QUARTERLY

OF

APPLIED MATHEMATICS

VOLUME X JANUARY • 1953 NUMBER 4

QUARTERLY
OF
APPLIED MATHEMATICS

This periodical is published quarterly by Brown University, Providence 12, R. I. For its support, an operational fund is being set up to which industrial organizations may contribute. To date, contributions of the following industrial companies are gratefully acknowledged:

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The subscription price for the QUARTERLY is $6.00 per volume (April-January), single copies $2.00. Subscriptions and orders for single copies may be addressed to: Quarterly of Applied Mathematics, Brown University, Providence 12, R. I., or to Box 2-W, Richmond, Va.

Entered as second class matter March 14, 1944, at the post office at Providence, Rhode Island, under the act of March 3, 1879. Additional entry at Richmond, Virginia.

WILLIAM BYRD PRESS, INC., RICHMOND, VIRGINIA

QUARTERLY

OF

APPLIED MATHEMATICS

VOLUME X 1952

Printed by the
William Byrd Press, Inc.
Richmond, Virginia

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Scarborough, 2nd edition, pp. 99–103, the mistake arising from identification of two ξ-values in the interval which may be distinct.)

Consider the given data $f(j) = j$ for $j = 0, 1, \cdots, n$, for which the polynomial approximation is $p(x) = x$. Can the error be bounded by any function of $x_0, \cdots, x_n$, $y_0 = f(x_0), \cdots, y_n = f(x_n)$? That it cannot is clear from the function $f(x) = x + k \sin \pi x$. We have $f(\frac{1}{2}) = \frac{1}{2} + k$, $p(\frac{1}{2}) = \frac{1}{2}$, and $f(\frac{1}{2}) - p(\frac{1}{2}) = k$. Since k is independent of x_i and $f(x_i)$, this error value cannot be bounded by any function of these without strong hypotheses on the function f

A NOTE ON MY PAPER

ON AXIALLY SYMMETRIC FLOW AND THE METHOD OF GENERALIZED ELECTROSTATICS*

Quarterly of Applied Mathematics, **10**, 197-213 (1952)

By L. E. PAYNE (*University of Maryland*)

It has been brought to the author's attention that the flow problem for a spindle was considered by E. W. Hobson ("On a class of spherical harmonics of complex degree with application to physical problems", Trans. Camb. Phil. Soc., **14**, 211-236 (1889)). Hobson used a method which is entirely different from that employed by the author, but unfortunately his solution is in error. The solution to the problem is given in a corrected and simplified form in this paper. The only reference to Hobson's solution which the author has found is given in the appendix of A. B. Basset's "Treatise on hydrodynamics", vol. 2 (1888). Basset, however, did not discuss the problem and consequently did not recognize the errors in Hobson's solution.

*Received July 21, 1952.

BOOK REVIEWS

Finite deformation of an elastic solid. By Francis D. Murnaghan. John Wiley & Sons, Inc., New York, 1951. viii + 140 pp. $4.00.

The book contains an unusually lucid exposition of the theory of finite elastic deformations, a field in which the basic physical ideas are often lost in the tangle of complex mathematical notations. The elegance of the present treatment is achieved by the consistent use of matrices. The first chapter serves as an introduction to vectors and matrices. The second chapter is concerned with the strain matrix, its behavior under transformations of the initial and final reference frames, its invariants, and the compatibility relations. The stress matrix and the general relations between stress and strain are discussed in Chapter 4, and for non-isotropic materials in Chapter 5. Chapters 6 and 7 are devoted to the application of the theory to specific problems such as simple shear, simple tension, or torsion of a circular cylinder.

Bringing clarity into a field where confusion is the rule, the volume will doubtless be welcomed by all students of mechanics of continua. In two respects, however, the book disappointed this reviewer. First, there is no reference whatsoever to the considerable volume of classical and recent work in this field. While one can readily understand the author's refusal to work his way through the maze of confusing notations and often conflicting results of these papers, it cannot be denied that a critical digest of at least the more important of these papers would have greatly increased the value of the present book. Secondly, while the basic assumptions of the theory are clearly stated, the author rarely discusses their physical justification.

W. PRAGER

Jacobian elliptic function tables. By L. M. Milne-Thomson. Dover Publications, Inc., 1950. xi + 132 pp. $2.45.

