—NOTES—

A SUFFICIENT CONDITION FOR AN INFINITE DISCRETE SPECTRUM*

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1. In the differential equation

$$x'' + f(t)x = 0, \tag{1}$$

let $f = f(t)$ denote a real-valued, continuous function on the half-line $0 \leqq t < \infty$. Both necessary and sufficient conditions in order that the equation (1) be oscillatory, so that every solution of (1) possesses an infinity of zeros on $0 \leqq t < \infty$ clustering at $+\infty$, are known; see, for instance, [10], [5]. The present note will deal primarily with the problem of obtaining a sufficient criterion in order that (1) be oscillatory, in the particular case that $f(t)$ satisfies the limit relation

$$f(t) \to 0, \qquad \text{as} \qquad t \to \infty. \tag{2}$$

The following will be shown:

(I) *If $f(t)$ satisfies* (2), *then the differential equation* (1) *is oscillatory whenever the inequality*

$$\limsup_{h \to +0} h\left(\limsup_{S \to \infty} S\left[\limsup_{T \to \infty} \int_S^T f(t)\,dt + \int_0^S f(t) \mid (t - \gamma)/(S - \gamma) \mid^{1+h} dt\right]\right) > \frac{1}{4} \tag{3}$$

holds for every fixed number $\gamma \geqq 0$.

Obviously, the inequality is satisfied in case the innermost "lim sup" is $+\infty$. (It should be pointed out here that if (2) holds and if $\lim_{T\to\infty}\int_0^T f(t)dt$ fails to exist either as a finite limit or as $-\infty$, then (1) is surely oscillatory; [3], p. 389. Cf. also [11] and [2].) It is noteworthy that the criterion furnished by (I) remains valid if the assumption (2) is replaced by certain other conditions; cf. the remark at the end of section 2.

The result (I) will have an implication concerning the discrete spectrum of the boundary value problem on $0 \leqq t < \infty$ ([7]) determined by the differential equation

$$x'' + (\lambda + f(t))x = 0 \tag{4}$$

and the boundary condition

$$x(0) \cos \alpha + x'(0) \sin \alpha = 0, \qquad 0 \leqq \alpha < \pi, \tag{5}$$

in the particular case that (2) holds. For every fixed α, the relation (2) implies that the half-line $\lambda \geqq 0$ belongs to the spectrum; [2]. Moreover, only a discrete spectrum (isolated eigenvalues) can exist for $\lambda < 0$ and there exists a finite number of eigenvalues $\lambda < 0$, or an infinity of such eigenvalues clustering only at $\lambda = 0$, according as (1) is not or is oscillatory; [7], p. 252. Accordingly, (I) provides a sufficient condition that (4) and (5), in the case (2), determine a boundary value problem with an infinity of negative eigenvalues clustering at $\lambda = 0$. (It should be pointed out that (4) may have

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a non-trivial solution $x(\neq 0)$ of class $L^2[0, \infty)$ for a positive value λ even when (2) holds; cf. the constructions of [8], pp. 394-395 and [9], pp. 268-269.)

In section 3 below, a specific application of the criterion (I) to the spectral problem mentioned above will be made. The boundary value problem to be considered will be of the type arising in the quantum mechanical treatment of the two particle problem. In this case, there is a singularity in the coefficient function of the differential equation at the origin but the nature of the problem remains essentially identical with that considered in connection with (4) above; cf. [4], pp. 154, 163.

That the assertion of (I) can become false if the strict inequality $>$ of (3) is relaxed to $\geqq$ can easily be shown by an example. In fact, if $f(t) = Ct^{-2}$, where, say, $0 < t < \infty$, it is readily seen that (3) reduces to the inequality $C > \frac{1}{4}$. However, if $C = \frac{1}{4}$, so that $f(t)$ becomes $\frac{1}{4}t^{-2}$, then (1) possesses the non-oscillatory solution $x = t^{1/2}$, for $0 < t < \infty$. (Needless to say, the criterion (I) actually requires the continuity of the function $f(t)$ only for large values t and the fundamental interval $0 \leqq t < \infty$ may be replaced by any half-line $T \leqq t < \infty$.)

