ON FINITE TWISTING AND BENDING OF CIRCULAR RING SECTOR PLATES AND SHALLOW HELICOIDAL SHELLS*

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1. Introduction. In the following we consider a thin circular ring sector plate under the action of two equal and opposite forces perpendicular to the plane of the plate, along the axis through the center of the ring (Fig. 1). The ring sector plate may be con-

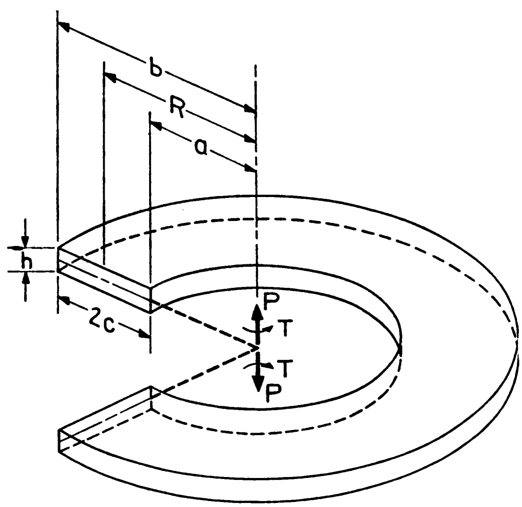

Fig. 1.

sidered as part of a winding of a close-coiled helical spring. The problem of the stress distribution in the twisted ring sector with rectangular cross section has first been considered by J. H. Michell[1] as a problem of three dimensional elasticity. A solution of this problem by means of the theory of thin plates will bear the same relation to Michell's solution as Kelvin and Tait's solution for the torsion of a rectangular plate bears to St. Venant's solution for the torsion of a beam with rectangular cross section.

The reason for the present note is the further observation that by treating the problem as a plate problem it becomes possible to analyse non-linear effects in a relatively simple manner by making use of the equations for finite deflections of thin plates.

In addition to the problem of non-linear effects for an originally flat plate we also consider the corresponding problem for a shallow helicoidal shell, thereby obtaining information concerning the influence of initial pitch on stresses and deformations in the spring.

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[1]J. H. Michell, Proc. London Math. Soc., **31**, 140-141 (1889).

In the consideration of both these problems one is led to a study of the simultaneous action of the pair of axial forces described above and of a pair of couples the axes of which coincide with the direction of these forces (Fig. 1). We determine in particular of what magnitude the couples have to be in order to prevent the association of circumferential displacements of the points of the ring plate with the axial displacements caused by the axial forces. We find that if these circumferential displacements are prevented the effect of non-linearity and the influence of initial pitch are much more pronounced than in the absence of the couples. We also encounter a problem of instability of the ring sector plate which is associated with the presence of the couples.

A further result which we find is to the effect that non-linearity is relatively more pronounced in regard to the magnitude of stresses than in regard to the axial force-displacement relation. A similar result is known for the problem of finite torsion of thin rectangular plates[2].

It seems worth noting that the present developments may also be considered as a contribution to a non-linear theory of dislocations.

Apart from the question of method the present treatment of the problems of the ring sector differs from that by means of the theory of thin rods[3] in the following respects.

Account is taken of a non-linear effect which is significant if the ratio of width of cross section to thickness of cross section is sufficiently large. For the helicoidal shell this effect is found to be present even in the range of applicability of the linearized theory. In the theory of torsion of rectangular plates this effect manifests itself through axial normal stresses proportional to the square of the angle of twist. These normal stresses have no axial resultant or bending couple but give rise to a twisting couple which is proportional to the cube of the angle of twist[2]. A corresponding result is here obtained for the problem of the ring sector.

While the results of the theory of thin rods are based on the assumption that the ratio of width of cross section to radius of center line of cross section is sufficiently small the present results hold for all possible values of this ratio.