The book opens with 38 pages of definitions, identities, and generally useful information concerning the elliptic functions $\mathrm{sn}(u, m)$, $\mathrm{cn}(u, m)$, $\mathrm{dn}(u, m)$, $z(u)$. The first three of these are tabulated to five places for values of u running from 0 to 3 in increments of .01, and for values of m from 0 to 1 in increments of .1. The function $z(u)$ is tabulated to seven places for u ranging from 0 to 3 in increments of .01. The complete elliptic functions are also tabulated and $\mathrm{cn}(u, 1)$, $\mathrm{dn}(u, 1)$ are given for $3 \leq u \leq 4$.

G. F. CARRIER

Tables of the error function and of its first twenty derivatives. By The Staff of the Computation Laboratory. Harvard University Press, Cambridge, Mass., 1952. xxvii + 276 pp. $8.00.

The function $(2\pi)^{-1/2} \int_0^x \exp(-u^2/2)\, du$ and its first twenty-one derivatives are tabulated. The argument increments are .004 for the function and its first nine derivatives, and are .002 for the remaining derivatives. The argument range is from zero in all cases to 6.468, 8.236, 9.610, and 10.902. These ranges refer respectively to: the error function and its first five derivatives, the next five derivatives, the twelfth to sixteenth derivatives, and the rest of the twenty-one derivatives.

G. F. CARRIER

Tables relating to Mathieu functions. Prepared by The Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards. Columbia University Press, New York, 1951. xlvii + 278 pp. $8.00.

This book is concerned with the solutions of the Mathieu equation, $u_{xx} + (b - s\cos^2 x)u = 0$, which have the period 2π, $= 0$. s is treated as a given parameter and b plays the role of the eigenvalue. The first thirty eigenvalues (15 corresponding to even solutions, 15 to odd) are tabulated for s ranging from 0 to 100. Since these functions have the representation $u = \Sigma D_n \cos nx$ (or $\sin nx$), the values of D_n are tabulated for various n, s, $b_r(s)$. Finally, the joining factors are tabulated over the appropriate range of s. All this is prefaced by that general information which renders the tables easy to use.

G. F. CARRIER

Activity analysis of production and allocation. Proceedings of a Conference. Edited by Tjalling C. Koopmans. John Wiley & Sons, Inc., New York, and Chapman and Hall, Ltd., London, 1951. xiv + 404 pp. $4.50.

As proceedings of a conference, this book is almost necessarily a compilation of articles and papers of uneven quality and a superficially great diversity of topics. Nevertheless, in this reviewer's opinion it portends a coming of age of economics mathematically, both in the sense of an *applied* mathematical economics, and in the sense of a discipline whose techniques may be employed profitably in some older "well-mathematized" areas. The crux of this is the brilliant theory of interacting productive aggregates of T. C. Koopmans (Chapter III) which furnishes a background for economic studies comparable in scope and utility to the linear steady state theory of electrical circuits in its realm.

Although the developments of this volume have profound implications for centralized planning, allocation, decentralization of authority, and orientation of research (profit-wise) in industry, they were largely stimulated by practical problems of military logistics. Thus the Army Air Forces group under M. K. Wood and G. B. Dantzig developed methods for (a) programming interdependent activities, e. g., the Berlin Airlift, (b) analysis of inter-industry commodity flows (generalizing and employing W. W. Leontief), (c) effective computation.

Mathematically, the central problem—the "linear programming" problem—is maximization of a linear functional (sometimes a vector) of non-negative variables subject to linear inequalities, e. g. the "transportation problem", to ship specified quantities of a product to n destinations from m origins under specified availability restrictions and travel costs so as to have least total shipment cost (resp. to find the "efficient" combinations of activities in production). Requisite theory of convex cones is developed *ab ovo* by Gerstenhaber, Gale and Arrow with interesting extensions by Gale, Kuhn and Tucker in the direction of equivalent maximal and minimal principles. There is close connection with 2-person game theory—every game can be written as a linear programming problem, and vice-versa usually, so that results and techniques in either bear on the other.

Computationwise (and for some theoretical questions) in linear programming, the finite step "simplex" method of Dantzig, which explicitly informs of arrival at a maximal solution, is one of the most effective tools yet developed—far outclassing, say, the Dines elimination technique. Heuristically employed up to the present, since Dantzig's proofs apply only to extremely special situations, recent results of the reviewer ("Mathematical Background of Linear Programming", Carnegie Tech-Air Forces Project in Intra-Firm Analysis, July 1951) provide for its applicability to every situation.

To supplement the misleadingly meager specific applications of linear programming detailed, current ones include: inversion of matrices, solution of simultaneous linear equations, optimal award of contracts to bidders, blending of aviation gasolines, and even plastic limit design of structures.

