

MINIMUM WEIGHT DESIGN AND THE THEORY OF PLASTIC COLLAPSE*

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Summary. This paper examines the problem of assigning economical sections to the members of a structure whose geometrical form is given. The criterion of failure is taken to be that of the plastic theory of collapse, and the criterion of minimum weight is employed to determine the best design. A geometrical analogue of the equations involved is used to clarify their significance, and such proofs as there are in the text, are cast into geometrical terms. A method of solution is suggested at the end of the paper, but the primary concerns of the paper are the general features of the problem.

1. Introduction. The problem which this paper considers may be stated as follows: given the geometrical form of a plane frame, together with a set of static loads which bear upon the frame, what cross sectional dimensions must the members be given in order that the loads may be just sustained and the weight of the structure as a whole be as small as possible? The members are assumed to have constant cross sections throughout their length.

In order to decide whether or not a frame will just sustain the loads, a criterion of strength must be assumed, and the criterion employed here is that of the theory of plastic collapse [1].** This theory is applicable to framed structures of ductile material which derive their strength from a bending action, e. g. rectangular building frames, pitched roof portals etc. It assumes that the bending moment in a member tends to a maximum limiting value as the curvature becomes indefinitely large. This limiting value of the bending moment is called the fully plastic moment, and if this moment is attained at a particular cross section a plastic hinge is said to have formed there. A plastic hinge allows a finite change of slope to occur at the cross section where it forms, and the fully plastic moment resists any further rotation of the hinge.

If the collapse of a structure is now defined as the condition when deformations increase indefinitely under constant loads, then, assuming that instability effects can be ignored, it can be shown that collapse occurs when a sufficient number of plastic hinges have formed to transform structure, or any part of it, into a mechanism. A load factor λ is usually employed to determine the correct collapse mechanism. Suppose, for instance, that a frame, whose members i have sections such that their fully plastic moments are β_i , is subjected to the loads W_k acting at the points k . (The notation used omits a summation symbol; all small suffixes should be read as repeating ones, capital suffixes refer to particular variables or equations). Then, if an arbitrary mechanism Y is assumed, an equation of virtual work may be written down

$$\lambda_Y W_k \delta_{kY} = \beta_i \theta_{iY}$$

where δ_{kY} are the displacements of the loads and θ_{iY} are the hinge displacements of the members, relevant to the mechanism Y . This equation determines the value of λ_Y and

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**Numbers in square brackets refer to the Bibliography at the end of the paper.

it can be proved that $\lambda_Y \geq \lambda_C$ where λ_C is the load factor for the correct collapse mechanism C of the frame. [2]. It can be seen that if the frame is to withstand the loads W_k the β_i must be such that $\lambda_C \geq 1$ and thus $\lambda_Y \geq 1$. This briefly is the concept of strength used in this paper, so the assumptions above are inherent in all that follows.

J. Heyman has already suggested a method of solving specific problems of this type [3] but the purpose of this paper is to clarify the nature of the problem, and to identify it with the problem of "activity analysis" or "linear programming" which occurs in Economics [4].

The text is divided into three sections. The first states the general problem and the second introduces a geometrical analogue of the equations involved which helps to clarify their significance. The last section suggests a method of solution, but this is included for the sake of completeness, it is not recommended as a practical design procedure.

I. The problem. Suppose it is required to design for minimum weight, a frame whose members i have lengths l_i when it is subjected to the loads W_k acting at the points k . If sections are assigned so that the fully plastic moments of the members are β_i it will be assumed that the weight, G , of the frame as a whole may be represented by the expression

$$G = l_i \beta_i .$$

The assumption here, of course, is that the cross sectional area A of a member is proportional to its fully plastic moment. In fact $A \propto \beta^{1/n}$ where $2 > n > 1$ for most sections, but the assumption that $n = 1$ does not seriously restrict the theory which follows and its implications will be pointed out at the end of the paper.

Now suppose that with these values of β_i a collapse mechanism Y is assumed for the frame. The equation of virtual work for this mechanism is

$$\lambda_Y W_k \delta_{kY} = \beta_i \theta_{iY}$$

and if the frame is not to fail in the mode Y ,

$$\lambda_Y \geq 1$$

$$i.e. \quad \frac{\theta_{iY}}{W_k \delta_{kY}} \beta_i \geq 1$$

This condition may be rewritten,

$$\alpha_{Yi} \beta_i \geq 1, \quad \text{where} \quad \alpha_{Yi} = \frac{\theta_{iY}}{W_k \delta_{kY}}$$

and, since the equation is one of virtual work, all terms are positive, and so all the α 's are positive.

