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SIMILARITY LAWS FOR SUPERSONIC FLOWS*

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Summary. The non-linear differential equation for the velocity potential of three-dimensional steady irrotational supersonic flow past wings of finite span has been investigated. It is found that the whole Mach number range from 1 to ∞ may be divided into two regions (not strictly divided), in each of which similarity laws are obtained, with two parameters $K_1 = (M^2 - 1)^{1/2}/\tau^n$ and $K_2 = A(M^2 - 1)^{1/2}$; τ is the non-dimensional thickness ratio, A the aspect ratio of the wing, M the Mach number of the uniform stream in which the wing is placed. The factor n is given explicitly as a function of M and τ; in the lower region of Mach numbers it tends to $1/3$ as $M \to 1$, for all τ, giving the ordinary transonic rule, and in the upper region it tends to -1 as $M \to \infty$, for all τ, as in the ordinary hypersonic rule.

It is shown that both two-dimensional flow and flow over a three-dimensional slender body, including axially symmetrical flow, are special cases of the present analysis, involving only one parameter K_1 in the similarity rules.

I. Introduction. One of the major difficulties in solving problems of flow of compressible fluid is the non-linearity of the differential equations that govern the flow. Over a certain range of Mach number, the differential equations of flow of compressible fluid can be linearized for many investigations of practical importance, and the resulting equations give valuable information on the flow field of a compressible fluid. However, there exist other ranges of Mach numbers, in particular the transonic flow ($M \cong 1$) and the hypersonic flow ($M \gg 1$), where the differential equations cannot be linearized. In these cases, we have to study non-linear equations. At the present time, there is no general method of solving these non-linear differential equations. For practical purposes the study of the effects of non-linearity on the flow must rely mainly on experimental results. In order to increase the value of either theoretical or experimental data, it is useful to find some similarity laws which will enable a family of possible solutions to be deduced from a single result.

In general similarity laws may be found for bodies moving in the compressible fluid, which have at least one dimension perpendicular to the direction of main flow small in comparison with that in the direction of main flow. For the linearized theory well-known similarity laws have been obtained by Glauert [1] and Prandtl [2] for subsonic flow, and by Ackeret [3] for supersonic flow. However, it will be shown later that the similarity

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laws for linearized theory are arbitrary. This fact has already been observed in the study of linearized axially symmetrical flow of compressible fluid [4, 5], where contradictory results have been obtained by the use of different similarity laws. Unique similarity laws can be determined only when non-linear terms are considered.

Von Kármán [6] was the first to obtain the similarity laws for transonic flows, where the fluid velocity is very near to the velocity of sound, for both the two-dimensional and the axially symmetrical cases. Spreiter [7] extended the transonic similarity laws to wings of finite span.

The similarity laws for hypersonic flow, where the fluid velocity is much larger than the local velocity of sound, were first obtained by Tsien [8], again both for two dimensions and axial symmetry. Hayes [9] extended the hypersonic similarity laws to three-dimensional slender bodies.

Spreiter [7] tried to combine the similarity laws of transonic flow with those of linearized theory. He was able to do so because of the arbitrariness of the similarity laws of linearized theory. Van Dyke [10] empirically generalized the hypersonic similarity laws to the supersonic flow where the fluid velocity is not much larger than the local velocity of sound. It is the object of this paper to investigate the possibility of determining uniquely similarity laws which link flows over more widely extended ranges of Mach number than are covered by the particular cases cited above.

We begin with the non-linear differential equation for the velocity potential of three dimensional steady irrotational supersonic flow for wings of finite span; two-dimensional and axially symmetrical flows, and flow past three-dimensional slender bodies, may be considered as special cases of this general flow. The relative importance of the various non-linear terms is discussed so that the ordinary transonic and hypersonic similarity laws are generalized for large Mach number range. It is found that from a known solution for a given Mach number and ratio of thickness to chord, a family of solutions may be determined; there is however a barrier dividing the Mach number range into two regions, so that a solution which falls into the 'generalized transonic range' cannot be used to give a flow in the 'generalized hypersonic range', and vice-versa.

