CONNECTION FORMULAS BETWEEN THE SOLUTIONS OF MATHIEU'S EQUATION*

BY

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Abstract. The problem of connecting the various types of solutions of Mathieu's equation is solved by the introduction of a new parameter Φ which is a function of the the two equation parameters a and q. This quantity Φ is introduced and enclosed between two very close analytic limits in section 2. In sections 3, 4, 5 precise definitions are given and information is collected for the three main types of functions which are to be connected. Section 6 contains the connection formulas. Section 7 reviews the status of knowledge achieved. Section 8 is an appendix on integral equations which are more general than those developed earlier in the text, but which appear to be of no use for the main purpose of this paper.

1. Introduction. A variety of different types of solutions have been written down for the Mathieu differential equation

$$\frac{d^2 f}{dx^2} + (a - 2q \cos 2x)f(x) = 0 \tag{1}$$

a general solution of which we shall call *me x*. The special solutions proposed stress different qualitative features of this general solution and suggest themselves in different types of applications. By general principles, there must exist a connection formula between any three such solutions. It is the purpose of this paper to write down these connection formulas for some of the more important solutions of equation (1).

In carrying out this program we shall assume q real, and once it is assumed real it may be assumed positive because the transformation

$$x \to \frac{\pi}{2} + x$$

will reverse the sign of q. We shall express this sometime by writing

$$q = k^2. \tag{2}$$

The parameter a will also be assumed real. The treatment is particularly designed for positive a; this becomes important in the discussion of the next section.

2. On the function *ke y*. Equation (1) contains three distinct real equations, one of which is (1) itself. The second is obtained by setting

$$x = iz$$

which yields

$$\frac{d^2 f}{dz^2} - (a - 2q \cosh 2z)f = 0 \tag{3}$$

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and the third by setting

$$x = \frac{\pi}{2} + iy$$

which yields

$$\frac{d^2f}{dy^2} - (a + 2q \cosh 2y)f = 0. \tag{4}$$

The study of equation (4) for f as a function of the real variable y will occupy the rest of this section.

Whenever $|y|$ is large equation (4) can be approximated closely by

$$\frac{d^2f}{dy^2} - (a + qe^{2|y|})f = 0 \tag{5}$$

which admits as solutions the modified Bessel Functions

$$f_1(y) = K_{a^{1/2}}(k\, e^{|y|}), \qquad f_2(y) = I_{a^{1/2}}(k\, e^{|y|}). \tag{6}$$

The solutions (6) give the asymptotic character of the solutions of (4); they show us that there must be one solution vanishing as $K_{a^{1/2}}(k\, e^{y})$ for positive y. We shall call it $ke\ y$; we shall normalize it by the prescription that

$$ke\ y \sim \frac{\exp(-k\, e^{y})}{(k\, e^{y})^{1/2}} \qquad \text{for} \qquad y \gg 0. \tag{7}$$

We now assume that in equation (4), we have the inequality

$$a + 2q > 0. \tag{8}$$

The solutions of (4) are then always curved away from the axis. This means for $ke\ y$ that it stays positive throughout, and that it cannot vanish exponentially for large negative y. Hence we must be able to write

$$ke\ y \sim \frac{\exp[k\, e^{|y|} + \Phi(a, k)]}{(k\, e^{|y|})^{1/2}} \qquad \text{for} \qquad y \ll 0 \tag{9}$$

with $\Phi\ (a, k)$ real.

We shall now show that under the restriction (8), $\Phi(a, k)$ is a smoothly varying function of a and k, that is, it does not partake in the oscillatory character of the parameter β introduced by Floquet's theorem. We shall establish this fact in the remainder of this section by enclosing Φ between narrow limits, neither one of which shows any oscillation. These limits are reproduced in Figs. 1 and 2; they furnish incidentally a close numerical approximation to Φ when desired. The structural discussion of Mathieu's equation will then be resumed in section 3.

We start out with the approximate value of Φ which is obtained by application of the Jeffreys' or WKB method[1] to equation (4). We set

$$ke\ y = A(y)e^{-S(y)} \tag{10}$$

[1]H. Jeffreys. Proc. Lond. Math. Soc. **23**, 428 (1924).

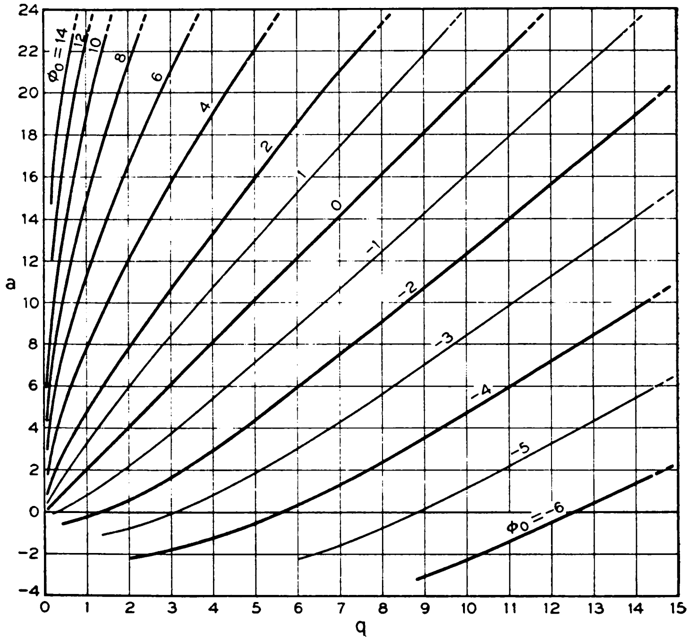

FIG. 1 MAP OF THE LOWER BOUND ϕ_0 OF $\phi(a,q)$. THIS LOWER BOUND RESULTS FROM THE JEFFREYS APPROXIMATION.

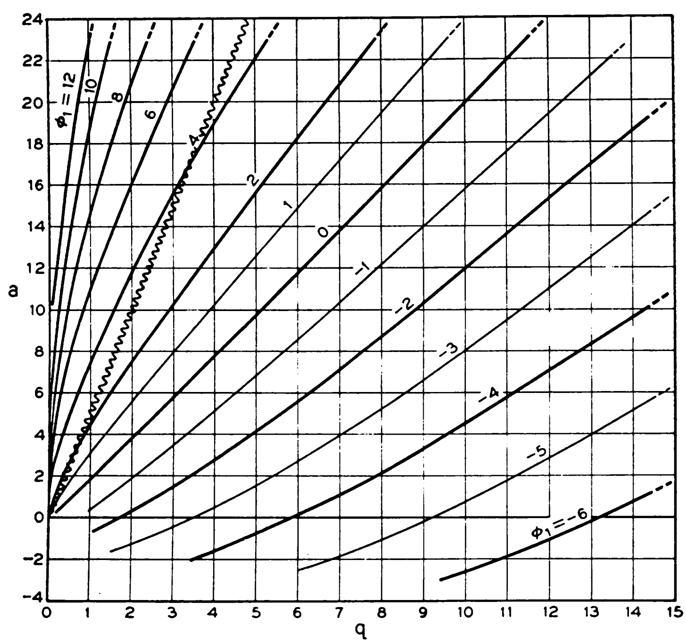

FIG. 2 UPPER BOUND ϕ_1 OF $\phi(a,q)$. THE BOUND IS NOT PROVED TO THE LEFT OF THE WAVY LINE, BUT PROBABLY HOLDS THERE ALSO.

and set

$$\left(\frac{dS}{dy}\right)^2 = a + 2q \cosh 2y. \tag{11}$$

This yields for A the equation

$$\frac{d^2 A}{dy^2} - 2\frac{dA}{dy}\frac{dS}{dy} - A\frac{d^2 S}{dy^2} = 0. \tag{12}$$

The standard solution is obtained by neglecting the first term in (12). We find in this case

$$ke\ y \approx \frac{\exp\left[\frac{1}{2}\Phi_0 - \int_0^y (a + 2k^2 \cosh 2\eta)^{1/2}\, d\eta\right]}{(a + 2k^2 \cosh 2y)^{1/4}}. \tag{13}$$

The term $\frac{1}{2}\Phi_0$ enters into the exponent in order to satisfy the asymptotic requirement (7); it obeys the relation

$$\frac{1}{2}\Phi_0 = \lim_{y=\infty}\left[\int_0^y (a + 2k^2 \cosh 2\eta)^{1/2}\, d\eta - 2k \sinh y\right]. \tag{14}$$

Equation (14) then represents the Jeffreys approximation to Φ as defined in (9):

$$\Phi \approx \Phi_0\ . \tag{15}$$

Φ_0 is easily evaluated in terms of complete elliptic integrals; we find

for $a \geqq 2k^2$

$$\Phi_0 = 2\frac{a - 2k^2}{(a + 2k^2)^{1/2}} D\left[\left(\frac{a - 2k^2}{a + 2k^2}\right)^{1/2}\right], \tag{16a}$$

for $a \leqq 2k^2$

$$\Phi_0 = -\frac{2k^2 - a}{k}\left\{K\left[\frac{(2k^2 - a)^{1/2}}{2k}\right] - D\left[\frac{(2k^2 - a)^{1/2}}{2k}\right]\right\}. \tag{16b}$$

The two expressions (16) do not indicate a break at $a = 2k^2$, for they are analytic continuations of each other.