2. Differential equations and boundary conditions of the problem. The differential equations for finite bending of thin plates of uniform thickness are in polar coordinate form[4]

$$D\nabla^2\nabla^2 w = p(r, \theta) + \frac{1}{r}\frac{\partial}{\partial r}\left(rN_r \frac{\partial w}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(N_\theta \frac{1}{r}\frac{\partial w}{\partial \theta}\right)$$

$$+ \frac{1}{r}\frac{\partial}{\partial r}\left(N_{r\theta}\frac{\partial w}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(N_{r\theta}\frac{\partial w}{\partial r}\right), \tag{1}$$

$$\nabla^2\nabla^2 F = Eh\left\{\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right]^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right\}, \tag{2}$$

where w is the transverse deflection, F Airy's stress function, and $D = Eh^3/12(1 - \nu^2)$.

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[2]S. Timoshenko, *Strength of materials*, Part 2, Van Nostrand, p. 301; A. E. Green, Proc. Roy. Soc. London (A) **154**, 430-455 (1936); **161**, 197-220 (1937).

[3]See A. E. H. Love, *Treatise on the mathematical theory of elasticity*, 4th Ed., Cambridge 1934, pp. 414-417 for reference to work by Kirchhoff, Kelvin and Tait, St. Venant, and Perry.

[4]K. Federhofer, Z. Angew. Math. Mech. **25/27**, 20 (1947).

Stress resultants and couples are given as follows

$$N_\theta = \frac{\partial^2 F}{\partial r^2}, \qquad N_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}, \qquad N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right), \tag{3}$$

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right), \qquad M_\theta = -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right),$$
$$M_{r\theta} = -(1-\nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right), \tag{4}$$

$$V_r = -D\frac{\partial \nabla^2 w}{\partial r}, \qquad V_\theta = -D\frac{\partial \nabla^2 w}{r\,\partial \theta}, \tag{5}$$

$$R_r = V_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} + N_r\frac{\partial w}{\partial r} + N_{r\theta}\frac{1}{r}\frac{\partial w}{\partial \theta},$$
$$R_\theta = V_\theta + \frac{\partial M_{r\theta}}{\partial r} + N_\theta\frac{1}{r}\frac{\partial w}{\partial \theta} + N_{r\theta}\frac{\partial w}{\partial r}. \tag{6}$$

In addition to this there occur concentrated corner forces of magnitude $2M_{r\theta}$ at corners formed by lines r = const. and θ = const.

Let $r = a$ and $r = b$ be the inner and outer circular edges of the plate and let $\theta = 0$ and $\theta = \alpha$ be the radial edges of the plate (Fig. 1). We prescribe the following boundary conditions, assuming for the present that the couples T indicated in Fig. 1 are absent,

$$r = a,\, b; \quad N_r = N_{r\theta} = R_r = M_r = 0 \tag{7}$$

$$\theta = 0,\, \alpha; \begin{cases} \int_a^b N_\theta\, dr = \int_a^b N_{r\theta}\, dr = \int_a^b rN_\theta\, dr = \int_a^b M_\theta\, dr = 0 \\ \int_a^b R_\theta\, dr + 2(M_{r\theta})_a - 2(M_{r\theta})_b = P \\ \int_a^b rR_\theta\, dr + 2(rM_{r\theta})_a - 2(rM_{r\theta})_b = 0 \end{cases} \tag{8}$$

The boundary conditions at the edges θ = const. are taken in such form that a semi-inverse procedure of solution becomes possible.

3. Solution of the boundary value problem. The fact that the solution to be obtained should give stress resultants and couples which are independent of the polar angle θ and a transverse displacement w which is proportional to θ and knowledge of the corresponding solution in three-dimensional linear elasticity suggests the following form of w and F,

$$w = k\theta, \qquad F = F(r), \tag{9}$$

Substitution of (9) in the differential equations (1) and (2), with $p(r, \theta) = 0$, shows that (1) is identically satisfied and (2) becomes

$$\nabla^2\nabla^2 F = Eh\frac{k^2}{r^4}. \tag{10}$$

From (10),

$$F = \tfrac{1}{8}Ehk^2[(\ln r)^2 + A \ln r + Br^2 + Cr^2 \ln r], \tag{11}$$

where A, B, C are constants of integration and an unessential additive constant has been omitted.