2. Fundamental equations and boundary conditions. If Φ is the velocity potential of a three-dimensional steady irrotational flow of compressible fluid, the differential equation for Φ is

$$(a^2 - \Phi_x^2)\Phi_{xx} + (a^2 - \Phi_y^2)\Phi_{yy} + (a^2 - \Phi_z^2(\Phi_{zz} - 2\Phi_x\Phi_y\Phi_{xy} - 2\Phi_y\Phi_z\Phi_{yz} - 2\Phi_z\Phi_x\Phi_{zx} = 0, \tag{1}$$

where x, y, z are the Cartesian coordinates, subscripts denote partial derivatives, i.e., $\Phi_x = \partial\Phi/\partial x$ etc., and a is the local sound speed which is determined by the equation:

$$a^2 + \frac{\gamma - 1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) = a_0^2 \, . \tag{2}$$

Here γ is the ratio of the specific heats and a_0 is the speed of sound in the gas at rest.

Now if a thin wing of finite span is placed in an otherwise uniform stream of velocity V in the x-direction, we may introduce a perturbed velocity potential ϕ such that

$$\Phi = V(x + \phi) \tag{3}$$

with

$$\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \ll 1.$$

Equation (2) may be written as follows:

$$\frac{a^2}{a_1^2} = 1 - \frac{\gamma - 1}{2} M^2(2\phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2), \tag{4}$$

where a_1 is the speed of sound corresponding to the free stream velocity V and $M = V/a_1$, the Mach number in the undisturbed stream.

Substituting equations (3) and (4) into (1) and retaining terms up to second order, one has, with $\lambda^2 = (\gamma - 1)/(\gamma + 1)$,

$$\begin{aligned} &\left\{1 - M^2 - (\gamma + 1)M^2\phi_x - \frac{\gamma - 1}{2} M^2(\phi_y^2 + \phi_z^2)\right\}\phi_{xx} \\ &\quad + \left\{1 - (\gamma - 1)M^2\phi_x - \frac{\gamma + 1}{2} M^2(\phi_y^2 + \lambda^2\phi_z^2)\right\}\phi_{yy} \\ &\quad + \left\{1 - (\gamma - 1)M^2\phi_x - \frac{\gamma + 1}{2} M^2(\lambda^2\phi_y^2 + \phi_z^2)\right\}\phi_{zz} \\ &\quad - 2M^2\phi_y\phi_{xy} - 2M^2\phi_z\phi_{xz} = 0. \end{aligned} \tag{5}$$

This is the fundamental differential equation for ϕ for the consideration of similarity laws, and is a generalization of Tsien's equation for two-dimensional flow [8].

In (5) we have retained terms which, while not important everywhere, may be so in certain regions. For example, in hypersonic flow ϕ_y^2, ϕ_z^2, ϕ_x are all of the same order of magnitude. Furthermore, while we retain in (5) terms which are usually second-order, we shall seek and use those terms which give an effectively first-order equation.

The first-order boundary conditions for ϕ are:

(1) at infinity

$$\phi_x = \phi_y = \phi_z = 0; \tag{6}$$

(2) on the surface of the wing, which is represented by

$$z = h(x, y), \tag{7}$$

the normal velocity component is zero, i.e.

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=0} = \frac{\partial h(x, y)}{\partial x} \tag{8}$$

3. General discussion of similarity rules. We are going to find conditions under which it is possible to reduce the differential equation for ϕ and the boundary conditions simultaneously into non-dimensional form. We write

$$x = \xi c, \qquad y = \eta b, \qquad z = \zeta l, \qquad \phi = \phi' m, \tag{9}$$

where c is the mean chord of the wing and b its span; thus, $A = b/c$ = aspect ratio. l and m are conversion factors which are to be determined.

Substituting equation (9) into (6) and (8), we have,

$$\phi'_\xi = \phi'_\eta = \phi'_\zeta = 0 \qquad \text{at infinity,} \tag{10}$$

$$\left(\frac{\partial \phi'}{\partial \zeta}\right)_{\zeta=0} = \frac{l\tau}{m} f_\xi(\xi, \eta), \tag{11}$$

where $Tf(\xi, \eta) = h(x, y)$ and $T = \tau c$ is the maximum thickness of the airfoil section; it will be supposed that $\tau \ll 1$.

