AN ADDITION TO PORITSKY'S SOLUTIONS OF A DIFFERENTIAL EQUATION OF TORSION*

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In the theory of circular shafts of variable cross section, the only stress components which do not vanish are the shearing stresses,

$$P_{x\theta} = \frac{G}{r^2} \frac{\partial \psi}{\partial r}, \qquad P_{r\theta} = -\frac{G}{r^2} \frac{\partial \psi}{\partial x} \tag{1}$$

the stress function ψ is a function of r and x alone, and the coordinate system is cylindrical (r, x, θ). Equations (1) satisfy the equilibrium equations and will satisfy compatibility if

$$\frac{\partial}{\partial r} \left(\frac{1}{r^3} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial x} \left(\frac{1}{r^3} \frac{\partial \psi}{\partial x} \right) = 0 \tag{2a}$$

or

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{3}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial x^2} = 0. \tag{2b}$$

In problems relating to cones or spheres, it will be convenient to express Eq. (2a) in spherical coordinates R, ω, ϕ, as

$$\frac{\partial}{\partial R} \left[\frac{1}{R^2} \frac{1}{(1-\mu^2)^2} \frac{\partial \psi}{\partial R} \right] + \frac{\partial}{\partial \mu} \left[\frac{1}{R^4} \frac{1}{(1-\mu^2)} \frac{\partial \psi}{\partial \mu} \right] = 0, \tag{3}$$

where R is the radius in spherical coordinates and $\mu = \cos \omega = x/R$.

The product solutions of (3) are:

$$\psi = (1-\mu^2)^2 P''_{n+1}(\mu) R^{-n}, \tag{4a}$$

$$\psi = (1-\mu^2)^2 P''_{n+1}(\mu) R^{n+3}, \tag{4b}$$

$$\psi = (1-\mu^2)^2 Q''_{n+1}(\mu) R^{-n}, \tag{4c}$$

$$\psi = (1-\mu^2)^2 Q''_{n+1}(\mu) R^{n+3}, \tag{4d}$$

where Pn and Qn are the Legendre functions of the first and second kind.

The solutions (4) do not give functions ψ varying as R, and only Eq. (4d) gives a solution varying as $R^2(n = -1)$. These missing solutions are:

$$\psi = (1+\mu^2)R^2 \qquad \psi = R^2\mu, \tag{5a}$$

$$\psi = (1+\mu^2)R \qquad \psi = R\mu. \tag{5b}$$

Solutions (4) are given by equations (79) and (80) of reference [1]. The solutions (5a) and (5b), however, do not agree with those given in Eqs. (107) and (108) of reference [1].

Although one is usually interested in torsion of shafts with stress free surfaces, it is

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interesting to consider the type of loading which results from Eqs. (1) if this requirement is relaxed. For a cone, consideration of the equilibrium of a small volume element at the surface of the cone will show that the surface stress consists of rings of shear with a magnitude

$$P_{\omega\phi} = \frac{1}{r^2} \frac{d\psi}{dR} \bigg|_{\mu=\text{const.}} = \frac{1}{R^2(1 - u^2)} \frac{d\psi}{dR} \bigg|_{\mu=\text{const.}} . \tag{6}$$

Equation (6) indicates that for a given shear distribution on the surface of the cone, the boundary condition is on $(d\psi/dR)_{\mu=\text{const.}}$ rather than on ψ. It is evident that although Eqs. (4) and (5) give a complete set of functions varying as any positive or negative power of R, this set does not include any function which has a derivative behaving as $1/R$. The two functions which have a derivative behaving as $1/R$ are, omitting arbitrary multipliers.

$$\psi_1 = \left[\frac{\mu^2}{2} + \frac{(3\mu - \mu^3)}{4} \log \frac{1 + \mu}{1 - \mu} + \frac{\log (1 - \mu^2)}{2} + \log R \right],$$

$$\psi_2 = \left[\frac{5\mu}{3} - \frac{\mu^3}{9} - \frac{1}{3} \log \frac{1 + \mu}{1 - \mu} - \frac{(3\mu - \mu^3)}{6} \log (1 - \mu^2) + \frac{(3\mu - \mu^3)}{3} \log R \right]. \tag{7}$$

