ij}^{(\xi)}; \quad (4)$$

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¹The effect of the linear displacements is ignored in Eqs. (1) and (2). This is in conformity with the experimental results given in Ref. [1] (see the bibliography at the end of the paper).

in Eqs. (4), the force systems $S_{ij}^{(r)}$ are restricted by the set of equilibrium equations

$$\sum_i S_{ij}^{(r)} \begin{Bmatrix} \cos \phi_{ij} \\ \sin \phi_{ij} \end{Bmatrix} + \delta_{0r} \begin{Bmatrix} X_i \\ Y_i \end{Bmatrix} = 0, \quad (r = 0, 1, \dots, m; i = 1, 2, \dots, n) \quad (5)$$

where δ_{pq} is the Kronecker Delta.

It is convenient to restrict the force systems $S_{ij}^{(r)}$ further by the requirement (which can always be fulfilled) that

$$\sum_k \frac{S_k^{(0)} S_k^{(r)} L_k}{E_k A_k} = p \delta_{0r}, \quad (r = 0, 1, 2, \dots, m) \quad (6a)$$

$$\sum_k \frac{S_k^{(r)} S_k^{(s)} L_k}{E_k A_k} = \delta_{rs}, \quad (r, s = 1, 2, \dots, m) \quad (6b)$$

where the summation extends over all the members of the truss.² Eqs. (6a) indicate that the force system $S_{ij}^{(0)}$ is the actual force system, i.e. the one corresponding to minimum strain energy, in the unstressed, unbuckled truss for $\lambda_0 = 1$.

The axial forces in the buckled state are further restricted by the "compatibility condition" that

$$\delta L_k = (u_i - u_j) \cos \phi_{ij} + (v_i - v_j) \sin \phi_{ij}, \quad (7)$$

in which δL_k is the change in the distance between joints i and j relative to the unloaded state and (u, v) are the cartesian components of the linear displacement vectors of the joints. If both sides of Eqs. (7) be multiplied by $S_k^{(r)}$ ($r = 1, 2, \dots, m$) and summed up over all the members of the truss, we are led, after rearranging the summations and in view of Eqs. (5), to the system of "virtual work" equations

$$\sum_k S_k^{(r)} \delta L_k = 0. \quad (r = 1, 2, \dots, m) \quad (8)$$

For the sake of generality it will be assumed that the truss has been prestressed, with the initial force system expressed by

$$S_k^* = \sum_{i=1}^m \lambda_i^* S_k^{(i)}. \quad (9)$$

On the other hand, δL_k can be written in the form

$$\delta L_k = \frac{(S_k - S_k^*) L_k}{E_k A_k} - \delta_k, \quad (10)$$

where δ_k represents the shortening of the chord length of the k -th member corresponding to its curvature in bending. If we substitute Eqs. (10) in (8) and consider Eqs. (6) and (9), we arrive at the following set of equations for any bent state:

$$\lambda_r - \lambda_r^* = \sum_k S_k^{(r)} \delta_k. \quad (r = 1, 2, \dots, m) \quad (11a)$$

It has been shown elsewhere [3] that, for an actual buckling mode,

$$\sum_k S_k^{(r)} \delta_k = \frac{1}{2} \sum_i \sum_j a_{ij,r} \theta_i \theta_j = \mu f_{,r} \quad (r = 0, 1, 2, \dots, m) \quad (11b)$$

² S_k, L_k, E_k, A_k are the axial force, length, modulus of elasticity and cross-sectional area, respectively, of the k -th bar.

in which a subscript, preceded by a comma, designates the partial derivative with respect to the corresponding parameter λ , or $f_{,r} \equiv \partial f / \partial \lambda_r$. Thus, during buckling,

$$\lambda_r - \lambda_r^* = \mu f_{,r}, \quad (r = 1, 2, \dots m) \quad (12a)$$

where $\mu > 0$ is a measure of the extent to which buckling has proceeded and is expressed by

$$\mu = \frac{1}{2} \frac{\sum_{i=1}^n \theta_i^2}{\sum_{i=1}^n A_{ii}}, \quad (12b)$$

in which A_{ii} is the cofactor of a_{ii} in Eq. (3). It follows immediately from Eqs. (3) and (12) that, as buckling commences, the load λ_0 will generally increase, and never decrease. Considering all parameters λ_r ($r = 0, 1, 2, \dots m$) to be functions of μ , Eq. (3) can be written in the form

$$F(\mu) \equiv f[\lambda_0(\mu), \lambda_1(\mu), \dots \lambda_m(\mu)] \equiv 0 \quad (13)$$

for all values of μ . Differentiating Eq. (13) with respect to μ , we obtain the identity

$$\frac{dF}{d\mu} \equiv f_{,0} \frac{d\lambda_0}{d\mu} + \sum_{\xi=1}^m f_{,\xi} \frac{d\lambda_\xi}{d\mu} \equiv 0. \quad (14)$$