Stress resultants and stress couples are now

$$\begin{aligned}
N_r &= \frac{1}{8} Ehk^2\left[2 \frac{\ln r}{r^2} + \frac{A}{r^2} + 2B + C(1 + 2 \ln r)\right],\\
N_\theta &= \frac{1}{8} Ehk^2\left[2 \frac{1 - \ln r}{r^2} - \frac{A}{r^2} + 2B + C(3 + 2 \ln r)\right],\\
N_{r\theta} &= 0, \qquad V_r = 0, \qquad V_\theta = 0, \\
M_r &= 0, \qquad M_\theta = 0, \qquad M_{r\theta} = (1 - \nu)D \frac{k}{r^2}, \\
R_r &= 0, \qquad R_\theta = -2(1 - \nu)D \frac{k}{r^3} + N_\theta \frac{k}{r} \cdot
\end{aligned} \tag{12}$$

Determination of the constants of integration A, B, C and k is effected by means of the boundary conditions (7) and (8), some of which are seen to be satisfied identically. The relevant remaining conditions are the first of equations (7) and the third and fifth of Eqs. (8). We have from (7)

$$\begin{aligned}
\frac{A}{a^2} + 2B + C(1 + 2 \ln a) &= -2 \frac{\ln a}{a^2}, \\
\frac{A}{b^2} + 2B + C(1 + 2 \ln b) &= -2 \frac{\ln b}{b^2} \cdot
\end{aligned} \tag{13}$$

The third of Eqs. (8) may be transformed as follows if account is taken of the fact that $F'(b) = F'(a) = 0$,

$$\int_a^b rN_\theta \, dr = \int_a^b rF'' \, dr = (rF')_a^b - \int_a^b F' \, dr = F(b) - F(a) = 0.$$

We have then

$$\begin{aligned}
A \ln (b/a) + B(b^2 - a^2) + C(b^2 \ln b - a^2 \ln a) &= -(\ln b)^2 + (\ln a)^2 \\
&= -\ln (b/a) \ln ab.
\end{aligned} \tag{14}$$

Equation (13) and (14) serve to determine A, B and C. The remaining fifth of Eqs. (8) is to be used for the determination of the relation between force P and deflection k. We may write

$$\int_a^b N_\theta r^{-1} \, dr = \int_a^b F''r^{-1} \, dr = (F'r^{-1})_a^b + \int_a^b F'r^{-2} \, dr = \int_a^b N_r r^{-1} \, dr.$$

Therewith, we have from this fifth of Eqs. (8),

$$(1-\nu)Dk\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\frac{1}{8}Ehk^3\left[\frac{1+2\ln a}{2a^2}-\frac{1+2\ln b}{2b^2}+\frac{A}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right.$$
$$\left.+2B\ln\frac{b}{a}+C\left(\ln\frac{b}{a}+(\ln b)^2-(\ln a)^2\right)\right]=P. \qquad (15)$$

After some transformations in which we make use of the following formulas

$$A\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=2C\ln\frac{b}{a}+2\left(\frac{\ln b}{b^2}-\frac{\ln a}{a^2}\right),$$

$$2B(b^2-a^2)=-C[b^2-a^2+2(b^2\ln b-a^2\ln a)]-2\ln\frac{b}{a}, \qquad (16)$$

$$C\left\{\frac{2a^2b^2}{b^2-a^2}\left(\ln\frac{b}{a}\right)^2-\frac{b^2-a^2}{2}\right\}=\ln\frac{b}{a}\left\{1-\frac{b^2+a^2}{b^2-a^2}\ln\frac{b}{a}\right\},$$