The approximation (16) to Φ is shown in Fig. 1; we shall now prove for it that the approximate identity (15) is an inequality, that is, that

$$\Phi > \Phi_0\ . \tag{17}$$

To prove this let us write equation (12) in the form

$$A^2\frac{d^2 S}{dy^2} + 2A\frac{dA}{dy}\frac{dS}{dy} - A\frac{d^2 A}{dy^2} = 0.$$

Subtracting $(dA/dy)^2$ on either side we get

$$\frac{d}{dy}\left(A^2\frac{dS}{dy} - A\frac{dA}{dy}\right) = -\left(\frac{dA}{dy}\right)^2 < 0.$$

Integrating this inequality between $-\infty$ and $+\infty$, and observing that $A(dA/dy)$ vanishes at either end, we get

$$\left(A^2\frac{dS}{dy}\right)_{y\to+\infty} - \left(A^2\frac{dS}{dy}\right)_{y\to-\infty} < 0.$$

If the Jeffreys' approximation were correct this difference would be zero; actually, the formulas (7), (9), (10), (11) and (14) yield

$$A(+\infty) \sim \frac{\exp\{\frac{1}{2}\Phi_0\}}{(dS/dy)^{1/2}}, \tag{18a}$$

$$A(-\infty) \sim \frac{\exp\{-\frac{1}{2}\Phi_0 + \Phi\}}{(dS/dy)^{1/2}}. \tag{18b}$$

These expressions reduce the inequality to

$$e^{\Phi_0} < e^{-\Phi_0+2\Phi}$$

which is equivalent to (17).

In order to gain an upper limit for Φ, we transform equation (12) by the substitution

$$\frac{1}{A}\frac{dA}{dy} = B \tag{19}$$

which yields the equation

$$\frac{d^2S}{dy^2} + 2B\frac{dS}{dy} = \frac{dB}{dy} + B^2,$$

and hence the inequality

$$\frac{dB}{dy} - 2B\frac{dS}{dy} - \frac{d^2S}{dy^2} < 0.$$

Multiplying with e^{-2S} and integrating we get from this

$$e^{-2S}B + \int_y^\infty e^{-2S}\frac{d^2S}{dy^2}\,dy > 0.$$

Returning to A by (19) we get

$$\frac{1}{A}\frac{dA}{dy} + e^{2S}\int_y^\infty e^{-2S}\frac{d^2S}{dy^2}\,dy > 0.$$

Integrating the second term by parts we get this in the form

$$\frac{d}{dy}\ln\left[A\left(\frac{dS}{dy}\right)^{1/2}\right] + \frac{1}{2}e^{2S}\int_y^\infty e^{-2S}\frac{d}{dy}\left(\frac{d^2S/dy^2}{dS/dy}\right)dy > 0.$$

With the help of the equations (18) this is finally transformed into

$$\Phi - \Phi_0 < \frac{1}{2}\int_{-\infty}^{+\infty} e^{2S(y)}\,dy\int_y^\infty e^{-2S(\eta)}\frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right)d\eta. \tag{20}$$

The inequality (20) is the desired upper limit and could be evaluated by numerical methods. However, we shall proceed instead to majorize the double integral by explicit analytic expressions.

In discussing (20) we observe first that the expression in the integrand

$$\frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right) = \frac{4q(a \cosh 2\eta + 2q)}{(a + 2q \cosh 2\eta)^2} \tag{21}$$

is positive everywhere and vanishes strongly at either infinity; the inner integral is therefore majorized by replacing $e^{-2S(\eta)}$ by its largest value $e^{-2S(y)}$. This procedure is of no use at this stage as it gives a divergent result. However, if we integrate the outer integral by parts according to the scheme

$$\int e^{2S(y)}\, dy \sim \int d(e^{2S(y)})\, \frac{1}{2(dS/dy)}$$

then we verify by this method that the integrated out part vanishes. The expression (20) then takes the form

$$\Phi - \Phi_0 < \frac{1}{4}\int_{-\infty}^{+\infty} \frac{1}{dS/dy}\,\frac{d}{dy}\left(\frac{d^2S/dy^2}{dS/dy}\right) dy$$
$$+ \frac{1}{4}\int_{-\infty}^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^2}\, e^{2S(y)}\, dy \int_y^{\infty} e^{-2S(\eta)}\, \frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right) d\eta.$$

The first integral is positive, as can be seen by another integration by parts:

$$\Phi - \Phi_0 < \frac{1}{4}\int_{-\infty}^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^3}\, dy$$
$$+ \frac{1}{4}\int_{-\infty}^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^2}\, e^{2S(y)}\, dy \int_y^{\infty} e^{-2S(\eta)}\, \frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right) d\eta. \tag{22}$$

The sign of the second integral is not immediately obvious, because the integrand changes its sign with d^2S/dy^2. An easy way to obtain an upper limit of Φ is to replace the integrand by 0 in the range in which it is negative; we find thus

$$\Phi - \Phi_0 < \frac{1}{4}\int_{-\infty}^{+\infty} \frac{(d^2S/dy^2)^2}{(dS/dy)^3}\, dy + \frac{1}{4}\int_0^{\infty} \frac{d^2S/dy^2}{(dS/dy)^2}\, e^{2S(y)}\, dy \int_y^{\infty} e^{-2S(\eta)}\, \frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right) d\eta.$$

Having now the integrand positive throughout, we can majorize it by the trick discussed above of replacing the exponent $S(\eta)$ by $S(y)$. Observing that

$$\lim_{\eta\to\infty} \frac{d^2S/d\eta^2}{dS/d\eta} = 1,$$

we get

$$\Phi - \Phi_0 < \frac{1}{4}\int_{-\infty}^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^3}\, dy + \frac{1}{4}\int_0^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^2}\left[1 - \frac{d^2S/dy^2}{dS/dy}\right] dy$$
$$= \frac{1}{4}\int_0^{\infty} \frac{(d^2S/dy^2)^2}{(dS/dy)^3}\, dy + \frac{1}{4}\,\frac{1}{(dS/dy)_{y=0}}\,.$$

By the observation that

$$\left|\frac{d^2S}{dy^2}\right| < \left|\frac{dS}{dy}\right|$$

the first term is seen to be smaller than the second. We get thus with (11) the inequality

$$\Phi < \Phi_2 \tag{23}$$

where

$$\Phi_2 = \Phi_0 + \frac{1}{2(a + 2q)^{1/2}}. \tag{24}$$

The upper limit (24) is shown in Fig. 3. This figure makes the limit appear rather close,

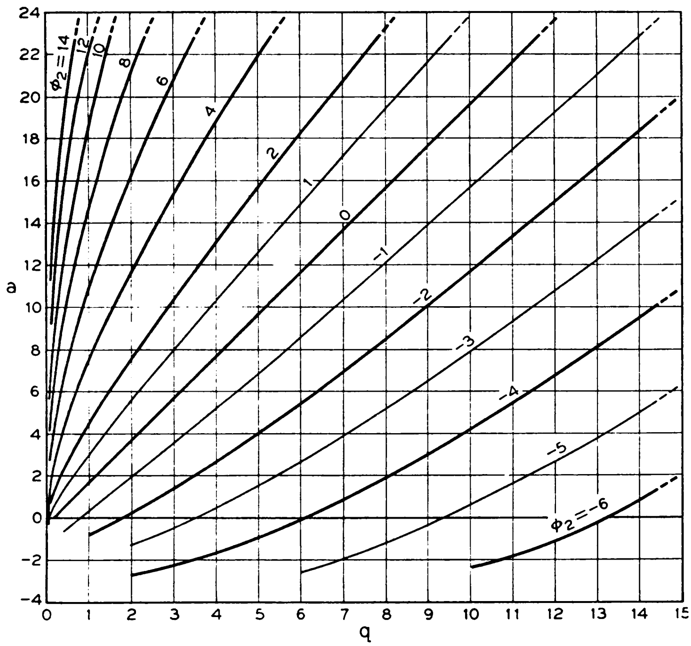

FIG. 3 UPPER BOUND Φ_2 OF $\Phi(a, q)$. THIS BOUND IS EASILY PROVED FOR ALL VALUES OF a AND q.

but a detailed study for small a and q shows deviations. We shall now establish a tighter limit by showing that the second integral in (22) is always negative and can be discarded, provided we introduce the restriction

$$a < 5q. \tag{25}$$

Introduce the abbreviation

$$E(y) = e^{2S(y)} \int_y^\infty e^{-2S(\eta)} \frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right) d\eta.$$

Because of (21), this function is positive everywhere and vanishes at $\pm\infty$. It is seen to obey the differential equation

$$\frac{dE}{dy} = 2\frac{dS}{dy}(E - E_{\mathrm{lim}})$$

where

$$E_{\mathrm{lim}} = \frac{1}{2(dS/dy)}\frac{d}{dy}\left(\frac{d^2S/dy^2}{dS/dy}\right).$$

In an E,y-plane as shown in Fig. 4, the differential equation above defines a slope at