On the other hand, by dividing Eq. (12a) by μ and letting μ approach zero, we are led to

$$\frac{d\lambda_r}{d\mu} (\mu = 0) = f_{,r} (\mu = 0) \quad (r = 1, 2, \dots m) \quad (15)$$

It follows, from Eqs. (14) and (15), that as the truss is on the verge of buckling

$$\frac{d\lambda_0}{d\mu} = -(f_{,0})^{-1} \sum_{\xi=1}^m (f_{,\xi})^2, \quad (16)$$

where all the terms in Eq. (16) are to be evaluated at $\mu = 0$.

It can be shown that $f_{,0}$ is negative. In fact, if λ_0^1 be the smallest positive buckling parameter, i.e. the one corresponding to neutral equilibrium, the positive definiteness of the matrix $[a_{ii}]$ implies $f(\lambda_0) > 0$ for $\lambda_0 < \lambda_0^1$. Since, by Eq. (3), $f(\lambda_0^1) = 0$, and if multiple roots be excluded, it follows that

$$f_{,0}(\lambda_0 = \lambda_0^1) < 0. \quad (17)$$

An inspection of Eqs. (16) and (17) shows that

$$\frac{d\lambda_0}{d\mu} (\mu = 0) \geq 0, \quad (18)^3$$

where the equality sign applies only to the special case

$$f_{,r}(\lambda_r = \lambda_r^*) = 0. \quad (r = 1, 2, \dots m)$$

It follows further from Eqs. (12) that for increasing μ , and provided the λ_r all stay

³Actually, this relationship applies to all values of μ , as can be demonstrated by means of energy considerations.

finite,⁴ the external and internal forces approach a state governed by the equations

$$\begin{aligned} f(\lambda_0^u, \lambda_1^u, \dots, \lambda_m^u) &= 0, \\ f_{,r}(\lambda_0^u, \lambda_1^u, \dots, \lambda_m^u) &= 0, \quad (r = 1, 2, \dots, m) \end{aligned} \quad (19)$$

in which the superscript designates limiting, or "ultimate" values of the parameters, with λ_0^u referred to as the "ultimate load".

The determination of λ_0^u is of interest to the engineer since it represents the ultimate carrying capacity of the truss. Since, however, the solution of Eqs. (19) presents formidable numerical obstacles, it is easier to estimate the value of λ_0^u by means of two theorems establishing lower and upper bounds to the ultimate load.

A stability criterion. Before proceeding to these two theorems, it is convenient to establish a stability criterion for the unbuckled truss, which is to be used in the later proof. This criterion is based on the assumption that, for a given system of external forces identified by the load parameter λ_0 , the potential energy corresponding to the unbuckled state is less than that of any neighboring, "geometrically consistent" bent state if the equilibrium of the truss is to be stable, or

$$V^* < V. \quad (20)$$

In what follows, a geometrically consistent bent state is defined as one in which the deflection curves of the bars are of sufficient degree of smoothness to make the discussion meaningful, and satisfy the geometric boundary conditions of continuity at the joints. Geometric consistency also implies the satisfaction of Eqs. (7) and (10); it is finally assumed that the force equations (4), (5), and (6), and therefore also Eqs. (8) and (11a), are valid.

With these definitions, the potential energy

$$V = U - W = \frac{1}{2} \sum_k \frac{S_k^2 L_k}{E_k A_k} + U_B - \lambda_0 \sum_i (X_i u_i + Y_i v_i),$$

in which U_B is the strain energy associated with the bending of the truss bars, can be expressed by Eqs. (4), (5), (6) and (7) in the form

$$V = \frac{1}{2} p \lambda_0^2 + \frac{1}{2} \sum_{\xi=1}^m \lambda_{\xi}^2 + U_B - \sum_k S_k \delta L_k;$$

in view of Eqs. (8), (9), and (10), this reduces to

$$V = -\frac{1}{2} p \lambda_0^2 + \frac{1}{2} \sum_{\xi=1}^m \lambda_{\xi}^2 + U_B + \lambda_0 \sum_k S_k^{(0)} \delta_k.$$

Similarly, the potential energy in the unbent state is given by

$$V^* = -\frac{1}{2} p \lambda_0^2 + \frac{1}{2} \sum_{\xi=1}^m \lambda_{\xi}^{*2}.$$

