ON THE EXPANSION OF FUNCTIONS IN TERMS OF THEIR MOMENTS*

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Summary. A general method is devised for the reconstruction of functions of a continuous variable from their moments. An analogue is given for functions of a discrete variable. An application is given to the solution of partial differential equations with a given initial condition.

1. Introduction. Recently, in some work connected with the lateral spread of showers of cosmic-ray particles in their passage through the atmosphere, the authors were faced with the problem of finding the radial distribution function of the particles with respect to the shower axis. This function was known to be the solution of a partial differential equation, from which, however, only the moments of the function could be conveniently determined. It was then necessary to devise a method for reconstructing the function from its moments. From the physical nature of the problem considered by the authors, it was known that such a function existed; however, situations could arise in which even this knowledge was not available. Criteria for the existence and uniqueness of distribution functions corresponding to a given set of moments having been the subject of extensive study [1], and analytical procedures for the determination of the distribution functions have been described. However, it has also been pointed out [2] that these methods have little value in practice. The only method known to the authors which is well adapted to application has been given by Spencer and Fano [3], but this was not of sufficient generality for our purpose. We therefore developed a method of considerably wider applicability.

The expansion of a function in terms of its moments is a problem which arises not only in methematical physics, but also in many branches of statistics. In the hope that the method may prove useful to workers in fields other than our own, we now give a brief account of the theory in its most general form.

2. Expansions in terms of the δ-function. The use of the δ-function has been well established in quantum mechanics [4] and pulse theory; however, the rigour of the mathematical procedures in which it is used has sometimes been questioned [5], and we therefore state at the outset the unambiguous meaning of an equation of the type

$$f^*(x) = w^*(x) \sum_{k=0}^{\infty} a_k(x) \; \delta^{(k)}(x), \tag{1}$$

where the superfix represents the number of differentiations of the δ-function with respect to the arbitrary variable x. The first k derivatives of the function $a_k(x)$ must exist for $x = 0$, but $w(x)$ may have any kind of singularity there. Then (1) will be held equivalent to the assertion

$$\int_{-a}^{b} \frac{dx\, q(x) f^*(x)}{w^*(x)} = \sum (-1)^k q_k^{(k)}(0),$$

$$q_k(x) = q(x) a_k(x) \tag{2}$$

*Received January 28, 1953.

where $q(x)$ is an arbitrary function regular at $x = 0$, and a and b are any positive numbers less than the radius of convergence ρ of the power series

$$q(x) = \sum_{k=0}^{\infty} q^{(k)}(0)x^k/k!. \tag{3}$$

Let

$$f(x) = f^*(x)/w^*(x); \tag{4}$$

then since

$$\int_{-a}^{b} q(x)f(x)\,dx = \sum_{k=0}^{\infty} q^{(k)}(0) \int_{-a}^{b} x^k f(x)\,dx/k!, \tag{5}$$

one has, by comparison with (1) and (2),

$$f(x) = \sum_{k=0}^{\infty} (-1)^k f_{(k)}\, \delta^{(k)}(x)/k!, \tag{6}$$

$$f_{(k)} = \int_{-a}^{b} x^k f(x)\,dx. \tag{7}$$

Thus, any function, integrable in the range $-a < x < b$ can be expanded in this range as a series of derivatives of the δ-function, with coefficients which are proportional to the moments (7) of the function $f(x)$. This result is easily extended to functions of any number of variables.

3. Expansions in orthogonal polynomials. The result (6) requires some elaboration in order to provide a practical method for the determination of $f(x)$ when the coefficients $f_{(k)}$ of the series are known. The method which we shall adopt is to make a formal expansion of $\delta^{(k)}(x)$ in series of orthogonal polynomials;* but first we obtain the expansion of the function $f(x)$ which anticipates the result. Let

$$f(x) = w(x) \sum_{n=0}^{\infty} f_n S_n(x); \tag{8}$$

where $S_n(x)$ is a set of polynomials which will be defined presently; and if $f(x)$ has singularities at $x = x_1, x_2, \cdots, x_t$, where $|x_t| < a$, $|x_t| < b$, let $w(x)$ be a "weight-function" with a singularities of the same type, so that $f(x)/w(x)$ is regular for $|x| < a$, $|x| < b$. The polynomials $S_n(x)$ are defined by

$$S_n(x) = \begin{vmatrix} w_{(0)} & w_{(1)} & \cdots & w_{(n)} \\ w_{(1)} & w_{(2)} & \cdots & w_{(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ w_{(n-1)} & w_{(n)} & \cdots & w_{(2n-1)} \\ 1 & x & \cdots & x^n \end{vmatrix} \quad \text{for} \quad n \geq 1, \tag{9}$$

*For the general theory of such polynomials, c.f. Szegö [6]. The cases of Legendre and Laguerre polynomials are known [1].