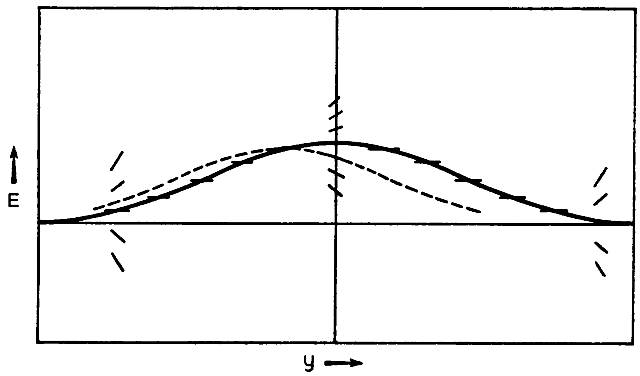

FIG. 4 DIAGRAM FOR THE STUDY OF THE AUXILIARY FUNCTION E(y). THE SHORT LINES INDICATE SLOPES, THE SOLID CURVE CONNECTS POINTS OF ZERO SLOPE. THE CURVE E(y) IS SHOWN DASHED. BY CONSTRUCTION, ITS VALUE FOR A POSITIVE y IS SMALLER THAN IT IS FOR −y.

at every point. The curve $E = E_{\mathrm{lim}}$ divides this plane in two parts. Above this curve, the slopes are positive, below negative. The curve $E = E_{\mathrm{lim}}$ is always crossed with zero slope. The inequality (25) is needed at this point because if it is satisfied then we find from (21) that E_{lim} has a slope whose sign is opposite to that of y. When (25) does not hold then E develops two humps as shown in Fig. 5. In the case of Fig. 4, E starts out by being 0 for $+\infty$, rises for positive decreasing values of y, but must stay below the line $E = E_{\mathrm{lim}}$ because of the slope requirement; thus E still rises as y becomes negative. At some negative y, E reaches its maximum as it crosses the line $E = E_{\mathrm{lim}}$; then this same slope requirement forces E to stay above the latter curve while going to zero. The result of this behavior is that we have for all y

$$0 < E(-|y|) < E(+|y|).$$

The second integral in (22) now takes the form

$$\frac{1}{4}\int_{-\infty}^{+\infty}\frac{d^2S/dy^2}{(dS/dy)^2}E(y)\,dy$$

or, for symmetry reasons

$$\frac{1}{8}\int_{-\infty}^{+\infty}\frac{d^2S/dy^2}{(dS/dy)^2}\{E(y) - E(-y)\}\,dy.$$

This last expression is negative because the curly bracket and d^2S/dy^2 have opposite sign, and thus the second integral in (22) is proved to be negative.

The proof given breaks down if the inequality (25) is reversed. The reason for this is shown on Fig. 5; as soon as $E_{\lim}$ develops a minimum at $y = 0$ instead of a maximum

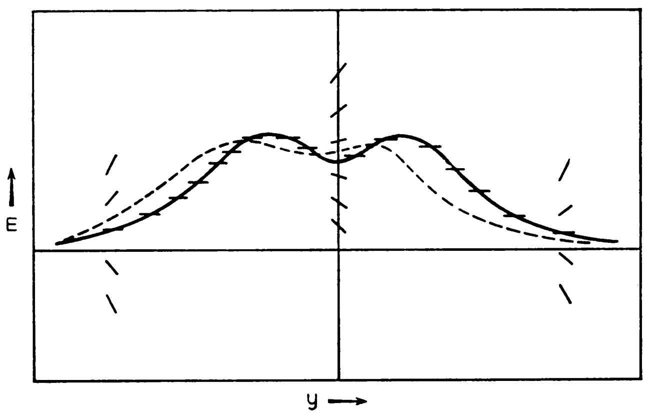

FIG. 5 DIAGRAM ANALOGOUS TO FIG. 4 FOR THE CASE WHEN INEQUALITY (24) IS REVERSED. THE FUNCTION $E(y)$ IS SHOWN IN DASHED OUTLINE. ITS MODE OF CONSTRUCTION NO LONGER IMPLIES THAT $E(|y|) \geqq E(-|y|)$.

the method of constructing E outlined above may give E a positive slope for small $|y|$. Thus the needed inequality does not hold everywhere. Nevertheless it still does hold in the greater part of the interval. Numerical checks actually indicate that the second integral in (22) is always negative.

We have thus found, at least under the restriction (25), and probably everywhere, that

$$\Phi < \Phi_1 \tag{26}$$

where

$$\Phi_1 = \Phi_0 + \frac{1}{4}\int_{-\infty}^{+\infty} \frac{(d^2S/dy^2)^2}{(dS/dy)^3}\, dy,$$

or explicitly

for $a \geqq 2k^2$

$$\Phi_1 = \left(2\frac{a - 2k^2}{(a + 2k^2)^{1/2}} - \frac{a}{3(a + 2k^2)^{3/2}}\right)D\left[\left(\frac{a - 2k^2}{a + 2k^2}\right)^{1/2}\right] + \frac{1}{3}\frac{1}{(a + 2k^2)^{1/2}}K\left[\left(\frac{a - 2k^2}{a + 2k^2}\right)^{1/2}\right], \tag{27a}$$

for $a \leqq 2k^2$

$$\Phi_1 = \left(-\frac{2k^2 - a}{k} + \frac{k}{3(2k^2 + a)}\right)K\left[\frac{(2k^2 - a)^{1/2}}{2k}\right] + \left(\frac{2k^2 - a}{k} + \frac{a}{6k(2k^2 + a)}\right)D\left[\frac{(2k^2 - a)^{1/2}}{2k}\right]. \tag{27b}$$

This upper limit is shown in Fig. 2, together with the line (25) above which the theorem (26) has not been proved. These established limits determine fairly closely the behavior of $\Phi(a, k^2)$.

It is possible without difficulty to come closer to the numerical value of Φ than Figs. 1 and 2 do. For this purpose, equation (12), which is linear in A, may be solved in successive approximations beginning with

$$2\frac{dA_0}{dy}\frac{dS}{dy} + A_0\frac{d^2S}{dy^2} = 0$$

which yields (13); successive additive corrections A_1, A_2, A_3, $\cdots$ are then obtained from the recursion system

$$2\frac{dA_n}{dy}\frac{dS}{dy} + A_n\frac{d^2S}{dy^2} = \frac{d^2A_{n-1}}{dy^2}.$$

If this calculation is carried out up to A_1, and if we take the corrected A in the form $A_0 \exp A_1/A_0$ rather than in the more obvious form $A_0 + A_1$, then a corrected Φ results which is exactly the arithmetic mean of the two limits Φ_0 and Φ_1:

$$\Phi \approx \tfrac{1}{2}(\Phi_0 + \Phi_1). \tag{28}$$

Thus a good approximation to the value of Φ is obtained by taking the arithmetic mean of the readings on Fig. 1 and Fig. 2.

3. The Lindemann-Stieltjes Functions. The following three sections define and discuss the solutions of (1) between which connection formulas are to be established. These discussions contain a good deal of information which is already available but which has to be combined with the new material to yield the desired results.

The Lindemann-Stieltjes functions[2] are based on the symmetry of the equation (1) about the points $x = 0$ and $x = \pi/2$ which are regular points of the equation. There must thus exist an even and an odd power series solution about either one of these points. We introduce the following definitions

I. $ce\ (x; a, q)$ shall be the even function about $x = 0$; its value at $x = 0$ shall be 1.
II. $se\ x$ shall be the odd function about $x = 0$; its derivative at $x = 0$ shall be 1.
III. $de\ x$ shall be the even function about $x = \pi/2$; its value there shall be 1.
IV. $te\ x$ shall be the odd function about $x = \pi/2$; its derivative there shall be -1.

ce and *se* always form a linearly independent pair, as do *de* and *te*. Whenever the periodic Mathieu functions $ce_n x$ or $se_n x$ exist they are respectively identical with the generalized functions $ce\ x$ and $se\ x$ defined here. Similarly, ce_{2n} or se_{2n+1}, when existing, are identical with $de\ x$, ce_{2n+1} or se_{2n} with $te\ x$. The simplest realization of these functions is by the power series method:

$ce\ x$ has a series in even powers of $\sin x$.
$se\ x$ " " " in odd " " "
$de\ x$ " " " " even " " $\cos x$.
$te\ x$ " " " " odd " " "

[2] E. T. Whittaker and G. N. Watson. *A course of modern analysis.* Cambridge University Press, Fourth Edition, Section 19.5.

The coefficients are obtained directly from (1) through term by term recursion with the first coefficient equal 1. However, the radius of convergence of these series is only 1, and this excludes the possibility of getting connection formulas by this procedure.

As these functions are constructed relatively easily from others not having their symmetry we will not investigate their structure further, but get it indirectly from the study of the other types.

4. The Mathieu analogues to the Hankel functions. We will now use the function $ke\ y$ of section 2 as an auxiliary function to define solutions of (1) having given asymptotic character. We define four such functions, namely $he^{(1)}x$, $he^{(2)}x$, $he^{(3)}x$, $he^{(4)}x$, by the supplementary prescriptions

$$he^{(1)}\left(\frac{\pi}{2} + iy\right) = ke\ y, \tag{29}$$

$$he^{(2)}\left(-\frac{\pi}{2} + iy\right) = ke\ y, \tag{30}$$

$$he^{(3)}\left(-\frac{\pi}{2} - iy\right) = ke\ y, \tag{31}$$

$$he^{(4)}\left(\frac{\pi}{2} - iy\right) = ke\ y, \tag{32}$$

y being taken as real initially. We now drop this restriction and continue the functions outside their defining lines. As equation (1) is free of singularities for finite x, equations (29)-(32) hold then everywhere and the four functions $he^{(i)}x$ are related to each other by symmetry operations of equation (1).