Substituting (9) into (5), we have

$$\begin{aligned}
&\left[(M^2 - 1)\left(\frac{l}{c}\right)^2 + (\gamma + 1)M^2\left(\frac{m}{c}\right)\left(\frac{l}{c}\right)^2 \phi'_\xi + \frac{\gamma - 1}{2} M^2\left(\frac{m}{c}\right)^2\left(\frac{l}{c}\right)^2\left(\frac{\phi'^2_\eta}{A^2} + \phi'^2_\zeta\left(\frac{c}{l}\right)^2\right)\right]\phi'_{\xi\xi} \\
&\quad + \left[-1 + (\gamma - 1)M^2 \frac{m}{c} \phi'_\xi + \frac{\gamma + 1}{2} M^2\left(\frac{m}{c}\right)^2\left\{\frac{\phi'^2_\eta}{A^2} + \lambda^2\left(\frac{c}{l}\right)^2 \phi'^2_\zeta\right\}\right]\phi_{\eta\eta}\left(\frac{l}{cA}\right)^2 \\
&\quad + \left[-1 + (\gamma - 1)M^2 \frac{m}{c} \phi'_\xi + \frac{\gamma + 1}{2} M^2\left(\frac{m}{c}\right)^2\left\{\lambda^2 \frac{\phi'^2_\eta}{A^2} + \left(\frac{c}{l}\right)^2 \phi'^2_\zeta\right\}\right]\phi'_{\zeta\zeta} \\
&\quad + 2M^2\left(\frac{m}{c}\right)\left(\frac{l}{cA}\right)^2 \phi'_\eta \phi'_{\xi\eta} + 2M^2\left(\frac{m}{c}\right)\phi'_\zeta \phi'_{\zeta\xi} = 0.
\end{aligned} \tag{12}$$

In order to get the parameters in the similarity laws we write

$$l = c\tau^{-n}, \qquad m = c\tau^{n'}, \tag{13}$$

where n and n' are factors to be determined.

Then, for similarity laws to be possible, considering only the linear terms in (12), the following necessary conditions are found:

$$\frac{M^2 - 1}{\tau^{2n}} = K_1^2, \tag{14}$$

$$A\tau^n = K'_2, \tag{15}$$

or, combining these,

$$A(M^2 - 1)^{1/2} = K_2 \tag{16}$$

where K_1, K'_2 and K_2 are constants.

From the boundary conditions, we have

$$\tau^{1-n-n'} = \text{constant}; \tag{17}$$

this can only be true for variable τ provided that

$$n + n' = 1, \tag{18}$$

and the value of the constant is unity.

Equations (14), (16) and (18) represent the conditions for similarity laws in the linearized theory of compressible flow. Only the parameter K_2 for the aspect ratio (16) is unique; the index n in the parameter K_1, is arbitrary, since the choice of n' is arbitrary. In order to have unique similarity laws it is necessary to study the non-linear terms. From (12) we see that the non-linear terms fall into two groups, one group being the more important for transonic-supersonic flow ($n > 0$), the other for supersonic-hypersonic flow ($n < 0$). In the transonic-supersonic flow region the important non-linear term is $(\gamma + 1)M^2(m/c)(l/c)^2\phi'_\xi$. For this term to be unchanged with variation of airfoil thickness, for example, in the immediate transonic region ($M \cong 1$) we must have $\tau^{n'-2n} =$ const., i.e.,

$$n' - 2n = 0. \tag{19}$$

From (18) and (19), we have

$$n = \tfrac{1}{3}; \qquad n' = \tfrac{2}{3}. \tag{20}$$

This is the well-known transonic similarity law due to von Kármán [6].

When M is very large, the important non-linear terms are in the second square bracket in (12); both are of the same order of magnitude as $(\gamma - 1)\ M^2(m/c)\ \phi_\xi'$; for similarity we require

$$M^2\tau^{n'} = \text{const}. \tag{21}$$

Thus, from (14) and (21),

$$n' = -2n. \tag{22}$$

Finally, from equations (18) and (22),

$$n = -1, \qquad n' = 2 \tag{23}$$

This is the well-known hypersonic similarity law due to Tsien.

It is clear, from an examination of (12), that in order to obtain similarity laws which are valid in the intermediate (supersonic) range of Mach numbers we must look for a variation of n with Mach number. It is found in the next section that n depends upon two parameters, M and τ (say), but that the value of n tends to 1/3 as $M \to 1$ and to -1 as $M \to \infty$, for all τ.