$$a=R-c,\qquad b=R+c,\qquad E=2(1+\nu)G. \qquad (17)$$

Eq. (15) appears in the form

$$k\left[1+\frac{3(1+\nu)}{4}f_1\left(\frac{c}{R}\right)\left(\frac{k}{h}\right)^2\right]=\frac{3PR^3}{2Gch^3}\left[1-\left(\frac{c}{R}\right)^2\right]^2, \qquad (18)$$

where

$$f_1(x)=\frac{\left\{\left[1-\left(\frac{1-x^2}{2x}\ln\frac{1+x}{1-x}\right)^2\right]^2-\left(\frac{1-x^2}{2x}\ln\frac{1+x}{1-x}\right)^2\left[\frac{1+x^2}{2x}\ln\frac{1+x}{1-x}-1\right]^2\right\}}{\left\{1-\left(\frac{1-x^2}{2x}\ln\frac{1+x}{1-x}\right)^2\right\}}. \qquad (19)$$

For sufficiently small values of x, practically up to $x\sim 1/4$, we have

$$f_1(x)\sim\frac{16}{45}x^4,\qquad x\lesssim\frac{1}{4} \qquad (20a)$$

For larger values of x we have the limiting relation

$$\lim_{x\to 1}f_1(x)=1 \qquad (20b)$$

Equations (18) and (20) indicate the way in which the nonlinear effect depends on deformation and dimensions of the plate. For wide plates for which $1-x\ll 1$ we have the usual result that for linearity the ratio of deflection per unit of circumferential angle to plate thickness must be small compared to unity. As the plate gets narrower larger and larger deflections lie within the linear range, the condition for linearity being that $(k/h)(c/R)^2$ be somewhat smaller than unity.

A further question of interest concerns the magnitude of the stresses in the plate. As we have neglected the effect of transverse shear deformation we cannot determine the magnitude of the *transverse* shearing stresses. But we can determine, within the accuracy of the results of plate theory the magnitude of the horizontal shearing stress $\tau_{r\theta} = 6M_{r\theta}/h^2$. We have from (12)

$$\tau_{r\theta} = (1 - \nu)D\,\frac{6k}{r^2h^2} \tag{21a}$$

and from this

$$\tau_{r\theta}(a) = \tau_{r\theta,\max} = \frac{E}{2(1+\nu)}\,\frac{kh}{a^2}\,. \tag{21b}$$

In the non-linear range we have in addition a stress $\sigma_\theta = N_\theta/h$. We limit ourselves here to a consideration of this stress along the edges $r = a$ and $r = b$ of the plate. We have from (12), taking account of the fact that $N_r(a) = N_r(b) = 0$,

$$\begin{aligned}\sigma_\theta(a) &= \frac{1}{8}\,Ek^2\left\{\frac{2}{a^2} - 4\,\frac{\ln b/a}{b^2 - a^2} + C\left[2 - \frac{4b^2}{b^2 - a^2}\ln\frac{b}{a}\right]\right\},\\ \sigma_\theta(b) &= \frac{1}{8}\,Ek^2\left\{\frac{2}{b^2} - 4\,\frac{\ln b/a}{b^2 - a^2} + C\left[2 - \frac{4a^2}{b^2 - a^2}\ln\frac{b}{a}\right]\right\},\end{aligned} \tag{22}$$

where C is given by (16).