A. Charnes

Thermodynamics of irreversible processes. By S. R. DeGroot. North-Holland Publishing Company, Amsterdam, and Interscience Publishers, Inc., New York, 1951. xvi + 242 pp. $4.00.

The subject of irreversible processes has developed in relatively complete form within the last decade. This book shows how the ideas of entropy flow and entropy production together with the Onsager reciprocal relations lead to a theory of the thermodynamics of irreversible processes. The first two chapters of the book give an account of the theory, especially the Onsager relations; the following seven chapters include many examples from physics and chemistry and, among other topics, discussion of the application of the theory to heat conduction, electrical conduction (with and without a magnetic field), relaxation phenomena in continuous single component systems, discontinuous systems (with and without chemical reactions), ordinary diffusion, thermal diffusion, viscosity, diffusion potentials, thermoelectricity, reaction rates, electrochemistry, electrokinetic effects, and interference of a chemical reaction and a relaxation phenomenon. The last two chapters of the book are concerned with special topics such as the definition and discussion of stationary states of various order and further discussion of the foundations of the Onsager relations and the general theory.

The author has marked sections of the book so that it can be read in three cycles, the first for a general idea of the thermodynamics of irreversible processes, the second includes the examples but omits the statistical basis of the theory, and the third includes the entire monograph. It seems clear that the book will be tremendously useful to most physical chemists and to many physicists. The book assumes a background in thermodynamics and physical statistics although much of it can be read without the latter.

Rohn Truell

Die Welt der Vektoren. Einführung in Theorie und Anwendung der Vektoren, Tensoren und Operatoren. By Franz Ollendorff. Springer-Verlag, Wien, 1950. viii + 470 pp. $9.50.

Dr. F. Ollendorff, other books by whom the reader has certainly used and enjoyed, is at present Professor at the Hebrew Technical College, Haifa (Israel). The present work is derived from a course of vector analysis, preparatory to modern physics, given to the students in their senior year. Its purpose is thus frankly utilitarian, but the book is penetrated through and through with the author's admiration for the beauty of its subject. This combination is so effective that the reader is often ashamed of voicing his objections to himself, as if he were interrupting a beautiful lecture.

The book contains eight chapters. Chapters 1, *Vectors and Scalars*, and 2, *Vector Fields*, contains the standard material and its application to geometry, mechanics, fluid mechanics and Maxwell electrodynamics, together with a few less common examples.

Chapters 3, *Vector Analysis in Affine Space*, 4, *Tensor Algebra*, and 5, *Tensor Analysis in Affine Space*, present the fundamental tools of tensor analysis applied to an affine space of z dimensions, with metric. (The possible absence of a metric is not discussed). There is an interesting treatment of pseudo-scalars and of axial vectors of both kinds, inspired by Brillouin but really distinct, and connected with the different definitions of the positive side of a surface element in Stokes' and Gauss' theorems. Chapter 6, *Minkowski Space*, applies this material to special relativity, as well as to de Broglie and Schrödinger waves.

Chapter 7, *Riemannian Space*, gives the essentials of parallel displacement, geodesics and covariant differentiation. This is then used in a short but beautifully clear presentation of the dynamics of general relativity.

Chapter 8, *Hilbert Space*, generalizes from real to complex scalars and from a finite to an infinite number of dimensions. The mathematical reader, who will have had misgivings during the treatment of tensor analysis, will here become quite restive. The author however keeps his goal firmly in mind, and rapidly reaches the pass from which he can lead his audience into the green pastures of linear integral equations, matrix mechanics, Hamilton and Pauli operators.

A feature of the book is its richness in physical examples, to which it is impossible to do justice in a review. As the author very well says in his Preface, a work on vectors cannot be at the same time a textbook on Physics. Still every writer on Vector Analysis is anxious to give examples, and some have given such a condensed treatment of several extended physical fields that they end by serving neither the beginner nor the advanced reader. It seems to us that Prof. Ollendorff has struck on this point a very happy balance. The treatment of the physical examples (with the exception perhaps of some at the very end) is sufficiently extended to give the reader, due in part no doubt to the author's skill, no feeling of undue condensation or hurry. Indeed any well-prepared student should find this book a most readable text, and it will introduce him to some of the most beautiful questions of mathematical physics beside those from his own special field. The Haifa Technical College should assuredly be proud of such a course, of the lecturer and of his book.

P. Le Corbeiller

Theory of perfectly plastic solids. By William Prager and Philip G. Hodge, Jr. John Wiley & Sons, Inc., New York and Chapman & Hall, Limited, London, 1951. x + 264 pp. $5.50.