Thus, criterion (20) now takes the form

$$V - V^* = \frac{1}{2} \sum_{\xi=1}^m (\lambda_{\xi} - \lambda_{\xi}^*)(\lambda_{\xi} + \lambda_{\xi}^*) + U_B + \lambda_0 \sum_k S_k^{(0)} \delta_k \geq 0$$

⁴For a discussion of this question see Ref. [3].

which, by Eqs. (11a) and by considering only states in the neighborhood of the unbuckled state, is finally expressed as

$$V - V^* = \sum_k S_k \delta_k + U_B \geq 0, \quad (21)$$

where the equality sign applies to the case of neutral equilibrium, with the bent state represented by the first buckling mode. The stability criterion (21) can also be demonstrated directly. In fact, if the arbitrary set of deflection curves $y_k(x)$ be developed in a Fourier series

$$y_k(x) = \sum_{\tau=1}^{\infty} c_{\tau} \eta_k^{\tau}(x).$$

in which $\eta_k^{\tau}(x)$ represents the normalized buckling mode associated with the positive load parameter λ_0^{τ} , it can be shown, by a series of partial integrations and by virtue of the orthogonality of the normal modes, that

$$\sum_k S_k \delta_k + U_B = \sum_{\tau=1}^{\infty} c_{\tau}^2 (\lambda_0^{\tau} - \lambda_0).$$

The right side of this equation is positive definite if $\lambda_0 < \lambda_0^1$, i.e. when the load parameter is smaller than the smallest buckling load.

Lower bounds to the ultimate load. After these preliminary remarks, we now proceed to state the first theorem. In what follows, we shall call a load parameter $\lambda_0 > 0$ "critical" if there exists a set of prestressing parameters λ_r^* ($r = 1, 2 \dots m$) such that the truss is in neutral equilibrium with its force system defined by $(\lambda_0, \lambda_1^*, \dots, \lambda_m^*)$. With this definition, which is analogous to that of a "statically admissible multiplier" in rigid-plastic limit design [4], we state

THEOREM I: *The ultimate load is the largest of all critical loads.*

From this theorem we deduce the following

COROLLARY: If an arbitrary internal force system, and an external load system identified by λ_0' , satisfy the force equations of equilibrium, and if the truss, with its forces so defined, is in stable equilibrium, then

$$\lambda_0'' > \lambda_0'. \quad (22)$$

It is clear that this corollary establishes an easily calculable lower bound to the ultimate load since the stability of a truss for a given force system can be determined by a number of stability criteria [5, 6].

PROOF: It can readily be verified that if the prestressing parameters be so selected that $\lambda_r^* = \lambda_r''$ ($r = 1, 2 \dots m$), Eqs. (3) and (12a) are identically satisfied by

$$\lambda_r \equiv \lambda_r'' \quad (r = 0, 1, 2, \dots m)$$

for all values of μ . In other words, in this "special case", the truss buckles similarly to a statically determinate truss under constant external loads and internal forces.⁵

⁵It has been shown previously [7] that this case corresponds to a stationary value of the buckling load. See also Eq. (18).

Inspection of Eq. (11a) shows that the buckling mode of the special case is such as to satisfy

$$\sum_k S_k^{(r)} \delta_k = 0. \quad (r = 1, 2, \dots, m) \quad (23)$$

If now the stability criterion (21) be applied to a truss in stable equilibrium under an external load λ'_0 , and if the arbitrary deflection curves be so selected as to coincide with the buckling mode of the "special case", it follows from Eqs. (23) that

$$V - V^* = \lambda'_0 \sum_k S_k^{(0)} \delta_k + U_B > 0.$$

Since, also by Eqs. (21) and (23),

$$\lambda_0'' \sum_k S_k^{(0)} \delta_k + U_B = 0,$$

and since further U_B is positive definite, it follows that

$$\lambda_0'' > \lambda'_0. \quad (22)$$

Upper bounds to the ultimate load. Before proceeding to the discussion of upper bounds to the ultimate load, it is useful to state the following

LEMMA I: For a given set of prestressing parameters, the value of the critical load of a truss is not reduced by increasing the moment of inertia of a bar.⁶

PROOF: As stated before, a truss is in stable equilibrium only if the matrix of the coefficients a_{ii} is positive definite, or

$$Q = \frac{1}{2} \sum_i \sum_j a_{ij} \theta_i \theta_j \geq 0 \quad (24)$$

where the equality sign applies only to the trivial case $\theta_i = 0$ ($i = 1, 2, \dots, n$). By re-arranging the terms in the double sum, this can be expressed [8] in the form

$$Q = \frac{1}{8} \sum_k S_k L_k [(\epsilon_k \coth \epsilon_k - 1)^{-1} (\theta_k + \theta'_k)^2 + (\epsilon_k)^{-1} (\theta_k - \theta'_k)^2 \coth \epsilon_k], \quad (25a)$$

where ϵ_k is defined by

$$\epsilon_k^2 = \frac{S_k L_k^2}{4 E_k I_k} > -\pi^2, \quad (25b)$$

and where θ_k and θ'_k are, respectively, the rotations of the two joints connected by the k -th bar of moment of inertia I_k .