2. The proof of (I) will depend upon an application of an oscillation criterion obtained in [5]. (A somewhat similar application was made in [6] in the case that $f(t)$ was periodic.) It was shown in [5] that (1) is oscillatory if and only if, for every $T \geqq 0$, there exists on $T \leqq t < \infty$ a continuous function $x = x(t)$ satisfying $x(T) = 0$ and possessing a piecewise continuous derivative $x'(t)$ such that each of the integrals

$$\int_T^\infty x^2(t)\,dt, \qquad \int_T^\infty x'^2(t)\,dt, \qquad \text{and} \qquad \int_T^\infty f(t)x^2(t)\,dt$$

is finite, while

$$\int_T^\infty (x'^2 - fx^2)\,dt < 0. \tag{6}$$

It will now be shown that relations (3) and (2) imply (6), and so, (I) will follow.

For every positive number T and for every number $n > \frac{1}{2}$, consider numbers $T_2 < T_3 < T_4$ such that $T = T_1 < T_2$, and define the function $x = x(t)$ on $T_1 \leqq t < \infty$ as follows:

$$x(t) = \begin{cases} (t - T_1)^n, & \text{for} \quad T_1 \leqq t \leqq T_2\,; \\ (T_2 - T_1)^n, & \text{for} \quad T_2 \leqq t \leqq T_3\,; \\ (T_2 - T_1)^n(T_4 - T_3)^{-1}(T_4 - t), & \text{for} \quad T_3 \leqq t \leqq T_4\,; \\ 0, & \text{for} \quad T_4 \leqq t < \infty. \end{cases}$$

If $T_2 - T_1 = A$ and $T_4 - T_3 = B$, direct calculation readily shows that the requirement (6) reduces to

$$A \int_{T_2}^{T_3} f\,dt + A^{1-2n} \int_{T_1}^{T_2} f(t - T_1)^{2n}\,dt + AB^{-2} \int_{T_3}^{T_4} f(T_4 - t)^2\,dt > n^2(2n-1)^{-1} + AB^{-1}.$$

Suppose that T_2 (hence A) is determined and then, for an arbitrary positive number ϵ, choose B so as to satisfy $AB^{-1} < \epsilon$. Next choose T_3 (hence T_4) so large that

$$\left| AB^{-2} \int_{T_3}^{T_4} f(T_4 - t)^2 \, dt \right| < \epsilon. \tag{7}$$

That this can be done is clear from (2) and from the fact that $| T_4 - t |^2 B^{-2} \leqq 1$ when $T_3 \leqq t \leqq T_4$ $(= T_3 + B)$. It now follows that (6) is surely satisfied if

$$A \int_{T_2}^{T_3} f \, dt + A^{1-2n} \int_{T_1}^{T_2} f(t - T_1)^{2n} \, dt > n^2(2n - 1)^{-1} + 2\epsilon. \tag{8}$$

Actually it will be shown that (8) holds for (certain) large values A, T_k (that is, for certain A, $T_k \rightarrow \infty$) as a consequence of (3).

Let $L(S)$ be defined by

$$L(S) = \limsup_{T \rightarrow \infty} \int_S^T f(t) \, dt, \tag{9}$$

and for a fixed value T_2, choose T_3 so large that

$$\int_{T_2}^{T_3} f \, dt > L(T_2) - T_2^{-2}.$$

It is clear that for a fixed value T_1, $AT_2^{-1} \rightarrow 1$ as $T_2 \rightarrow \infty$, and so, for certain large A, T_k, the relation (8) will hold if T_1 is fixed and

$$T_2 L(T_2) + T_2 \int_{T_1}^{T_2} f[(t - T_1)/A]^{2n} \, dt > n^2(2n - 1)^{-1} + 2\epsilon$$

holds for certain large T_2. However, this last relation will certainly hold for certain large T_2 (T_1 fixed) if

$$\limsup_{S \rightarrow \infty} \left[SL(S) + S \int_0^S f \mid (t - T_1)/(S - T_1) \mid^{2n} dt \right] > n^2(2n - 1)^{-1} + 2\epsilon. \tag{10}$$