Equations (22) may be written in the alternate form

$$\sigma_\theta(a) = \frac{1}{4}\,\frac{Ek^2}{R^2}\,\frac{f_2(c/R)}{(1 - c/R)^2}\,, \qquad \sigma_\theta(b) = \frac{1}{4}\,\frac{Ek^2}{R^2}\,\frac{f_3(c/R)}{(1 + c/R)^2}\,, \tag{23}$$

where

$$\begin{aligned}f_2(x) &= 1 - \frac{(1-x)^2}{2x}\ln\frac{1+x}{1-x} + CR^2(1-x)^2\left[1 - \frac{(1+x)^2}{2x}\ln\frac{1+x}{1-x}\right],\\ f_3(x) &= 1 - \frac{(1+x)^2}{2x}\ln\frac{1+x}{1-x} + CR^2(1+x)^2\left[1 - \frac{(1-x)^2}{2x}\ln\frac{1+x}{1-x}\right],\\ CR^2 &= \frac{1}{2x}\ln\frac{1+x}{1-x}\left\{\frac{1+x^2}{2x}\ln\frac{1+x}{1-x} - 1\right\}\bigg/\left\{1 - \left(\frac{1-x^2}{2x}\ln\frac{1+x}{1-x}\right)^2\right\}.\end{aligned} \tag{24}$$

Of the two stresses $\sigma_\theta(a)$ and $\sigma_\theta(b)$ the larger one is the stress $\sigma_\theta(a)$ at the inner edge of the plate. For the sake of comparison we may write

$$\frac{\sigma_\theta(a)}{\tau_{r\theta}(a)} = \frac{1+\nu}{2}\,f_2\!\left(\frac{c}{R}\right)\frac{k}{h} \tag{25}$$

For sufficiently small values of c/R, practically for $c/R < 1/4$, we have

$$f_2\!\left(\frac{c}{R}\right) \sim \frac{4}{3}\left(\frac{c}{R}\right)^2, \qquad \frac{c}{R} \lesssim \frac{1}{4} \tag{26a}$$

For larger values of c/R we have the limiting relation

$$\lim_{c/R\to 1} f_2\!\left(\frac{c}{R}\right) = 1 \tag{26b}$$

Comparison of the stress ratio (25) with the load deflection relation (18) indicates that non-linearity affects the stresses much more strongly than the deflection characteristics of the plate. As long as $c/R < 1/4$ we have for example that when $\sigma_\theta(a)/\tau_{r\theta}(a) = 1/4$ then non-linearity is responsible for a three-percent correction only to the stress displacement relation and when $\sigma_\theta(a)/\tau_{r\theta}(a) = 1/2$ then this correction amounts to only about twelve percent.

4. Interaction between pure twisting and pure bending. A further problem within the present context is the problem of the effect of a moment T about the line of action of the force P on the relation between force P and deflection k. Differential equations and boundary conditions are as before with one exception, which consists in replacing the boundary condition $\int_a^b rN_\theta\, dr = 0$ by the condition

$$\int_a^b rN_\theta\, dr = T. \tag{27}$$

The solution for the stress function F now consists of two parts, one due to non-linearity in k and with $T = 0$, and the other without the non-linearity in k but with $T \neq 0$. Since the first part of the solution is given in the previous section we may now restrict attention to the second part of the solution. We write for this second part

$$F = T\{A^* \ln r + B^* r^2 + C^* r^2 \ln r\}. \tag{28}$$

The boundary conditions $N_r(a) = N_r(b) = 0$ are satisfied by setting

$$\begin{aligned} A^* a^{-2} + 2B^* + C^*(1 + 2 \ln a) &= 0, \\ A^* b^{-2} + 2B^* + C^*(1 + 2 \ln b) &= 0. \end{aligned} \tag{29}$$

The condition $\int_a^b rN_\theta\, dr = T$ becomes

$$A^* \ln (b/a) + B^*(b^2 - a^2) + C^*(b^2 \ln b - a^2 \ln a) = -T. \tag{30}$$

From (29) and (30) follows

$$\begin{gathered} A^* \frac{b^2 - a^2}{a^2 b^2} = 2C^* \ln \frac{b}{a}, \qquad -B^* = \left(\frac{1}{2} + \frac{b^2 \ln b - a^2 \ln a}{b^2 - a^2}\right) C^*, \\ C^* = \frac{2}{b^2 - a^2}\left[1 - \left(\frac{2ab}{b^2 - a^2} \ln \frac{b}{a}\right)^2\right]^{-1}. \end{gathered} \tag{31}$$