The argument used to establish (6) can be repeated along any line parallel to the imaginary axis. As soon as we are sufficiently far away from the real axis, the general solution $me\ x$ will behave as

$$me\ x \sim \frac{\exp\ [\pm 2ik\ \cos x]}{[2k\ \cos x]^{1/2}}. \tag{33}$$

However, a particular asymptotic form

$$\frac{1}{[2k\ \cos x]^{1/2}}\,(A\ \exp\ [2ik\ \cos x] + B\ \exp\ [-2ik\ \cos x])$$

can define a given function only within a strip of limited width parallel to the imaginary axis. One way to see this is by observing that the above expression is formally periodic in x with period 4π, while by Floquet's theorem equation (1) has generally no such solution.

In order to obtain this range we start out by proving the integral relation

$$\int_{-i\infty}^{+i\infty} \exp\ [-2k \sinh y\ \cos u] me\ u\ du = A\ ke\ y \tag{34}$$

where $me\ u$ is an *arbitrary* solution of (1), $ke\ y$ the *special* solution of (4) defined by

(7), and A is a number whose value depends on the choice of *me u*. As integrals of the type (34) have been discussed in the literature[3] we need not dwell upon the formal steps necessary to prove (34); we have only to find out for what range of y this particular combination of limits and functions is chosen correctly. From the asymptotic formula (33), it follows that the integral exists as long as y is real and not negative. However, in the formal steps necessary to prove that the integral obeys equation (4), factors such as $\cosh y \cdot \cos u$ appear which demand that the exponential produce convergence; hence the integral (34) defines some solution of (4) only as long as $y > 0$. Our next observation is that the solution is always the same regardless of the choice of *me u*. This is so because, for symmetry reasons, we have that

$$\int_{-i\infty}^{+i\infty} \exp\,[-2k \sinh y \cos u]\, se\, u\, du = 0.$$

Thus, only one linearly independent solution of (1) is left in (34), producing always the same solution of (4) on the right hand side. That this function is just *ke y* is seen by evaluating (34) for large y by the saddle point method. The saddle point is at the origin, which permits us to write

$$\exp\,[-2k \sinh y \cos i\, v] \sim \exp\,[-2k \sinh y(1 + \tfrac{1}{2}v^2)].$$

Hence

$$\int_{-i\infty}^{+i\infty} \exp\,[-2k \sinh y \cos u]\, me\, u\, du$$

$$\sim i\; me(0) \exp\,[-2k \sinh y] \int_{-\infty}^{+\infty} \exp\,[-k\, v^2 \sinh y]\, dv$$

$$= i\pi^{1/2}\; me(0)\, \frac{\exp\,[-2k \sinh y]}{(k \sinh y)^{1/2}}.$$

This asymptotic behavior is the same as the one of the definition (7) and hence equation (34) is proved for positive y.

We now use equation (34) to continue the function *ke y* analytically outside this original range of definition. We start out by permitting values of y slightly off the positive real axis. As long as this deviation x is less than $\pi/2$ this goes without difficulty; for the convergence producing factor in (34) is changed from its previous value to

$$\exp\,[-2k \sinh y \cos x \cos u]$$

which will produce convergence as long as $\cos x > 0$ and $\sinh y > 0$. Using the definition (29) we arrive thus at the equation

$$\int_{-i\infty}^{+i\infty} \exp\,[-2ik \cos x \cos u]\, me\, u\, du = A\; he^{(1)}\, x \tag{35a}$$

which holds as long as

$$\mathscr{I}(x) > 0, \tag{35b}$$

$$0 < \mathscr{R}(x) < \pi. \tag{35c}$$

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[3] N. W. McLachlan. *Theory and Application of Mathieu Functions.* Clarendon Press, 1947, Chapter X.

This range of definition can be extended further if we allow the path of integration to be deformed. The imaginary axis along which the above integral is running is one of a series of valleys along which it could be taken to produce a convergent result. The other valleys are separated from this one by distances 2π, 4π, etc., as shown in Fig. 6. These valleys have a width π, as shown by (35c); they are separated by ridges of the same width for which the integral diverges. Now as $\Re(x)$ changes the valleys of exp $[-2ik \cos x \cos u] = \mathcal{K}(x, u)$ shift. Set $x \to x + iy$, $u \to u + iv$ then we get for the real part of the exponent in $\mathcal{K}(x, u)$

$$\exp\left[-2k\,(\cos x \cosh y \sin u \sinh v + \sin x \sinh y \cos u \cosh v)\right]$$

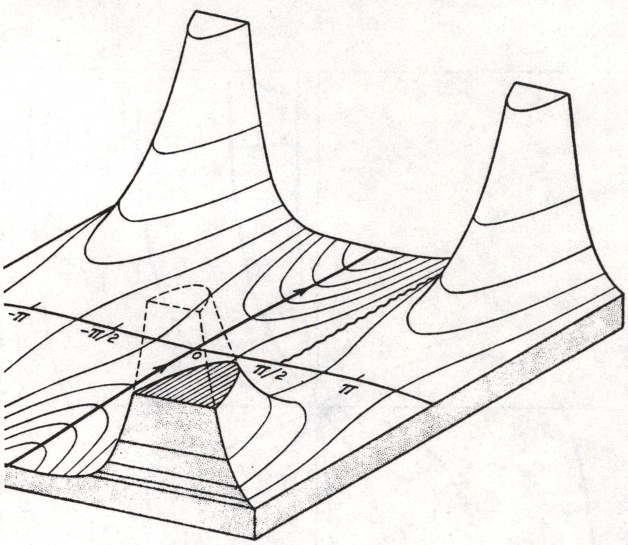

FIG. 6 PERSPECTIVE DRAWING IN THE U-PLANE SHOWING THE ABSOLUTE VALUE OF THE INTEGRAND AND THE PATH OF INTEGRATION FOR THE INTEGRAL (35). THE FACTOR ME U IS NOT INCLUDED AND X MUST LIE ON THE WAVY LINE NOT FAR FROM THE REAL AXIS

which, for v positive and large, becomes approximately

$$\exp\left[-ke^{v}\,(\cos x \cosh y \sin u + \sin x \sinh y \cos u)\right].$$

Clearly, if y is also large, the exponent contains $\sin(x + u)$ which is kept at its maximum value by setting

$$x + u = \frac{\pi}{2}. \tag{36a}$$

If y is not large then the bottom of the valley is given by

$$\tanh y \tan x \tan u = 1. \tag{36b}$$

The movement of x as a function of u does not differ essentially between (36a) and (36b). As $\Re(x)$ decreases from $\pi/2$, $\Re(u)$ increases by an amount which is essentially equal to this decrease. The movement becomes gradually more jerky as y becomes smaller and ceases to function for $y = 0$. Similarly for v negative and large we find for y large

$$x - u = \frac{\pi}{2}, \tag{37a}$$

and for general y

$$\tanh y \tan x \tan u = -1. \tag{37b}$$

The movement of the valleys is thus in the opposite directions on the two sides of the real axis. This means that for decreasing $\mathfrak{R}(x)$ the path of the integral (35) deforms as shown in Fig. 7.

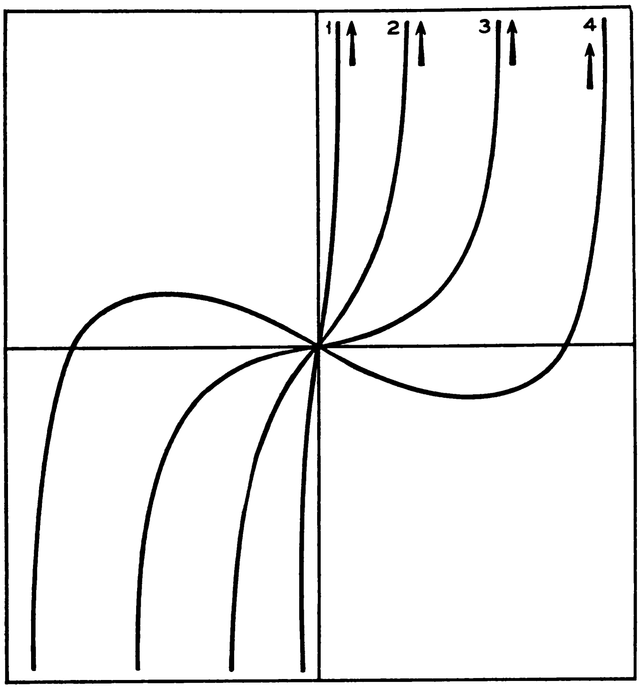

FIG. 7 DEFORMATION OF THE PATH IN THE U-PLANE FOR THE INTEGRAL (35). SUCCESSIVE NUMBERS ARE FOR R(X) DECREASING, STARTING FROM $\frac{\pi}{2}$.

We have thus found, for the entire half plane for which $\mathfrak{I}(x) > 0$, the solution of (1) which has, along the line parallel to the imaginary axis and passing through $\pi/2$, the behavior prescribed by (29) and (7); the solution is given to us in the form of an integral.

$$A\, he^{(1)}x = \int_{u_1 - i\infty}^{u_2 + i\infty} \exp\left[-2ik \cos x \cos u\right] me\, u\, du \tag{38a}$$

where

$$u_1 \text{ obeys (37)}, \tag{38b}$$

$$u_2 \text{ obeys (36)}, \tag{38c}$$

$$\mathcal{I}(x) > 0. \tag{38d}$$

Fig. 8 illustrates the path for the special case $\mathcal{R}(x) = 0$.