where $S_0(x) = 1$, and

$$w_{(k)} = \int_{-a}^{b} w(x)x^k \, dx. \tag{10}$$

Then these polynlmials satisfy the orthogonality relations

$$\int_{-a}^{b} w(x)S_m(x)S_n(x) \, dx = N_n\delta_{mn} , \tag{11}$$

where the normality constants are given by

$$N_n = \Delta_n\Delta_{n-1} ,$$

$$\Delta_n = \begin{vmatrix} w_{(0)} & w_{(1)} & \cdots & w_{(n)} \\ w_{(1)} & w_{(2)} & \cdots & w_{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ w_{(n)} & w_{(n+1)} & \cdots & w_{(2n)} \end{vmatrix} \quad \text{for} \quad n \geq 0, \tag{12}$$

$$\Delta_{-1} = 1.$$

There is obviously an expansion of x^k of the form

$$x^k = \sum_{l=0}^{k} \xi_{k,l} S_l(x); \tag{13}$$

hence, for values of x less than the radius of convergence ρ of the power series

$$f(x)/w(x) = \sum_{k=0}^{\infty} \varphi_k x^k \tag{14}$$

which exceeds both a and b, there will exist a convergent expansion of the type (8), with coefficients given by

$$f_n = \sum_{k=n}^{\infty} \varphi_k \xi_{k,n} \tag{15}$$

The coefficients f_n are most readily obtained by multiplying (8) by $S_m(x)$, and integrating from $-a$ to b; thus:

$$N_n f_n = \int_{-a}^{b} f(x)S_n(x) \, dx$$

$$= \begin{vmatrix} w_{(0)} & w_{(1)} & \cdots & w_{(n)} \\ w_{(1)} & w_{(2)} & \cdots & w_{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ w_{(n-1)} & w_{(n)} & \cdots & w_{(2n-1)} \\ f_{(0)} & f_{(1)} & \cdots & f_{(n)} \end{vmatrix} \quad \text{for} \quad n \geq 1, \tag{16}$$

$$N_0 f_0 = f_{(0)} .$$

The function $f(x)$ has then been expanded in a series of orthogonal polynomials, the coefficients of which are linear combinations of its moments $f_{(k)}$. It is necessary that the quantities Δ_n defined by (12) should be positive, if a monotonic function $f(x)$ exists. The conditions for this have been discussed in detail by Shohat and Tomarkin [1], Chap. 1.

The result (8) may also be deduced from (1) by expanding $\delta^{(k)}(x)$ in terms of the orthogonal polynomials $S_n(x)$, thus:

$$\delta^{(k)}(x) = w(x)(-1)^k k! \sum_{l=k}^{\infty} \Delta_{l(k)} \, S_l(x)/N_l \tag{17}$$

where $\Delta_{l(k)}$ is the minor of $w_{(l+k)}$ in the last row or column of the determinant Δ_l. Substituting (17) into (1), and interchanging the order of the summations, one then obtains (8), which is therefore one way of interpreting the formula (1).

This procedure is easily adapted to distribution functions $F(n)$ of a discrete variable n. Here we consider expansions of the type

$$F(n) = W(n) \sum_{r=0}^{\infty} F_r S_r(n) \tag{18}$$

where

$$S_r(n) = \begin{vmatrix} W_{(0)} & W_{(1)} & \cdots & W_{(r)} \\ \cdots & \cdots & \cdots & \cdots \\ W_{(r-1)} & W_{(r)} & \cdots & W_{(2r-1)} \\ 1 & n & \cdots & n^r \end{vmatrix} \qquad \text{for} \qquad r \geq 1, \tag{19}$$

$$S_0(n) = 1,$$

and

$$W_{(r)} = \sum_{n=0}^{\infty} W(n) n^r. \tag{20}$$

It is necessary for the existence of the $W_{(r)}$ that $W(n)$ should decrease at least exponentially for large values of n.

The determinants $S_r(n)$ satisfy the orthogonality relations

$$\sum_{n=0}^{\infty} W(n) S_q(n) S_r(n) = N_q \delta_{q,r}, \tag{21}$$

where the N_q are defined by equations precisely analogous to (12);

The coefficients F_r in (18) are determined by the relation

$$\sum_{n=0}^{\infty} S_q(n) F(n) = N_q F_q \tag{22}$$

assuming the convergence of the series. Substituting for $S_q(n)$ from (19), this gives

$$N_q F_q = \begin{vmatrix} W_{(0)} & \cdots & W_{(q)} \\ \vdots & & \vdots \\ W_{(q-1)} & \cdots & W_{(2q-1)} \\ F_{(0)} & \cdots & F_{(q)} \end{vmatrix} \qquad \text{for} \qquad q \geq 1, \tag{23}$$

$$N_0 F_0 = F_{(0)} ,$$

where

$$F_{(q)} = \sum_{n=0}^{\infty} F(n) n^q \tag{24}$$

are the moments of the distribution, assumed to be finite.