Since ϵ can be chosen arbitrarily small, the 2ϵ appearing on the right side of (10) may be deleted. If now $h = 2n - 1$, one sees that (6) holds for every T $(= T_1 = \gamma)$ if the relation

$$h \limsup_{S \rightarrow \infty} S\left[L(S) + \int_0^S f(t) \mid (t - \gamma)/(S - \gamma) \mid^{1+h} dt \right] > (1 + h)^2/4 \tag{11}$$

holds for some $h > 0$ (where possibly $h = h(\gamma)$) and γ is an arbitrary non-negative number. However, relations (3) and (9) clearly imply that (11), for every $\gamma \geqq 0$, is valid for some positive h and thus the assertion (I) is proved.

Remark. The assumption (2) was used in order to obtain (7). It is clear though from the proof given above that (3) will imply (6) if

$$\left| AB^{-2} \int_{T_3}^{T_3+B} f^-(t)(T_3 + B - t)^2 \, dt \right| < \epsilon, \qquad f^- = \begin{cases} f, & f \leq 0 \\ 0, & f > 0 \end{cases}$$

can be obtained for A, B fixed and for (certain) large T_5. Thus, the criterion furnished by (I) will be valid if, for instance, the restriction (2) is replaced by any one of the three assumptions (i) $f(t) \geqq 0$, or even (ii) $f^-(t) \to 0$ as $t \to \infty$, or

$$\text{(iii)} \qquad \left| \int_0^\infty f^-(t)\, dt \right| < \infty .$$

3. In order to obtain an application of (I), consider the radial portion of the separated form of the quantum mechanical wave equation of the two particle problem, namely,

$$R'' + c(\lambda - V(r) - l(l+1)/cr^2)R = 0;$$

cf. [4], p. 150. Here, c, λ and l are constants and the prime denotes differentiation with respect to r. It will be assumed that $V(r) \to 0$ as $r \to \infty$; cf. [4], p. 152. It is clear from an earlier remark that, as far as concerns the application of (I), only the continuity of the function $V(r)$ for large r, say for $1 \leqq r < \infty$, is needed. (As is customary, it will be assumed that the singularity of $V(r)$ at $r = 0$ is of a suitably restricted type; cf., e.g., [4], pp. 152, 163.) It follows from (I) and the calculation of section 1 that the equation

$$R'' + c(-V(r) - l(l+1)/cr^2)R = 0 \tag{12}$$

is oscillatory if, for all $\gamma \geqq 0$,

$$\limsup_{h \to +0} h\left(\limsup_{S \to \infty} S\left[\limsup_{T \to \infty} \int_S^T - cV(r)\, dr + \int_1^S - cV(r) \mid (r - \gamma)/(S - \gamma) \mid^{1+h} dr\right]\right)$$
$$> \tfrac{1}{4} + l(l+1). \tag{13}$$

Thus, if the last inequality is satisfied, there exists an infinity of negative energy levels clustering at $\lambda = 0$. In the case of the hydrogen atom, $V(r) = kr^{-2}$, where $k < 0$, so that the bracketed portion of the left side of (13) is $+\infty$ and hence (12) is oscillatory for all values l; cf., e.g., [4], pp. 157 ff. In many cases however, the potential $V(r)$ appears to be unknown (cf. [4], p. 156 and [1], pp. 30 ff.) and the relation (13) offers a property of $V(r)$ guaranteeing the existence of an infinite (negative) discrete spectrum.

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