A relation between P and k follows again from $\int_a^b R_\theta\, dr - 2[M_{r\theta}]_b^a = P$, in the form

$$(1 - \nu) Dk \frac{b^2 - a^2}{a^2 b^2} + Tk\left[\frac{A^*}{2} \frac{b^2 - a^2}{b^2 a^2} + 2B^* \ln \frac{b}{a} + C^*(1 + \ln ab) \ln \frac{b}{a}\right] = P. \tag{32}$$

Introduction of A^*, B^* and C^* from (28) gives after some transformations

$$k\left[1 - (1 + \nu) \frac{T}{Eh^3} f_4\left(\frac{c}{R}\right)\right] = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2, \tag{33}$$

where

$$f_4(x) = \frac{6\left(\dfrac{1 - x^2}{2x}\right)^2 \ln \dfrac{1 + x}{1 - x}\left[\dfrac{1 + x^2}{2x} \ln \dfrac{1 + x}{1 - x} - 1\right]}{\left[1 - \left(\dfrac{1 - x^2}{2x} \ln \dfrac{1 + x}{1 - x}\right)^2\right]}. \tag{34}$$

Equation (33) shows that a positive moment T, that is a moment which tends to close an open ring sector, reduces the stiffness of the ring plate in so far as the effect of the axial force P is concerned. In contrast to this a negative moment T increases the transverse stiffness of the ring plate. For sufficiently large positive T we have instability in the sense that we may have $k \neq 0$ when $P = 0$. It may, however, be that instability occurs in other modes of deformation for values of T which are lower than that given by (33).

When c/R is sufficiently small, we have the approximation

$$f_4(x) \sim 3\,\frac{1}{x}, \qquad x \lesssim \frac{1}{4}\,. \tag{34a}$$

In the range of applicability of (34a) we may further write

$$T = \frac{2}{3}\,\sigma_T\, c^2 h, \tag{35}$$

where $\sigma_T \sim -\sigma_\theta(a) \sim \sigma_\theta(b)$. With this we have

$$\frac{T}{Eh^3}\, f_4\!\left(\frac{c}{R}\right) \sim 2\,\frac{\sigma_T}{E}\,\frac{cR}{h^2}, \qquad \frac{c}{R} \lesssim \frac{1}{4}\,. \tag{36}$$

The present section may be concluded by listing the form of the relation between k and P which holds when both the effect of T and non-linearity is considered. We have then

$$k\left[1 + \frac{3(1+\nu)}{4}\, f_1\!\left(\frac{c}{R}\right)\!\left(\frac{k}{h}\right)^2 - (1+\nu) f_4\!\left(\frac{c}{R}\right)\frac{T}{Eh^3}\right] = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2. \tag{37}$$

We make the following further observation. Finite transverse deflections of the ring plate will be associated, in the absence of a moment T, with circumferential displacements parallel to the undeflected middle surface of the plate. Such displacements are also caused by the moment T. This means that if the conditions of load application are such that circumferential displacements are prevented T will have a definite value, proportional to k^2 and the non-linear correction term in the transverse load-deflection relation for this form of load application will differ from the result (18) which holds when $T = 0$.

The problem of the determination of this modified correction term will be considered in the last section of this work, in conjunction with the problem of the ring plate with initial deflection.

5. Pure twisting and bending of an initially deflected plate. We now consider an initially deflected plate, or helicoidal shell, with middle-surface equation

$$W = K\theta. \tag{38}$$

We assume that K is sufficiently small for the shell to be considered shallow. It seems that for practical purposes we may admit values of K up to about $(R - c)/\pi$.