In order to ensure that the series (18) should converge, it is necessary to impose rather stringent conditions on $W(n)$, depending on the nature of the function $F(n)$. In practice, if the weight-function $W(n)$ is chosen to approximate fairly closely to $F(n)$, the series is either convergent, or else is asymptotic and equally suitable for computational purposes.

If one has no previous knowledge of the nature of the function $f(x)$ or $F(n)$, but is given only the moments $f_{(k)}$ or $F_{(r)}$ respectively, defined for the range $0 = a \leq x$ (or n) $\leq b \leq \infty$, the weight-function should be chosen in the following manner. The asymptotic behaviour of $f_{(k)}$ for large k is compared with that of the moments of a weight-function of the form

$$w(x) = \exp(-Bx^A) \qquad (B > 0,\ A > 0), \tag{25}$$

namely

$$w_{(k)} = (AB^{(k+1)/A})^{-1}\Gamma\{(k+1)/A\} \sim (k/BeA)^{k/A} \qquad \text{for} \qquad b = \infty, \qquad a = 0 \tag{26}$$

$$w_{(k)} \sim b^{k+1} e^{-Bb^A}/(k+1), \qquad b \text{ finite}, \qquad a = 0$$

A numerical comparison of these expressions with the known values enables one to choose a proper value of A, and thus to determine the nature of the singularity of $f(x)$, if any, at the origin. Any other singularity will be revealed by a failure of the series (8) to converge for large values of x; and in difficult cases it may be necessary to change the origin. A similar procedure may be followed to identify the "singularities" for a discrete variable.

The value of B in (25) should be determined as follows. Assuming that numerical values are available for the first $m + 1$ moments, the series (8) is terminated at the mth term, and B chosen so as to give correctly the value of the $(m + 1)$th moment, thus:

$$f_{(m+1)} = \sum_{n=0}^{m} f_n \begin{vmatrix} w_{(0)} & \cdots & w_{(n)} \\ \cdots & \cdots & \cdots \\ w_{(n-1)} & \cdots & w_{(2n-1)} \\ w_{(m+1)} & \cdots & w_{(m+n+1)} \end{vmatrix} . \tag{27}$$

This procedure for the determination of B requires only the solution of an algebraic equation at the $(m + 1)$th degree. If the weight-function is well chosen, it is our experience that a good approximation to $f(x)$ will result from the use of the first few moments only.

4. Solution of differential equations. We shall demonstrate the utility of the method described above in the solution of partial differential equations with a special type of boundary condition.

Consider the general equation

$$O_t f(t, x) = O_x f(t, x), \tag{28}$$

where O_t and O_x are operators of the form

$$O_t = \frac{1}{p(t, x)} \frac{\partial}{\partial t} q(t, x)$$

$$O_x = \sum_{k=0}^{\infty} a_k(x) \frac{\partial^k}{\partial x^k}, \tag{29}$$

with the initial condition

$$f(0, x) = \delta(x) \tag{30}$$

By an iteration procedure a solution can be found of the form

$$f(t, x) = \sum_{k=0}^{\infty} \Psi_k(t, x) \tag{31}$$

where

$$\frac{\partial}{\partial t} \{q(t, x)\Psi_{n+1}(t,x)\} = p(t, x) \sum_{k=0}^{\infty} a_k(x) \frac{\partial^k}{\partial x^k} \Psi_n(t, x). \tag{32}$$

One may choose $\psi_0(t, x) = 0$; then $q(t, x)\, \psi_1(t, x)$ must be independent of t and, according to (30),

$$\Psi_1(t, x) = q(0, x)\ \delta(x)/q(t, x) \tag{33}$$

Furthermore,

$$\Psi_2(t, x) = \{q(t, x)\}^{-1} \int_0^t p(t, x) \sum_{k=0}^{\infty} a_k(x) \frac{\partial^k}{\partial x^k} \{q(0, x)\ \delta(x)/q(t, x)\}\, dt, \tag{34}$$

etc.

The solution of (28) obtained in this manner is of the form (1), and can be reduced to the form (6), by use of the general formula

$$a_k(x)\ \delta^{(k)}(x) = \sum_{l=0}^{k} \frac{k!(-1)^l}{l!(k-l)!} a_k^{(l)}(0)\ \delta^{(k-l)}(x). \tag{35}$$

Thus, the moments of the function $f(t, x)$ with respect to the variable x are obtained immediately as functions of t; and, using the procedure described in the previous section for the reconstruction of a function from its moments, an explicit solution is obtained of the differential equation which satisfied the boundary condition (30).

The solution $f'(t, x)$ of (28) for the initial condition

$$f'(0, x) = \lambda(x) \tag{36}$$

is then easily obtained in the form

$$f'(t, x) = \int f(t, x - x')\lambda(x')\, dx'. \tag{37}$$

An almost identical method of solution may be devised for equations of the type

$$O_t f(t, x_1, \cdots, x_m) = O_{x_1, \cdots, x_m} f(t, x_1, \cdots, x_m)$$

which, in the application to cosmic ray shower theory, we have actually [7] succeeded in solving.

References

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