ON THE GAPS IN THE SPECTRUM OF THE HILL EQUATION*

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1. Let $f = f(t)$ be a real-valued, continuous, periodic function of period 1, so that

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n \exp(2\pi \text{ int}), \qquad (c_{-n} = \bar{c}_n), \tag{1}$$

and consider the Hill equation

$$x'' + (\lambda + f(t))x = 0, \qquad (\lambda \text{ real}; \ ' = d/dt). \tag{2}$$

It is known that (if $f \not\equiv 0$) there exists a sequence of closed intervals I_k: $\lambda_k \leq \lambda \leq \lambda^k$ (region of stability), where $\lambda_k < \lambda^k < \lambda_{k+1}$ and $k = 1, 2, \cdots$, with the property that (2) has some solution $x \not\equiv 0$ which is bounded on $-\infty < t < \infty$ if and only if λ belongs to the closed set $S = \sum I_k$; cf. [7], p. 14. The complementary set of S consists of a half-line $-\infty < \lambda < \lambda_1$ and the sequence of open intervals $J_k : \lambda^k < \lambda < \lambda_{k+1}$, $k = 1, 2, \cdots$. In several recent papers, various lower bounds for the value λ_1, the least point of the set S, in terms of the Fourier coefficients c_n of $f(t)$, have been obtained; [11], [5], [3]. The present note will be devoted to the problem of obtaining estimates (upper bounds) of the lengths $\lambda_{k+1} - \lambda^k$ of the "gaps" J_k of the set S in terms of these Fourier coefficients.

It follows from [4], p. 613, that the length of every gap J_k is surely not greater than

$$\limsup_{t\to\infty} f(t) - \liminf_{t\to\infty} f(t) \leq 4 \sum_{n=1}^{\infty} | c_n |. \tag{3}$$

In addition, asymptotic estimates, as $\lambda^k \to \infty$, for these gaps are known; [2]. In fact, since $f(t)$ is uniformly continuous on $0 \leq t < \infty$, the lengths $\lambda_{k+1} - \lambda^k$ of the intervals J_k tend to zero as $\lambda_{k+1} \to \infty$; *loc. cit.*, p. 850. Furthermore, additional regularity conditions on $f(t)$ result in more refined estimates. It should be pointed out here that the investigations of [2] related to singular boundary value problems ([8]) on the half-line $0 \leq t < \infty$ determined by (2) and a linear, homogeneous boundary condition at $t = 0$, and were not confined to the special case that $f(t)$ be periodic.

Let $m(\lambda)$, for $-\infty < \lambda < \infty$, be defined to be the distance from λ to the set S considered above, so that

$$m(\lambda) = \text{g.l.b.}\ | \lambda - \mu |, \qquad \mu \text{ in } S. \tag{4}$$

It will be shown in section 2 below that $m(\lambda)$ satisfies the inequality

$$m^2(\lambda) \leq 2 \sum_{n=1}^{\infty} | c_n |^2, \qquad \text{provided} \qquad \lambda \geq -c_0\ . \tag{5}$$

As a consequence of (4) and (5), one readily sees that the lengths $\lambda_{k+1} - \lambda^k$ of the gaps J_k satisfy

$$\lambda_{k+1} - \lambda^k \leq 2\left(2 \sum_{n=1}^{\infty} | c_n |^2\right)^{1/2}, \qquad \text{provided} \qquad \frac{1}{2}(\lambda_{k+1} + \lambda^k) \geq -c_0\ . \tag{6}$$

It will remain undecided whether (6) actually must hold for *all* gaps J_k, so that the first inequality of (6) would hold without the proviso of the second inequality. In any

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case, it is readily seen that the estimate of (6), when it applies, is an improvement over that of (3), namely $4 \sum_{n=1}^{\infty} | c_n |$.

In this connection, it should be pointed out that Kato [3], by an adaptation of a relation used by Wintner [11], has obtained the inequality

$$\lambda_1 \geq -c_0 - \left(\frac{1}{8}\right) \sum_{n=1}^{\infty} | c_n |^2,$$

for the least point λ_1 of the set S. (Wintner had previously shown that $\lambda_1 \geq -c_0 - 2 \cdot \sum_{n=1}^{\infty} | c_n |^2$.) Consequently, it is easily seen that the first inequality of (6) is surely valid for *all* gaps J_k if, for instance, the inequality

$$\left(\frac{1}{8}\right) \sum_{n=1}^{\infty} | c_n |^2 \leq \left(2 \sum_{n=1}^{\infty} | c_n |^2\right)^{1/2}$$

holds. (If one normalizes f so that its mean value is zero, hence $c_0 = 0$, this last inequality is equivalent to $\int_0^1 f^2 \, dt \leq 256$).