Let us now assume that a truss T is in stable equilibrium for a given set of internal and external forces, and consider the equilibrium, under the same force system, of a truss T' which is identical with T as to its geometric and elastic properties, except that the moment of inertia of its m -th bar has been increased, i.e.

$$I'_m > I_m.$$

If this bar be a tension member, it follows from Eq. (25b) that

$$(\epsilon'_m)^2 < \epsilon_m^2$$

⁶This lemma, which has immediate physical appeal, is analogous to a similar principle of Rayleigh dealing with the natural frequencies of vibration of elastic systems.

and, by inspection of Eq. (25a), that the value of the m -th term in the series has not decreased. The same can readily be demonstrated for compression members, which correspond to imaginary values of ϵ , and for unstressed members. Consequently, for any given set of joint rotations,

$$Q' \geq Q;$$

in view of Eq. (24), the truss T' is therefore also in stable equilibrium.

As a next step, it is easy to establish the following

LEMMA II: The value of the ultimate load of a truss is not lowered by increasing the moment of inertia of a bar.

PROOF: This is a direct consequence of the previous lemma and of Theorem I. In fact, if the prestressing parameters λ_r^* ($r = 1, 2 \dots m$) be so selected as to make the critical load λ_0 equal to the ultimate load λ_0^u for truss T , it follows from Lemma I that, for the same λ_r^* , the critical load λ_0' of truss T' satisfies the relationship

$$\lambda_0' \geq \lambda_0 = \lambda_0^u.$$

On the other hand, it follows from Theorem I that the ultimate load λ_0'' of truss T' is the largest of all critical loads of T' , or

$$\lambda_0'' \geq \lambda_0'$$

which, in view of the foregoing, implies

$$\lambda_0'' \geq \lambda_0^u. \quad (26)$$

An upper bound to the ultimate load can now be established by means of the fact that a truss of m -th degree of redundancy can be converted into a rigid-link mechanism of one degree of freedom by considering it pin-jointed and by removing l members, where $l \leq m + 1$. In general, it will be possible⁷ to select these l members in such a way that the mechanism so created is capable of joint velocities corresponding to a shortening of all the eliminated bars.

If now these l members be assigned compressive forces

$$S_k = -4\pi^2 E_k I_k (L_k)^{-2} \quad (k = 1, 2, \dots, l) \quad (27)$$

and if the remaining bar forces be so chosen as to satisfy the force equations of equilibrium (2a) for $\lambda_0 = \lambda_0'' > 0$, then, by

THEOREM II:

$$\lambda_0^u \leq \lambda_0''. \quad (28)$$

In other words, the load parameter found by the method described above represents an upper bound to the ultimate load.

The proof of this follows from the second lemma. In fact, since Eq. (27) defines the buckling load of a column which is fixed at both ends, it is apparent that λ_0'' can be interpreted as the ultimate load of a truss which is identical with the truss under

⁷Otherwise, the truss does not exhibit an ultimate load, i.e., the load increases indefinitely during buckling.

consideration except that all bars other than the selected l bars described above have infinite moment of inertia. Thus the second lemma implies the relationship (28).

It should be pointed out that the practical usefulness of the upper bound defined by Theorem II is sharply limited. A more detailed discussion of this question can be found elsewhere [3]; suffice it to state here that the theorem is much less broad in scope than a similar one establishing upper bounds to the collapse load of a plastic-rigid frame. [4] This is due to the fact that it has not been possible to construct an infinity of readily calculable "kinematically admissible" collapse modes by any simple process analogous to the insertion of a sufficient number of yield hinges. Fortunately, except for the purpose of estimating the degree of accuracy attained, the interest of the engineer is focused on the lower bound only, which can be made to approach the exact value of the collapse load as closely as desired.

Conclusion. Two principles have been derived which establish lower and upper bounds to the ultimate load sustained by a buckled redundant truss. Of these, the lower bound permits an approach to the exact value from below within any desired degree of accuracy.

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