The differential equations for shallow shells which take the place of the flat-plate equations (1) and (2) have been given by Marguerre.[5] The necessary changes consist

[5]K. Marguerre, Proc. Vth Int. Congress Appl. Mech. Cambridge 1938, p. 98.

in replacing the operation $\mathfrak{D}(w)$ on the right of (2) by $\mathfrak{D}(W + w) - \mathfrak{D}(W)$ and in replacing w on the right of (1) by $W + w$.

We may take as before

$$w = k\theta, \qquad F = F(r) \tag{9}$$

which reduces the two differential equations for w and F to the one equation

$$\nabla^2\nabla^2 F = Eh\,\frac{2Kk + k^2}{r^4}\,. \tag{39}$$

Equations (3), (4) and (5) which define stress resultants and couples remain unchanged while in equations (6) w must be replaced by $W + w$.

The form of the boundary conditions (7), (8) and (27) remains unchanged and this means that we have now

$$F = \tfrac{1}{8}Eh(2Kk + k^2)[(\ln r)^2 + A\ln r + Br^2 + Cr^2\ln r] + T[A^*\ln r + B^*r^2 + C^*r^2\ln r], \tag{40}$$

where A, B and C and A^*, B^* and C^* are given by (16) and (31), respectively.

In determining the relation between force P and deflection k according to the fifth of Eqs. (8) it must be observed that R_θ of (12) is now changed to

$$R_\theta = -2(1 - \nu)D\,\frac{k}{r^3} + N_\theta\,\frac{K + k}{r}\,. \tag{41}$$

A comparison of (40) and (41) with the corresponding earlier results shows that Eq. (37) for P as a function of k and T is changed into

$$k\left[1 + \frac{3(1+\nu)}{4}\,f_1\!\left(\frac{c}{R}\right)\frac{(k + 2K)(k + K)}{h^2} - (1 + \nu)f_4\!\left(\frac{c}{R}\right)\frac{T}{Eh^3}\right] - K\left[(1 + \nu)f_4\!\left(\frac{c}{R}\right)\frac{T}{Eh^3}\right] = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2. \tag{42}$$

As long as $k \ll K$ equation (42) reduces to the *linear* relation

$$k\left[1 + \frac{3(1+\nu)}{4}\,f_1\!\left(\frac{c}{R}\right)\frac{2K^2}{h^2}\right] - (1 + \nu)f_4\!\left(\frac{c}{R}\right)\frac{KT}{Eh^3} = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2. \tag{43}$$

In view of the nature of the distribution of stresses σ_θ which give rise to the terms accounting for the initial deflection of the plate the term with K^2 can be obtained only by taking account of plate action and is consequently not incorporated in the classical theory of curved beams.

We note further that while for the initially undeflected plate the non-linear correction term varies as the cube of the deflection we have, for the initially deflected plate, correction terms varying both as the square and as the cube of the additional deflection caused by the forces P.

6. Consideration of circumferential displacement. We now consider in addition to the transverse displacement w radial and circumferential displacement components

u and v. Components of finite strain for the middle surface of the shallow shell are of the form

$$\epsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial(W + w)}{\partial r}\right)^2 - \left(\frac{\partial W}{\partial r}\right)^2\right],$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2}\left[\left(\frac{\partial(W + w)}{r\,\partial \theta}\right)^2 - \left(\frac{\partial W}{r\,d\theta}\right)^2\right], \tag{44}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \left[\frac{\partial(W + w)}{\partial r}\frac{\partial(W + w)}{r\,\partial \theta} - \frac{\partial W}{\partial r}\frac{\partial W}{r\,\partial \theta}\right].$$