Formula (38) will now be used to find the range to which the asymptotic formula (7) is applicable. If we decrease $\mathcal{R}(x)$ from $\pi/2$ to some smaller value the two ends of the path displace as shown in Fig. 7; this displacement leaves undisturbed the location $u = 0$ of the saddle point while at the same rotating its orientation clockwise; this rotation

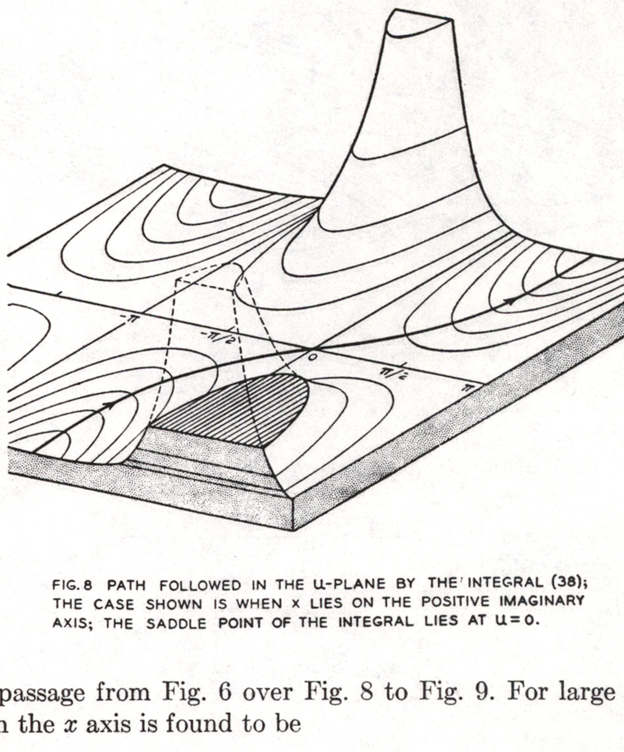

FIG. 8 PATH FOLLOWED IN THE U-PLANE BY THE INTEGRAL (38); THE CASE SHOWN IS WHEN X LIES ON THE POSITIVE IMAGINARY AXIS; THE SADDLE POINT OF THE INTEGRAL LIES AT U = 0.

shows up in the passage from Fig. 6 over Fig. 8 to Fig. 9. For large $\mathcal{I}(x)$, the angle α of the saddle with the x axis is found to be

$$\alpha = \frac{\pi}{4} + \frac{1}{2}\mathcal{R}(x) \tag{39}$$

The integral (38) yields then

$$he^{(1)}x \sim \frac{\exp\left[-2ik \cos x - i\pi/4\right]}{(2k \cos x)^{1/2}} \tag{40a}$$

with the restriction

$$\mathcal{I}(x) \gg 0 \tag{40b}$$

However, when $\mathcal{R}(x)$ drops below $-\pi/2$ the situation is altered. The two valleys are now so far removed that the path has to proceed over three saddles, lying at $-\pi$, 0, π. This situation is illustrated in Fig. 9. Of the three contributions, the one at the origin retains its analytical form (40a) but the two others will add to it and thus invalidate it. A similar modification must be applied when $\mathcal{R}(x)$ increases from $+\pi/2$ beyond $+ 3/2\pi$. The asymptotic expansion (40a) is therefore valid within the range

$$-\frac{1}{2}\pi \leqq \mathcal{R}(x) \leqq \frac{3}{2}\pi. \tag{40c}$$

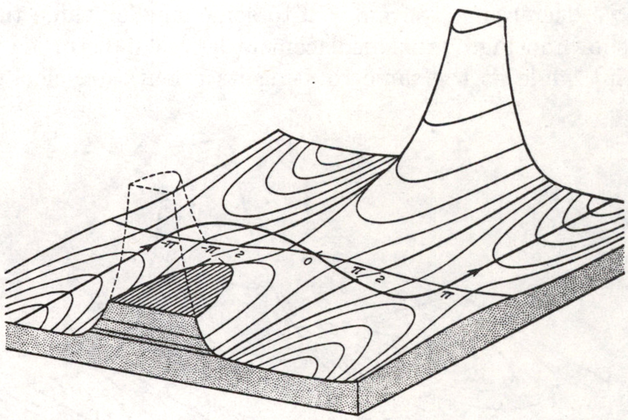

FIG. 9 PATH FOLLOWED IN THE U-PLANE BY THE INTEGRAL (38) AFTER R(x) HAS DROPPED SOMEWHAT BELOW $-\frac{\pi}{2}$. THE ORIGINAL SADDLE POINT REMAINS AT THE ORIGIN, ROTATING CLOCKWISE; IN ADDITION, TWO NEW SADDLE POINTS, AT $-\pi$ AND $+\pi$ MAKE THEIR APPEARANCE.

In the same way we obtain from equation (30)

$$he^{(2)}x \sim \frac{\exp\,[+2ik\cos x + i\pi/4]}{(2k\cos x)^{1/2}} \tag{41a}$$

if

$$-\frac{3}{2}\pi \leqq \mathcal{R}(x) \leqq +\frac{1}{2}\pi, \tag{41b}$$

$$\mathcal{I}(x) \gg 0, \tag{41c}$$

and from (31)

$$he^{(3)}x \sim \frac{\exp\,[-2ik\cos x - i\pi/4]}{(2k\cos x)^{1/2}} \tag{42a}$$

if

$$-\frac{3}{2}\pi \leqq \mathcal{R}(x) \leqq +\frac{1}{2}\pi, \tag{42b}$$

$$\mathcal{I}(x) \ll 0, \tag{42c}$$

and finally from (32)

$$he^{(4)}x \sim \frac{\exp\,[+2ik\cos x + i\pi/4]}{(2k\cos x)^{1/2}} \tag{43a}$$

if

$$-\frac{1}{2}\pi \leqq \mathcal{R}(x) \leqq +\frac{3}{2}\pi, \tag{43b}$$

$$\mathcal{I}(x) \ll 0. \tag{43c}$$

This defining range gives to (40) and (41) a common domain of existence and also to (42) and (43). In order to give a common range to three functions we use equation (9). We get from it and (29) that, for positive large y

$$he^{(1)}\left(\frac{\pi}{2} - iy\right) \sim \frac{\exp\,[2k\cosh y + \Phi]}{(2k\cosh y)^{1/2}}$$

while from (42) and (43)

$$he^{(3)}\left(\frac{\pi}{2} - iy\right) \sim \frac{\exp\,[2k\cosh y - i\pi/2]}{(2k\cosh y)^{1/2}},$$

$$he^{(4)}\left(\frac{\pi}{2} - iy\right) \sim \frac{\exp\,[-2k\cosh y]}{(2k\cosh y)^{1/2}}.$$

We know in addition that a universally valid linear relation must exist between these three functions. In the range considered, the first two are asymptotically large, while the third is asymptotically small. Hence, the relation must read

$$he^{(1)}x = i\,e^{\Phi}\,he^{(3)}x + (\text{unknown factor})\cdot he^{(4)}x. \tag{44}$$

Equation (44) is half a connection formula; we shall now see that the other half can be picked up from consideration of Floquet's theorem.

5. The Floquet Function. According to Floquet's theorem, there exists at least one solution of (1) which is multiplied with a constant factor when we apply the translational symmetry operation of equation (1)

$$x \rightarrow x + \pi.$$

This constant is usually written in the form $e^{i\pi\beta}$. It is known that $e^{i\pi\beta}$ oscillates back and forth from the unit circle to the real axis, having alternately positive and negative sign in the latter case. Whenever $e^{i\pi\beta}$ is real we shall use the supplementary definition

$$e^{\pi b} = \pm\, e^{i\pi\beta} \tag{45}$$

the sign being chosen so as to make the real quantity $e^{\pi b}$ positive. The lines along which $e^{i\pi\beta} = \pm 1$ have been studied intensively; a reproduction of the published results is shown in Fig. 10.[4]. Figure 11 shows a reproduction of a map of β published by McLachlan. This type of information is still rather incomplete at this time.

We now define as $je\ x$ or more specifically $je^+\ x$ the solution of equation (1) which obeys

$$je^+(x + \pi) = e^{i\pi\beta} je^+ x. \tag{46}$$

A second, generally independent, solution $je^-\ x$ is then obtained through

$$je^- x = je^+(-x) \tag{47}$$

which yields the identity

$$je^-(x + \pi) = e^{-i\pi\beta} je^- x. \tag{48}$$

The normalization constant will be left for later disposal. A very important property of $je\ x$ is that whenever $ce\ x$ equals $ce_n x$, je also equals $ce_n x$, and similarly for $se_n x$.