4. Generalized similarity law for transonic-supersonic flow. As M increases from unity, the non-linear term of lowest order is the second term in the first square bracket in (12).

In order to preserve the form of (12) unchanged up to and including this term, when the Mach number is not restricted to be in the immediate neighborhood of unity, the parameters required are

$$\frac{(M^2 - 1)}{\tau^{2n}} = K_1^2, \qquad A(M^2 - 1)^{1/2} = K_2,$$

with $n + n' = 1$ from the boundary condition as before, and

$$M^2\tau^{n'-2n} = K_3^2. \tag{24}$$

By elimination of τ, we derive

$$n[4 \log (K_3/M) + 3 \log \{(M^2 - 1)/K_1^2\}] = \log \{(M^2 - 1)/K_1^2\}. \tag{25}$$

This shows clearly that $n \to 1/3$ as $M \to 1$. Furthermore, by differentiating and proceeding to the limit, it may be shown that

$$\lim_{M\to 1(+0)} \left[\frac{dn}{dM}\right] = \begin{cases} 0 & (K_3 = 1), \\ \infty & (K_3 > 1), \\ -\infty & (K_3 < 1). \end{cases}$$

It is plausible to reject infinite values of dn/dM at $M = 1$, in view of the established position of the transonic similarity law which requires $n = 1/3$ to be approximately valid in the neighborhood of $M = 1$. Then $K_3 = 1$, and the law of variation of n with M is

$$n = \frac{1}{3} \log \left(\frac{M^2 - 1}{K_1^2} \right) \Big/ \log \left(\frac{M^2 - 1}{K_1^2 M^{4/3}} \right). \tag{26}$$

Examples of the variation of n with M, for three different values of the parameter K_1, are shown in Figure 1.

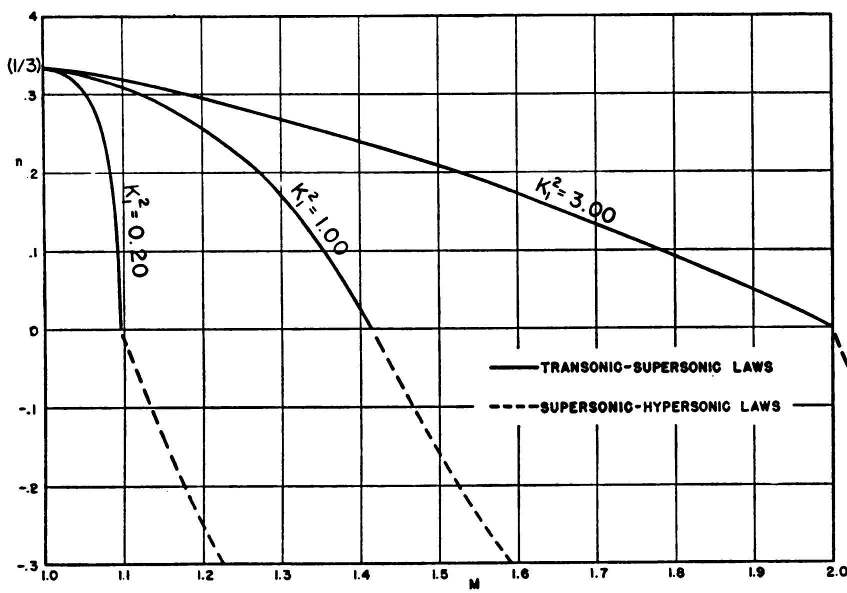

FIG. 1.

We are now able to deduce similarity laws in the sense that a solution $\phi(\xi, \eta, \zeta)$ of the differential equation of motion may be used to give an infinity of flows in the transonic-supersonic region. Given values of the parameters K_1, K_2, if we choose a value for M the corresponding values of τ, A, n are found.

It will be noticed that the solid curves have been stopped at $n = 0$. As $n \to 0$ the relative importance of the non-linear terms in the second square bracket in (12) increases, and at $n = 0$ these terms are of the same order of magnitude as the "transonic" term which has been the crux of the above discussion.