This equation is in effect a constraining condition on the possible values of the β_i , and similarly every conceivable mechanism of the frame will yield one such constraining condition. Hence, if all conceivable mechanisms form a set of conditions j the constraints of the problem can be expressed by the sets of inequalities.

$$\alpha_{ji} \beta_i \geq 1$$

Mathematically then, the problem is that of minimising the linear function

$$G = l_i \beta_i$$

when the variables β_i are subject to the restriction

$$\alpha_i \beta_i \geq 1$$

all terms in the expressions being positive or zero.

Considered as a problem in linear algebra, the problem is difficult for the reason that the significance of the constraining inequalities is difficult to grasp, but if the problem is translated into geometrical language their meaning becomes clear.

2. The analogue. The basis of this geometrical analogue is to give the variables β_i the significance of coordinate variables in an n -dimensional space. But before proceeding to the general case it would be as well, perhaps, to examine a simple two variable problem from this point of view.

Example. Let it be required to minimise the weight of the continuous beam shown in Fig. 1(a) when it has to bear the loads shown. The loads, and the lengths of the spans, in the problem have been divided by convenient factors in order to obtain simple numbers.

Let the left hand beam have a fully plastic moment of β_1 and the right hand beam one of β_2 .

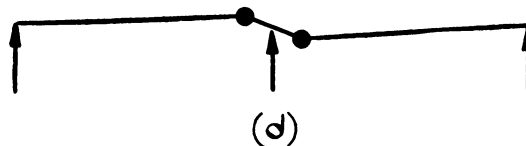
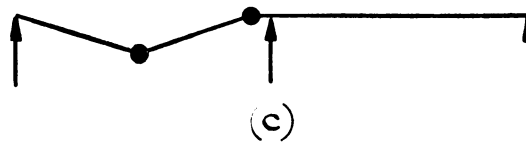
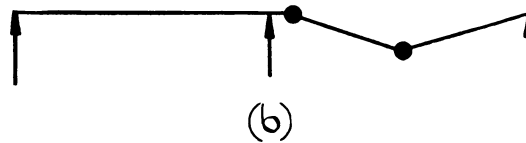
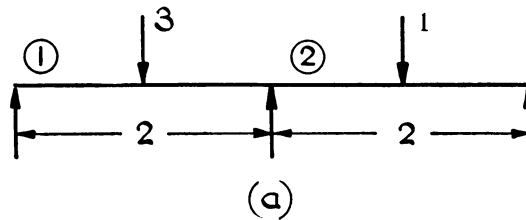


FIG. 1.

In this example there are four possible hinge positions, and there is one redundancy. Hence there are three basic mechanisms [5]. These are sketched in fig. 1, and it is easily seen that six different mechanisms can be obtained by superposing these three in various ways. There are thus six conceivable mechanisms in this example and the load factor expressions for these mechanisms are as follows:

$\lambda_A = \frac{3\beta_2}{1},$	as in Fig. 1(b)	}	$\beta_1 \geq \beta_2$
$\lambda_B = \frac{2\beta_1 + 4\beta_2}{3 + 1},$	both beams collapsing,		
$\lambda_C = \frac{2\beta_1 + \beta_2}{3},$	left hand beam collapsing,		
$\lambda_A^* = \frac{3\beta_1}{3},$	as in Fig. 1(c),	}	$\beta_2 \geq \beta_1$
$\lambda_B^* = \frac{4\beta_1 + 2\beta_2}{3 + 1},$	both beams collapsing,		
$\lambda_C^* = \frac{\beta_1 + 2\beta_2}{1},$	right hand beam collapsing.		