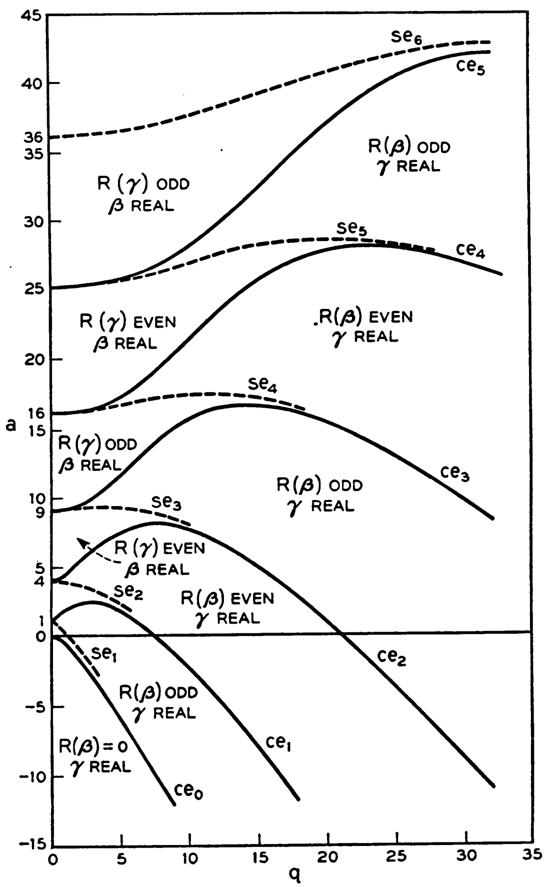

FIG. 10 "BUTTERFLY DIAGRAM" FOR THE MATHIEU FUNCTIONS IN THE a-q PLANE. THE LINES SHOW THE COMBINATION OF VALUES YIELDING PERIODIC SOLUTIONS. INFORMATION CONCERNING THE TWO PHASE ANGLES β AND γ IS ADDED.

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[4]See for instance S. Goldstein, Trans. Camb. Phil. Soc. **23**, 303 (1927); E. L. Ince. Proc. Roy. Soc. Edinburgh **46**, 20 (1925); **46**, 316 (1926); **47**, 294 (1927). Also the textbook of McLachlan. *loc. cit.*

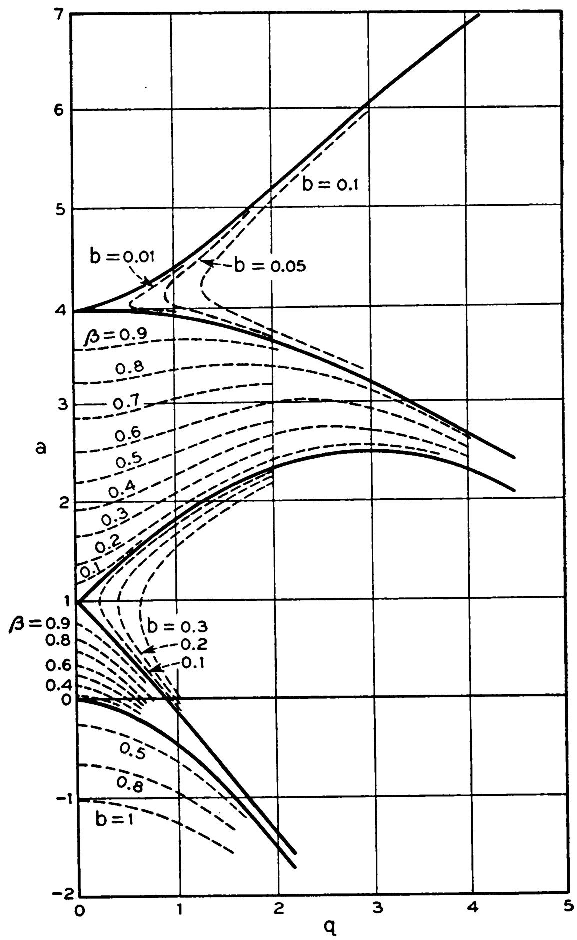

FIG. 11 DIAGRAM SHOWING CURVES OF CONSTANT β IN THE a-q PLANE (McLACHLAN). INFORMATION OF THIS TYPE IS MUCH MORE RESTRICTED THAN THE TYPE SHOWN IN FIG. 10

The asymptotic behavior of the Floquet function was first obtained by Dougall.[5] We shall derive it more quickly by our integral equation method developed in the last section. The pair of equations to be established is

$$B^{+}\, je^{+}\, x = \int_{u_2-2\pi+i\infty}^{u_2+i\infty} \exp\left[-2ik \cos x \cos u\right] je^{-}\, u\, du, \tag{49a}$$

$$B^{-}\, je^{-}\, x = \int_{u_2-2\pi+i\infty}^{u_2+i\infty} \exp\left[-2ik \cos x \cos u\right] je^{+}\, u\, du. \tag{49b}$$

u_2 is defined by (38c). The resultant path is shown in Fig. 12 for the case $\Re(x) = 0$.[6]

[5] J. Dougall, Proc. Edinburgh Math. Soc., **34**, 4 (1916); **41**, 26 (1923); **44**, 57 (1926).

[6] The path is identical with the one for the generalized Bessel integral for Bessel functions. The integral permits thus derivation of the Bessel expansion by substituting the Fourier expansion under the integral sign. These relationships suggested to the author the symbol *je* for that type of function.

It is obvious from the discussion preceding equation (38) that as long as $\mathcal{I}(x) > 0$ the two integrals (49) exist and define two solutions of Mathieu's equation. To complete the proof we have to show only that they also obey Floquet's theorem. This is seen as follows. Both terminals of the path lie on the positive imaginary side of the real axis; therefore if $\mathcal{R}(x)$ increases from an initial value, 0 say, the valleys in which the path terminates move in the sense contrary to x, in accordance with equation (36). When x has been increased by 2π the path has been shifted without distortion by an amount -2π. This shift leaves the kernel $\mathcal{K}(x, u)$ of the integral equation invariant, but multiplies the factor $je^- \, u$ in the integrand of (48a) with $e^{2\pi i\beta}$; hence the integral has been multiplied with this same factor and thus obeys Floquet's theorem with the factor $e^{\pi i\beta}$. The same procedure establishes (49b). B^+ and B^- are constants which will be discussed below.

The asymptotic expansion of $je^+ \, x$ and $je^- \, x$ is obtained from (48) by applying the saddle point method discussed in the previous section. The two saddle points lie at $-\pi$ and 0, as shown on Fig. 12. Each saddle point furnishes one of the exponentials

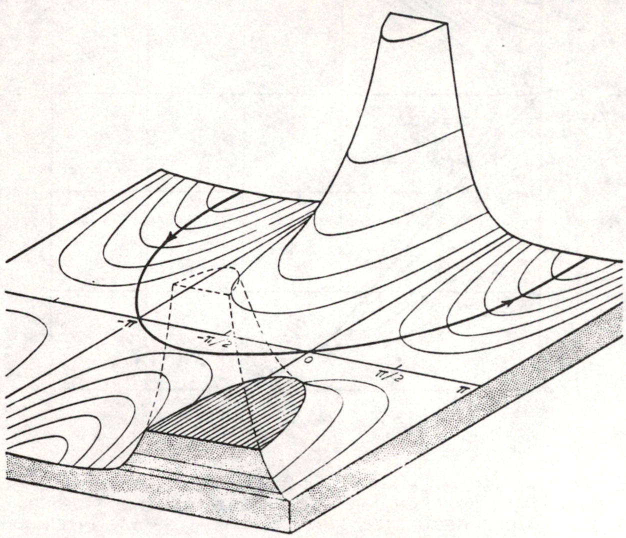

FIG. 12 PATH OF INTEGRATION FOR THE INTEGRALS (49) WHEN x LIES ON THE POSITIVE IMAGINARY AXIS

(33); they can be combined into a single term because we obtain the relative magnitude of the terms from (46) or (48). The resultant expressions are thus found to be

$$B^+ \, je^+ \, x \sim 2 \, je \, 0 \exp\left(\tfrac{1}{2}i\pi\beta\right)\left(\frac{\pi}{k\cos x}\right)^{1/2}\cos\left(2k\cos x - \frac{\pi}{4} + \frac{1}{2}\pi\beta\right), \tag{50a}$$

$$B^- \, je^- \, x \sim 2 \, je \, 0 \exp\left(-\tfrac{1}{2}i\pi\beta\right)\left(\frac{\pi}{k\cos x}\right)^{1/2}\cos\left(2k\cos x - \frac{\pi}{4} - \frac{1}{2}\pi\beta\right), \tag{50b}$$

with the restriction

$$\mathcal{I}(x) \gg 0. \tag{50c}$$

In the real direction the formulas (50) are also limited. The domain of validity is obtained by the methods used to establish (40c). We find

$$-\frac{\pi}{2} \leqq \Re(x) \leqq +\frac{\pi}{2}. \tag{50d}$$

We have, however, in this case the exceptionally favorable situation that the formulas (46) and (48) just supplement (50d) so as to furnish the asymptotic expansion of $je\ x$ for all values of $\Re(x)$. One simple way to express this is by saying that under the condition (50c) alone $je^+\ x$ is asymptotically equal to $J_{-\beta}(ke^{-ix})$, and $je^-\ x$ equals $J_{+\beta}(ke^{-ix})$.

The equations (50) would yield the connection formulas between the functions je and he were it not for the two undetermined constants B^+ and B^-. This indeterminacy can be partially removed by inspection, as follows.

(a) As the B's in (50) are factors in the asymptotic expansion of the *same* function in two different regions of the complex plane, their ratio can never be zero or infinite.

(b) When $je\ x \propto se_n x$ both B's are zero; this follows from symmetry considerations on (49). The formulas (50) do not lose their meaning however, because $je\ 0$ also vanishes. In these equations symmetry demands that the ratio B^+/B^- approach -1 as je approaches se_n.

(c) When $je\ x \propto ce_n x$, the B's in equation (50) cannot vanish, because $ce\ 0$ does not vanish. Symmetry or the equations (49) demands that the B's be equal. This means $B^+/B^- = +1$.