5. Generalized similarity law for supersonic-hypersonic flow. When n is negative, the most important non-linear term is the second term in the second square bracket in (12). The flows for which this term is retained we call 'generalized hypersonic flows'. In order to have coefficients in (12) which are independent of Mach number and thickness ratio we must have

$$M^2 \tau^{n'} = \text{constant} = K_4^2 (n' \to 2 \text{ as } M \to \infty) \tag{27}$$

in addition to (14), (16) and (18).

Consideration of (27) and (14) as $M \to \infty$ shows that $n \to -1$ and $K_4 \to K_1$. But K_4 and K_1 are constants; hence $K_4 = K_1$. Elimination of τ yields finally

$$n = -\frac{\log \{(M^2 - 1)/K_1^2\}}{\log \{M^4/K_1^2(M^2 - 1)\}}. \tag{28}$$

The variation of n with M is shown for three values of K_1 in Figure 2. As before the reliability of the approximations made falls off as $n \to 0$.

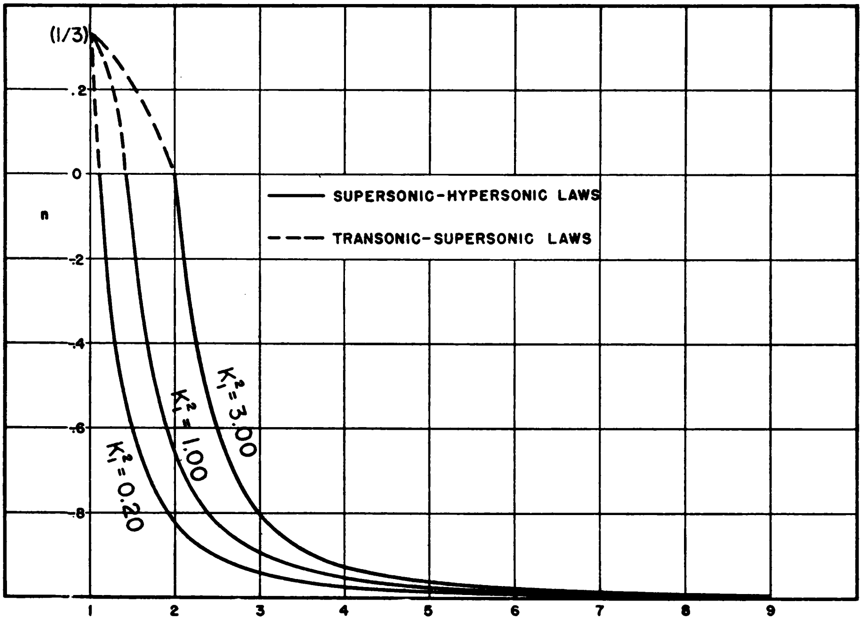

FIG. 2.

6. Conclusions. Both the differential equation for the velocity potential and the boundary conditions for the steady irrotational supersonic flow over a thin wing of finite aspect ratio A can be put into a non-dimensional form containing two parameters only, when only the most important of the non-linear terms are retained. The parameters are $K_1 = (M^2 - 1)^{1/2}/\tau^n$ and $K_2 = A(M^2 - 1)^{1/2}$, where M is the Mach number, τ the ratio of maximum thickness to chord and n is a function of any two of M, τ and K_1. The similarity laws thus hold for such profiles and Mach numbers as correspond to the same parameters K_1, K_2. There are, however, two important restrictions. The most important non-linear term being different in the two cases $n > 0$, $n < 0$, the line $n = 0$ divides the n,M-plane into two fundamentally different regions; it is therefore not possible to use a result which corresponds to positive n to deduce a flow which corresponds to negative n, or vice-versa. There is a region around $n = 0$ in which neglected and included terms are of approximately the same order of magnitude, so that the reliability falls off appreciably near $n = 0$. Further, the use of the rules should not be attempted for large variations of Mach number, e.g. from a supersonic flow to a genuinely hypersonic flow, as the latter requires the inclusion of a term which is neglected in the present discussion.

For two-dimensional flow $A \to \infty$; the similarity laws then contain only one parameter K_1.

For flow over a three-dimensional slender body, including the axially symmetrical flow, $A \to 0$; here also the similarity laws contain only the parameter K_1.

In practical cases, K_1 is usually larger than one for the transonic-supersonic region and smaller than one for the supersonic-hypersonic region, when τ is small.

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