For the present problems we have $W = K\theta$, $w = k\theta$, $\partial u/\partial\theta = 0$ and $\gamma_{r\theta} = 0$. From this it follows that

$$v = \frac{\omega}{2\pi}\theta r, \tag{45}$$

where ω is the relative angular displacement of the ends of one winding of the plate. The expressions for ϵ_r and ϵ_θ reduce to

$$\epsilon_r = \frac{du}{dr}, \qquad \epsilon_\theta = \frac{u}{r} + \frac{\omega}{2\pi} + \frac{Kk + \frac{1}{2}k^2}{r^2}, \tag{46}$$

leading to a compatibility relation of the form

$$\frac{dr\,\epsilon_\theta}{dr} - \epsilon_r = \frac{\omega}{2\pi} - \frac{Kk + \frac{1}{2}k^2}{r^2}. \tag{47}$$

We introduce the relations

$$Eh\epsilon_\theta = N_\theta - \nu N_r, \qquad Eh\epsilon_r = N_r - \nu N_\theta \tag{48}$$

and express N_θ and N_r in terms of the stress function F by means of (3), taking account of the fact that here $\partial F/\partial\theta = 0$. This leads to the result that

$$\frac{1}{Eh}\left[r\frac{d^3F}{dr^3} + \frac{d^2F}{dr^2} - \frac{1}{r}\frac{dF}{dr}\right] = \frac{\omega}{2\pi} - \frac{Kk + \frac{1}{2}k^2}{r^2} \tag{49}$$

Into (49) we introduce F from Eq. (40), and this gives us after some cancellations the relation

$$\frac{1}{2}(2Kk + k^2)C + \frac{4T}{Eh}C^* = \frac{\omega}{2\pi}, \tag{50}$$

where C and C^* are defined by (16) and (31).

If we wish the axial force-displacement relation under the assumption of vanishing circumferential displacement we find from (50), with $\omega = 0$, as appropriate value of the couple T,

$$\frac{T}{Eh} = -\frac{(2Kk + k^2)C}{8C^*} = -\frac{2Kk + k^2}{8}\left[\frac{b^2 + a^2}{b^2 - a^2}\ln\frac{b}{a} - 1\right]\ln\frac{b}{a} \tag{51}$$

Introduction of this value of T into Eq. (42) gives after some transformations the following relation between k and P,

$$k\left[1 + \frac{3(1+\nu)}{4} f_5\left(\frac{c}{R}\right) \frac{(K+k)(2K+k)}{h^2}\right] = \frac{3PR^3}{2Gch^3}\left[1 - \left(\frac{c}{R}\right)^2\right]^2, \tag{52}$$

where

$$f_5(x) = 1 - \left(\frac{1-x^2}{2x} \ln \frac{1+x}{1-x}\right)^2 \tag{53}$$

and

$$f_5(x) \sim \frac{4}{3} x^2, \qquad x \lesssim \frac{1}{2}, \qquad \lim_{x \to 1} f_5(x) = 1. \tag{54}$$

It may be seen that in the range $c/R < 1/4$ the effect of non-linearity and of initial deflection is much more pronounced when $\omega = 0$ than it is when $T = 0$, a factor $(4/15) \cdot (c/R)^4$ which occurs when $T = 0$ being replaced by a factor $(c/R)^2$ when $\omega = 0$. On the other hand, when c/R is sufficiently near to unity the effect of non-linearity and of pretwist is the same for $T = 0$ and for $\omega = 0$.

TABLE I. Values of functions occurring in load displacement relations and stress ratio $\sigma_\theta/\tau_{r\theta}$.

x	f_1	f_2	f_4	f_5
0	0	0	∞	0
0.1	0.000036	0.0159	29.08	0.0135
0.2	0.000574	0.0465	14.32	0.0530
0.3	0.00328	0.0836	8.97	0.1186
0.4	0.01024	0.1691	6.14	0.208
0.5	0.0265	0.255	4.31	0.321
0.6	0.0600	0.359	2.98	0.453
0.7	0.1248	0.483	1.94	0.600
0.8	0.2484	0.631	1.106	0.756
0.9	0.4923	0.809	0.427	0.903
1.0	1.0	1.0	0	1.0