(d) Suppose now we are in a region where $e^{i\pi\beta}$ is real. We then enter into equation (1) with the substitution suggested by (45)

$$me\ x \qquad e^{bx}p(x).$$

The resultant equation in $p(x)$ is real; its periodic solution must be real because otherwise there would be two linearly independent periodic solutions. Hence $je\ x$ is a real function of x, and for real $je\ 0$, the two expressions (50) must be conjugate complex. This gives

$$\frac{B^+}{B^-} = e^{\pi(b+i\gamma)} \tag{51a}$$

where b is given by (45) and γ is some real number which is integer at the two limiting lines of the region and increases (or decreases) by 1 as we proceed from the line $ce_n\ x$ to $se_n\ x$.

(e) Now let $e^{i\pi\beta}$ be on the unit circle. We then enter into (3) with the substitution

$$me\ x = e^{\beta z}p(z).$$

The resultant equation in z is real. If there is to be only one periodic solution $p(z)$ it must be real along the z direction, and hence the asymptotic expansions (50) must be real. It follows that

$$\frac{B^+}{B^-} = \pm e^{\pi(i\beta+c)} \tag{51b}$$

where c is some real number which vanishes for integer β. The undetermined sign is fixed in each of the separate regions of real β being $+$ between ce_{2n} and se_{2n+1} and $-$ between ce_{2n+1} and se_{2n}.

(f) We can sum up this information by writing

$$\frac{B^+}{B^-} = e^{i\pi(\beta+\gamma)} \tag{52}$$

where β and γ are functions of a and q which are fixed up to an even integer and the sign. The value of β and γ at the boundary lines of Fig. 10 is shown in Table I. These lines divide up the

Table I Values of β and γ for the Mathieu functions of the first kind

Funtion	β	γ
ce_{2nx}	even	even
ce_{2n+1x}	odd	odd
se_{2nx}	even	odd
se_{2n+1}	odd	even

a-q-plane into "wings" and "gaps". Between these two there is a reciprocal behavior of β and γ. In the wings, β is real and changes by 1; γ does the same thing in the gaps. Inversely, the real part of β is fixed in the gaps, and it has an imaginary part ib which varies; this behavior is duplicated by γ in the wings, where it has a variable imaginary part ic introduced by (51b). This variation is exhibited in Fig. 10.

6. The Connection Formulas. We start out by writing a *formal* connection formula with the help of the parameters β and γ of the last section. We dispose of the normalization factor by setting in accordance with (52)

$$je\,0 = \frac{B^+ \exp\,[-\tfrac{1}{2}i\pi(\beta+\gamma)]}{2(2\pi)^{1/2}} = \frac{B^- \exp\,[\tfrac{1}{2}i\pi(\beta+\gamma)]}{2(2\pi)^{1/2}}. \tag{53}$$

This reduces (50) to

$$je^+x \sim \frac{\exp\,(-\tfrac{1}{2}i\pi\gamma)}{(2k\,\cos x)^{1/2}} \cos\left(2k\,\cos x - \frac{\pi}{4} + \frac{1}{2}\,\pi\beta\right), \tag{54a}$$

$$je^-x \sim \frac{\exp\,(+\tfrac{1}{2}i\pi\gamma)}{(2k\,\cos x)^{1/2}} \cos\left(2k\,\cos x - \frac{\pi}{4} - \frac{1}{2}\,\pi\beta\right). \tag{54b}$$

The domain of validity is given by (50); it is contained in the larger domains (40) and (41). We may therefore write down the connection formulas

$$je^+x = \tfrac{1}{2}i\,\exp\,(-\tfrac{1}{2}i\pi\gamma)[\exp\,(-\tfrac{1}{2}i\pi\beta)he^{(1)}x - \exp\,(+\tfrac{1}{2}i\pi\beta)he^{(2)}x], \tag{55}$$

$$je^-x = \tfrac{1}{2}i\,\exp\,(+\tfrac{1}{2}i\pi\gamma)[\exp\,(+\tfrac{1}{2}i\pi\beta)he^{(1)}x - \exp\,(-\tfrac{1}{2}i\pi\beta)he^{(2)}x]. \tag{56}$$

Further, reversing the sign of x with (29), (30), (31), (32) and (47)

$$je^+x = \tfrac{1}{2}i\,\exp\,(\tfrac{1}{2}i\pi\gamma)[\exp\,(\tfrac{1}{2}i\pi\beta)he^{(3)}x - \exp\,(-\tfrac{1}{2}i\pi\beta)he^{(4)}x], \tag{57}$$

$$je^-x = \tfrac{1}{2}i\,\exp\,(-\tfrac{1}{2}i\pi\gamma)[\exp\,(-\tfrac{1}{2}i\pi\beta)he^{(3)}x - \exp\,(\tfrac{1}{2}i\pi\beta)he^{(4)}x]. \tag{58}$$

Solving (55) and (56) for $he^{(1)}x$ we get

$$he^{(1)}x = \operatorname{cosec}\pi\beta[\exp\,\{\tfrac{1}{2}i\pi(\gamma-\beta)\}je^+x - \exp\,\{-\tfrac{1}{2}i\pi(\gamma-\beta)\}je^-x]. \tag{59}$$

Eliminating $je^+ x$ and $je^- x$ from (57), (58) and (59) we get a relation between $he^{(1)} x$, $he^{(3)} x$ and $he^{(4)} x$ which must be identical with the previously derived incomplete equation (44). Comparing coefficients, we find thus

$$i \frac{\sin \pi\gamma}{\sin \pi\beta} = e^{\Phi}, \tag{60}$$

and the completed connection formula (44)

$$he^{(1)}x = i\, e^{\Phi} he^{(3)}x - (i\, e^{\Phi} \cos \pi\beta + \cos \pi\gamma) he^{(4)}x. \tag{61}$$

With equation (60), the formal connection formulas (55)-(59) become actual ones, with coefficients expressible in terms of β and Φ, which, in turn, are known functions of a and q. By the use of the subsidiary definitions (45) and (51) we can show up (60) as an equation between real quantities. Inside the wings of Fig. 10 we get

$$\sinh \pi c = e^{\Phi} \mid \sin \pi\beta \mid, \tag{62a}$$

and in the gaps

$$\sinh \pi b = e^{-\Phi} \mid \sin \pi\gamma \mid. \tag{62b}$$

Actually (60) contains a little more than (62); in the wings for instance, it tells us for each one of the two β's on which side of the real axis the corresponding γ is to be found; a similar sign is determined in the gaps.

Equation (61) effectively terminates the search for continuation formulas because it tells us how to continue the simple exponential asymptotic behavior (40) on the other side of the real axis. A large number of other formulas are derivable from the ones obtained, such as

$$je^+x = \tfrac{1}{2} \exp\left(\tfrac{1}{2}i\pi\beta - \Phi\right)\left[\exp\left(\tfrac{1}{2}i\pi\gamma\right)he^{(1)}x + \exp\left(-\tfrac{1}{2}i\pi\gamma\right)he^{(4)}x\right], \tag{63}$$

$$he^{(2)}x = -i\, e^{\Phi} he^{(4)}x + (i\, e^{\Phi} \cos \pi\beta - \cos \pi\gamma) he^{(3)}x, \tag{64}$$

and so forth.

We now come to the connection formulas for the Lindemann-Stieltjes functions. This task has two stages of difficulty. By symmetry alone we can write down relations such as

$$ce\, x \propto je^+x + je^-x,$$

$$ce\, x \propto he^{(1)}x + he^{(3)}x,$$

$$se\, x \propto je^+x - je^-x,$$

$$de\, x \propto \exp\left(-\tfrac{1}{2}i\pi\beta\right)je^+x + \exp\left(+\tfrac{1}{2}i\pi\beta\right)je^{-x}$$

$$de\, x \propto he^{(1)}x + he^{(4)}x,$$

$$te\, x \propto he^{(1)}x - he^{(4)}x.$$

In order to make connection formulas out of these proportions it is necessary to have quantitative information for the points $x = 0$ and $x = \pi/2$. This paper contains no such information for the point $x = 0$. For the point $x = \pi/2$, on the other hand, the necessary results were obtained incidentally in section 2. We will pursue this only to the zero stage of the Jeffreys approximation in which we get from (13)

$$ke\,0 \approx \frac{\exp\left(\frac{1}{2}\Phi_0\right)}{(a + 2k^2)^{1/4}}, \tag{65}$$

$$ke'0 \approx -\exp\left(\tfrac{1}{2}\Phi_0\right)(a + 2k^2)^{1/4}. \tag{66}$$

Hence we may write more precisely

$$de\,x = \frac{1}{2\,ke\,0}\,(he^{(1)}x + he^{(4)}x), \tag{67}$$

$$te\,x = \frac{i}{2\mid ke'0 \mid}\,(he^{(1)}x - he^{(4)}x). \tag{68}$$

The connection formulas which exhibit the asymptotic properties of the Lindemann-Stieltjes functions follow from the ones above by application of (61). We find

$$ce\,x \propto (1 + i\,e^{\Phi})he^{(1)}x - (i\,e^{\Phi}\,\cos\pi\beta + \cos\pi\gamma)he^{(2)}x, \tag{69}$$

$$se\,x \propto (1 - i\,e^{\Phi})he^{(1)}x + (\cos\pi\gamma + i\,e^{\Phi}\,\cos\pi\beta)he^{(2)}x. \tag{70}$$