Before proceeding to the proof of (5), it can be noted that the first inequality of (5) surely becomes false if the restriction $\lambda \geq -c_0$ is dropped. In fact, if $f(t) \equiv c_0$, so that (2) becomes the differential equation of the harmonic oscillator, then $\sum_{n=1}^{\infty} | c_n |^2 = 0$, and (5) yields the known result that $m(\lambda) \equiv 0$ for $\lambda \geq -c_0$. However, $m(\lambda) > 0$ for $\lambda < -c_0$, since S is the half-line $-c_0 \leq \lambda < \infty$.

2. The proof of (5) will depend upon certain results obtained in [6]. Let $g_1(t), g_2(t), \cdots$, denote a sequence of functions possessing continuous second derivatives on $0 \leq t < \infty$, satisfying

$$g_n(0) = g_n'(0) = 0, \tag{7}$$

and such that $g_n(t) \to 0$ uniformly on every finite t-interval $[0, T]$. Then, if g_n and $L(g_n)$ (where $L(x) \equiv x'' + fx$) are of class $L^2 [0, \infty)$, the inequality

$$m^2(\lambda) \liminf_{n \to \infty} \int_0^{\infty} g_n^2 \, dt \leq \liminf_{n \to \infty} \int_0^{\infty} (L(g_n) + \lambda g_n)^2 \, dt \tag{8}$$

holds. This follows readily by a method analogous to that given in [6], p. 580. (It is to be noted that the set S considered above is identical with the invariant spectrum (Weyl [8], p. 251) associated with the differential equation (2); [9], [1]. Moreover, the investigations of [6] related to the Weyl theory of singular boundary value problems, alluded to in section 1.)

Next, let $\mu > 0$, and let $g_n = y_n h$, where $h = \sin (\mu^{\frac{1}{2}} t)$ or $h = \cos (\mu^{\frac{1}{2}} t)$, and the $y_n = y_n(t)$ are functions possessing continuous second derivatives on $0 \leq t < \infty$. In addition, suppose that $y_n(0) = y_n'(0) = 0$, so that (7) certainly holds. and that y_n and $L(y_n)$ belong to $L^2(0, \infty)$. Finally, suppose that the y_n are such that the "lim inf" appearing on the left side of the inequality (8) can be replaced by "lim" for both $h = \sin (\mu^{\frac{1}{2}} t)$ and $h = \cos (\mu^{\frac{1}{2}} t)$.

It follows from (8) that

$$m^2(\lambda) \lim_{n \to \infty} \int_0^{\infty} y_n^2 h^2 \, dt \leq \liminf_{n \to \infty} \int_0^{\infty} ([y_n'' + (\lambda - \mu + f) y_n] h + 2 y_n' h')^2 \, dt.$$

If now the y_n satisfy

$$\int_0^{\infty} y_n'^2 \, dt \to 0, \qquad \int_0^{\infty} y_n''^2 \, dt \to 0, \qquad (n \to \infty),$$

it is seen that

$$m^2(\lambda) \lim_{n\to\infty} \int_0^\infty y_n^2 h^2 \, dt \leq \liminf_{n\to\infty} \int_0^\infty (\lambda - \mu + f)^2 y_n^2 h^2 \, dt. \tag{9}$$

Since (9) holds for both functions h, addition of the two corresponding inequality relations yields, in view of the fact that $\liminf A + \liminf B \leq \liminf (A + B)$, the inequality

$$m^2(\lambda) \lim_{n\to\infty} \int_0^\infty y_n^2 \, dt \leq \liminf_{n\to\infty} \int_0^\infty (\lambda - \mu + f)^2 y_n^2 \, dt. \tag{10}$$

Let $T > 0$ and define the function $Y_T(t)$ on $0 \leq t < \infty$ so that the graph of $Y_T(t)$ on $0 \leq t \leq T$ consists of three line segments joining, in order, the four points $(0, 0)$, $(1, T^{-\frac{1}{2}})$, $(T - 1, T^{-\frac{1}{2}})$, and $(T, 0)$. On $T < t < \infty$, let $Y_T(t) \equiv 0$. It is clear that the corners of this function can be smoothed out so as to obtain a function $y_T(t)$ satisfying the conditions imposed upon the y_n above. Furthermore, it is clear that if $y_n = y_{T_n}$, where $T = T_n \to \infty$ as $n \to \infty$, one can arrange that the functions y_n be such as to make (10) imply

$$m^2(\lambda) \leq \liminf_{S\to\infty} S^{-1} \int_0^S (\lambda - \mu + f)^2 \, dt, \qquad (\mu \geq 0). \tag{11}$$

(It is clear that the inequality $\mu \geq 0$ in (11), and not merely $\mu > 0$, can be allowed.) Now suppose that $\lambda \geq -c_0$ and choose $\mu \geq 0$ so that $\lambda - \mu = -c_0$. Then (11), (1), and the Parseval relation yield

$$m^2(\lambda) \leq \int_0^1 (-c_0 + f)^2 \, dt = 2 \sum_{n=1}^\infty | c_n |^2,$$

so that the relation (5) is now proved.

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