The student and research worker of today is in a fortunate position, as compared with his predecessor a generation ago, when he wants to penetrate a new field of science without turning to the scattered original literature. Instead of comprehensive text books, covering whole subjects such as Hydrodynamics or the Theory of Elasticity, he may choose among a great variety of more specialized monographs which make it possible for him to avoid devouring a lot of unnecessary material. The accompanying disadvantages of this process of development are obvious, but are probably overestimated by the older generation of scientists. The development has been actively promoted by American publishers.

The book under review forms an outstanding example of this type of modern monographs. It pretends to be a first introductory treatment, written on an intermediate level, and should be easily understood by the better students, having taken the basic courses in mathematics and mechanics of most engineering schools. With this principle in view, it might be said that the complementary mathematical treatment, now given in appendices, as well could have been incorporated in the main text, at least from the point of view of a European reader. To facilitate the penetration of the concentrated subject, ample references to elementary textbooks, even with indication of page numbers, are given.

Faithful to the scope covered by its title, the book gives a rather complete treatment of the theory of perfectly plastic solids according to v. Mises or Prandtl-Reuss. Interesting applications to engineering problems are given. The chapters are: 1. Basic concepts, 2. Trusses and beams, 3. Torsion of cylindrical or prismatic bars, 4. Plane strain: problems with axial symmetry, 5. Plane strain: general theory, 6. Plane strain: specific problems, 7. Plane strain: contained plastic deformation. Limit analysis, 8. Extremum principles. Cartesian tensors with subscript notation and summation conventions do appear only in chapter 8. Each chapter contains exercises and a list of references. The text and figures are distinguished by their clarity and stringency.

Those who look for physical aspects of the theory of plasticity, such as details as to recent development of the flow theory as compared with the so-called deformation theory of plastic solids, are referred to other publications.

FOLKE K. G. ODQVIST

Operational calculus based on the two-sided Laplace integral. By Balth. Van Der Pol and H. Bremmer. University Press, Cambridge, 1950. xiii + 415 pp. $10.00.

The material in this book evolved from a series of lectures by the authors in the period from 1938 to 1940. Written up and extended during the German occupation of the Netherlands, the text was later translated from Dutch to English.

The book is concerned with the application of operational calculus to mathematics, physics and engineering, and the stated aim is a modern treatment of the subject.

The chapter headings are as follows: I. General Introduction, II. The Fourier Integral as Basis of the Operational Calculus, III. Elementary Operational Images (transforms), IV. Elementary Rules, V. The Delta of Impulse Function, VI. Questions Concerning the Convergence of the Definition Integral, VII. Asymptotic Relations and Operational Transposition of Series, VIII. Linear Differential Equations with Constant Coefficients, IX. Simultaneous Linear Differential Equations with Constant Coefficients; Electric Circuit Theory, X. Linear Differential Equations with Variable Coefficients, XI. Operational Rules of More Complicated Character, XII. Step Functions and Other Discontinuous Functions, XIII. Difference Equations, XIV. Integral Equations, XV. Partial Differential Equations in the Operational Calculus of One Variable, XVI. Simultaneous Operational Calculus, XVII. Grammar (Operational Rules), XVIII. Dictionary (of transforms).

There are a number of excellent features of this book. One chapter, and other sections of the book, on the treatment of the delta function seem particularly good; they deal at some length with various features of the delta function, including applications in connection with differential and integral equations. The impulse function is formulated in terms of the Stieltjes integral; the transform of the delta function is discussed; functions approximating the delta function are considered. The use of the impulse function in obtaining the Green function of inhomogeneous differential equations is discussed. The sections treating integral equations, partial differential equations, and the transformation of functions of more than one variable by means of multiple Laplace integrals are also interesting, for one reason, because of the use of the delta function techniques in connection with these topics.

The treatment of the operational calculus given in this book is based on the two-sided Laplace transform, i.e., with limits $-\infty$ and ∞ instead of the usual limits 0 and ∞. This treatment demands that the separate one-sided integrals must have a common region of convergence otherwise the two-sided Laplace transform cannot exist. The overlapping of the convergence regions or the strip of convergence must be ascertained and may require somewhat more care and attention than the usual one-sided integrals. Explicit formulae for the convergence strip for any given original function $h(t)$ are obtained and discussed in the chapter on convergence of the definition integral.

The book seems to be very carefully written and it should be of real value to the student of operational methods.

ROHN TRUELL