From (40) and (41) it is evident that these two functions always are of the form

$$\frac{1}{(2k\,\cos x)^{1/2}}\cos\left(2k\,\cos x - \frac{\pi}{4} + \psi\right) \tag{71a}$$

with

$$-e^{2i\psi} = \frac{\text{second coefficient}}{\text{first coefficient}}. \tag{71b}$$

Perusal of (60) shows that the ψ so defined is always real (not alternating as in (54)). Simple phase shifts of 0 or $\pi/2$ result from (71) when

$$\cos\pi\beta = \pm\cos\pi\gamma = \pm 1$$

in the combination circumstances warranted by Table I; the shifts are then identical with the ones in (54). The remaining relation of the pair (69) and (70) gives us then the asymptotic expansion of the Mathieu function of the second kind whose phase shift comes out to be given by

$$\tan\psi = (-)^{\gamma-1}e^{(-)^{\beta-1}\Phi}.$$

The same formula (61) yields for (67) and (68)

$$de\,x = \frac{1}{2\,ke\,0}\,[-i\,e^{\Phi}he^{(2)}x + (1 - \cos\pi\gamma + i\,e^{\Phi}\,\cos\pi\beta)he^{(1)}x], \tag{73}$$

$$te\,x = \frac{1}{2\mid ke'0 \mid}\,[i\,e^{\Phi}he^{(2)}x + (1 + \cos\pi\gamma - i\,e^{\Phi}\,\cos\pi\beta)he^{(1)}x]. \tag{74}$$

These equations yield a real phase shift only in the special circumstances warranted by Table I and equation (54).

The special information (65) and (66) regarding the point $x = \pi/2$ also produces new information about the Floquet function at this point. Using (63), we get

$$je\,\frac{\pi}{2} = \exp\left(\frac{1}{2}\,i\pi\beta - \Phi\right)\cdot ke\,0\cdot\cos\frac{1}{2}\,\pi\gamma, \tag{75a}$$

$$je'\,\frac{\pi}{2} = -\exp\left(\frac{1}{2}\,i\pi\beta - \Phi\right)\cdot|\,ke'0\,|\cdot\sin\frac{1}{2}\,\pi\gamma. \tag{75b}$$

From (75), the connection formula between *je*, *de* and *te* is readily derived.

7. Concluding Remarks. This study is based on the notion that the Floquet parameter β is a known function of a and q. That this is partly a convenient fiction is seen from Fig. 11. It is a surprise that the results of this paper do furnish some new information concerning β. When we traverse the gap between wings from one bounding curve to the other, the exponential damping constant b is related to γ by (62b). On such a path γ changes from one integer to the next and $|\sin\gamma|$ passes therefore through its maximum once. We get therefore the relation for such a path

$$\text{Max }[e^{\Phi}\sinh\pi b] = 1. \tag{76}$$

This relation was checked from the Figs. 1, 2 and 11 for a stretch where both quantities are known. The result is Fig. 13 which confirms the prediction (76). In general, $e\Phi$ varies sufficiently slowly so that (76) can be used to determine a rough upper limit for b.

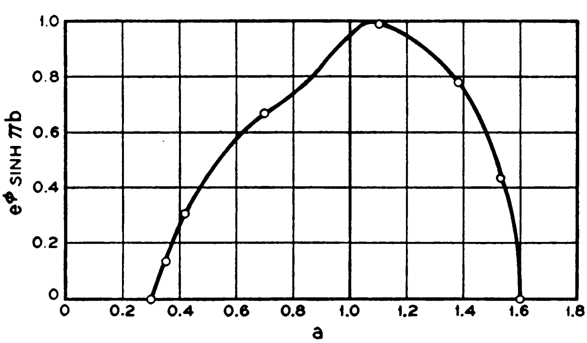

FIG. 13 PLOT OF e^{Φ} SINH πb AGAINST a FOR $q = 0.65$. ACCORDING TO EQUATION (76) THIS PRODUCT IS BOUNDED BY THE VALUE 1 WHICH IS ACTUALLY ATTAINED AT SOME POINT. THE CRUDENESS OF THE GRAPH REFLECTS THE LIMITED INFORMATION OF FIG. 11.

Notwithstanding this small bit of supplementary information concerning β, there remains the fact that the two natural parameters for the connection formulas are β and γ, and that their dependence on the equation parameters a and q is erratic and not expressible in closed form. It is not likely at this stage that an analytic relation will ever be found connecting β and γ to a and q. For this reason, this paper proceeds instead to find slowly varying and easily determined functions of a and q, from which β and γ can be determined by analytic means. One such parameter is Φ, whose determination was carried out in section 2, and for which the analytic relation to β and γ is equation (60). It is obvious that the accomplishment of the program calls for another such parameter, to replace β. This parameter has not yet been found. It is interesting to note as a possibility that $\beta + \gamma$ and $\beta - \gamma$ are simpler in their behavior than either one of them alone. In the meantime, the connection formulas of this paper must be used in conjunction with whatever published information is available concerning β.

8. Appendix on Integral Equations. The following integral equations are new, to my knowledge, but proved to be of no use in deriving connection formulas. They may, however, be useful in the hands of others.

Let z, ζ, w be complex numbers whose real parts are x, ξ, u and whose imaginary parts are y, η, v. Then the formulas are

$$B^{-}je^{-}z\, je^{+}\zeta = je\ 0 \int \exp\left[-2ik \cos z \cos \zeta \cos w - 2k \sin z \sin \zeta \sin w\right] je^{+} w\, dw, \tag{77a}$$

$$B^{+}je^{+}z\, je^{-}\zeta = je\ 0 \int \exp\left[-2ik \cos z \cos \zeta \cos w - 2k \sin z \sin \zeta \sin w\right] je^{-} w\, dw. \tag{77b}$$

The formulas are generalizations of (49) to which they reduce for the case $\zeta = 0$. The path and the range are best discussed in two stages. If

$$x - \xi = 0, \tag{77c}$$

then we need

$$y - \eta > 0, \tag{77c$'$}$$

and the path is exactly the one shown in Fig. 12. If

$$x - \xi \text{ arbitrary}, \tag{77d}$$

then we need

$$y - \eta > 0, \qquad \cosh^2(y - \eta) > \frac{4}{3}, \tag{77d$'$}$$

and the abscissa u of the terminal valleys is given by the generalization of (36b)

$$\tan (x - \xi) \tanh (y - \eta) \tan u = 1. \tag{77d$''$}$$

We set out immediately to prove d and will get c as a special case. We follow earlier proofs quite closely. The formal equivalence of the integral to a product of solutions of the Mathieu equation is found in the literature.[3] Although the integral exists when the exponents cancel, we need a negative real part in the exponent in order to implement the formal steps. This exponent reads

$$\exp = -2ik \cos z \cos \zeta \cos w - 2k \sin z \sin \zeta \sin w \pm 2ik \cos w.$$

The last term arises from the contribution (33) of the Floquet function; the difficult sign is the positive one. By an obvious transformation this becomes

$$\exp = -ike^{-iw} \cos (z - \zeta) - ike^{+iw} \cos (z + \zeta) + 2ik \cos w.$$

The second term is of no importance because v is to be positive and large. Introducing real and imaginary parts this becomes

$$\exp = -ike^{v}e^{-iu}[\cos (x - \xi) \cosh (y - \eta) - i \sin (x - \xi) \sinh (y - \eta)] + ike^{v}e^{-iu},$$

and the exponent's real part is

$$\mathcal{R}(\exp) = -ke^v[\cos(x-\xi)\cosh(y-\eta)\sin u + \sin(x-\xi)\sinh(y-\eta)\cos u - \sin u]. \tag{77d''}$$

We now introduce the choice of u indicated by (77d″). This means

$$\sin u = \frac{\cos(x-\xi)\cosh(y-\eta)}{(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2}},$$

$$\cos u = \frac{\sin(x-\xi)\sinh(y-\eta)}{(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2}}.$$

We thus get

$$\mathcal{R}(\exp) = -ke^v\left[(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2} - \frac{\cosh(y-\eta)\cos(x-\xi)}{(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2}}\right]$$

$$= \frac{-ke^v}{(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2}}[\cosh^2(y-\eta) + \cos^2(x-\xi) - 1 - \cosh(y-\eta)\cos(x-\xi)].$$

As stated in (77c), the curly bracket is positive when $\cos(x-\xi) = 1$ (and also for $\cos(x-\xi) = 0$); in the general case we form the perfect square

$$[\cos(x-\xi) - \tfrac{1}{2}\cosh(y-\eta)]^2$$

and then pull through on the remainder with (77d′). My surmise is that a more thought-out estimate could prove (77c′) all the time. Having proved the character of the function under these restrictions we can determine the particular nature of the left hand side of (77) by first making $\zeta = 0$ and z large and positive, to get the function of z; and then $z = 0$ and ζ large and negative to get the function of ζ. The formulas (77) are thus established. If we reverse (77c′) and set instead $y - \eta < 0$ the roles of y and η are reversed because the integral is formally symmetric in z and ζ.

What makes the integral (77) interesting is that one can pass with it from positive to negative imaginary values provided (77c′) is maintained. The difficulty in getting asymptotic expressions is in the location of the saddle points; one saddle point of (77a), for instance, lies at $w = \zeta$; this leads to a trivial cancellation and a confirmation of formula (50b). The fact remains, nevertheless, that formula (77) permits us to cross the real axis; this the simpler equations (38) and (49) do not permit us to do.