To obtain these functions the following procedure was used:

1. Equation (3) is multiplied by R^4 and differentiated with respect to R giving

$$\frac{\partial}{\partial R} \left[R^4 \frac{\partial}{\partial R} \left(\frac{1}{R^2} \frac{1}{(1 - \mu^2)^2} \psi_R \right) \right] + \frac{\partial}{\partial \mu} \left[\frac{1}{(1 - \mu^2)} \frac{\partial \psi_R}{\partial \mu} \right] = 0. \tag{8}$$

2. The desired solutions are those with ψ_R behaving as $1/R$. Thus ψ_R must be of the form.

$$\psi_R = \frac{f(\mu)}{R}. \tag{9}$$

Equation (8) becomes an ordinary differential equation in $f(u)$, and shows that

$$\psi_R = \left[A \left(\mu - \frac{\mu^3}{3} \right) + B \right] \frac{1}{R} \tag{10}$$

and therefore

$$\psi = \left[A \left(\mu - \frac{\mu^3}{3} \right) + B \right] \log R + g(\mu), \tag{11,}$$

where A and B are constants of integration

3. With

$$\psi_1 = B \log R + g_1(\mu), \qquad \psi_2 = A \left(\mu - \frac{\mu^3}{3} \right) \log R + g_2(\mu), \tag{12}$$

substitution into Eq. (3) will give ordinary differential equations for the unknown functions $g_1(u)$ and $g_2(u)$. These may be integrated, giving

$$g_1(\mu) = \frac{3}{4} B\mu^2 + \frac{3}{4} B \int (1 - \mu^2) \log \frac{1 + \mu}{1 - \mu} d\mu,$$

$$g_2(\mu) = A\mu - \frac{A}{2} \int (1 - \mu^2) \log (1 - \mu^2) \, d\mu. \tag{13}$$

If the integration indicated in Eq. (13) is performed and the values of $g_1(\mu)$ and $g_2(\mu)$ used in Eq. (12), the expressions given in Eq. (7) are obtained.

REFERENCES

1. H. Poritsky, *Stress fields of axially symmetric shafts in torsion and related fields*, Proc. Symposia Appl. Math., **3**, 163-186 (1951).

BOOK REVIEWS

Description of a magnetic drum calculator. By The Staff of the Computation Laboratory. Harvard University Press, Cambridge, 1952. 318 pp. $8.00

This book is one of a series put out by the staff of the Computation Laboratory of Harvard University and describes the Mark III calculator. This machine was completed in March of 1950 and was then moved to the Naval Proving Ground at Dahlgren, Virginia.

The book itself is a detailed description of this machine and is of principal interest to those persons who are immediately associated with the machine. It combines both an engineering and a mathematical description of the device.

The text is extremely well illustrated both with photographs and with schematics of the principal organs of the machine.

To illustrate the coding of problems for the machine there is a chapter which contains among other things the programming for four illustrative examples.

The text is undoubtedly an invaluable aid to those immediately concerned with the operation and programming of problems for the Mark III calculator.

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Introduction to the theory of plasticity for engineers. By Oscar Hoffman and George Sachs. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1953. xib + 276 pp. $6.50.

As one would expect on the basis of the authors' well deserved reputation their book contains an excellent exposition of the technological application of plasticity. Major emphasis is placed on the approximate solutions to problems of rolling, extruding, drawing, etc. in which the material is assumed to be ideally plastic and the true three dimensional character of the flow is not taken into account. Appreciable space is also devoted to the elementary classical problems of the thick-walled shell and tube and to the rotating cylinder and disk. Although the text opens with a discussion of stress and strain tensors and considers a stress space, the discussion of stress-strain relations and experimental data is brief and is essentially confined to the maximum shear stress and octahedral shear stress criteria. The extensive modern literature on stress-strain relations in the plastic range is, in the main, ignored. Also, except for a short section on two-dimensional plastic flow problems, little of the classical mathematical theory of plasticity is treated. No mention is made of plastic waves nor of the theorems or applications of limit analysis and design. The latter would be especially useful in evaluating some of the results obtained in the approximate solutions which are treated.

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