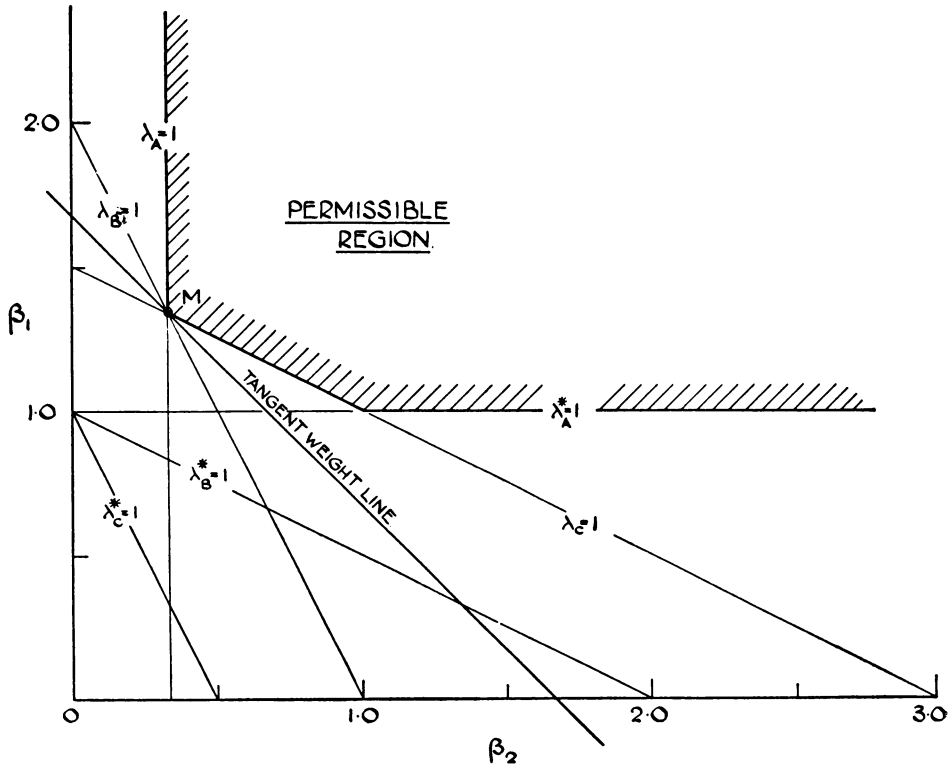


FIG. 1.(e)

Now if β_1 and β_2 are regarded as coordinate variables, the various load factor expressions:

$$\alpha_{v1}\beta_1 + \alpha_{v2}\beta_2 = \lambda_v$$

will form sets of lines whose perpendicular distances from the origin will be proportional to λ_v , and if the λ_v are set equal to unity, Fig. 1(e) is the result. Now since all load factors must be greater or equal to unity, the values of β_1 and β_2 which will allow the loads to be carried are given by points which lie above and to the right of all these load factor lines, in the region marked "permissible". This permissible region is outlined by the load factor lines which correspond to the most critical mechanisms for the beam, and any load factor line which does not touch this region, e. g. λ_B^* and λ_C^* corresponds to an impossible mechanism, a mechanism which, regardless of what values β_1 and β_2 are given, will never occur. The reason for this is that modes such as λ_B^* always violate the yield conditions, or, in mathematical terms, the inequality $\lambda_B^* \geq 1$ is dominated by other inequalities such as $\lambda_C \geq 1$.

The expression for the weight of any particular design,

$$G = 2\beta_1 + 2\beta_2$$

is again a line on the diagram whose perpendicular distance from the origin is proportional G and so the comparative weights of structures given by various points on the diagram, can be judged by their perpendicular distance from the line through the origin

$$2\beta_1 + 2\beta_2 = 0.$$

It is clear then that a minimal beam is given by the point, or points, of tangency when the line

$$G = 2\beta_1 + 2\beta_2$$

is tangent to the permissible region. In this case the single point M is this point of tangency, and the values of β_1 and β_2 at this point are

$$\beta_1 = 1\frac{1}{3}, \quad \beta_2 = \frac{1}{3}.$$

It is possible in some cases to have a line of tangency instead of a single point. This happens in the example of fig. 2 and, in this instance a range of values of β_1 and β_2 will give minimal frames.

It can be seen from the above figures that the permissible region is purely convex toward the tangent weight line, there are no reentrant angles to the region, and hence there is only one minimal position for the weight line.

3. General theory. In general if the β_i are regarded as coordinate variables in an n -dimensional space, the equations

$$\alpha_{ji}\beta_i = 1$$

will form a set of flats which will divide the positive section into a number of regions. If the external side of a flat is defined as that side which does not contain the origin, then a region can be defined, called the "permissible subspace," which is external to all flats. This subspace contains all the points which assign values to the β_i such that

$$\alpha_{ji}\beta_i \geq 1,$$

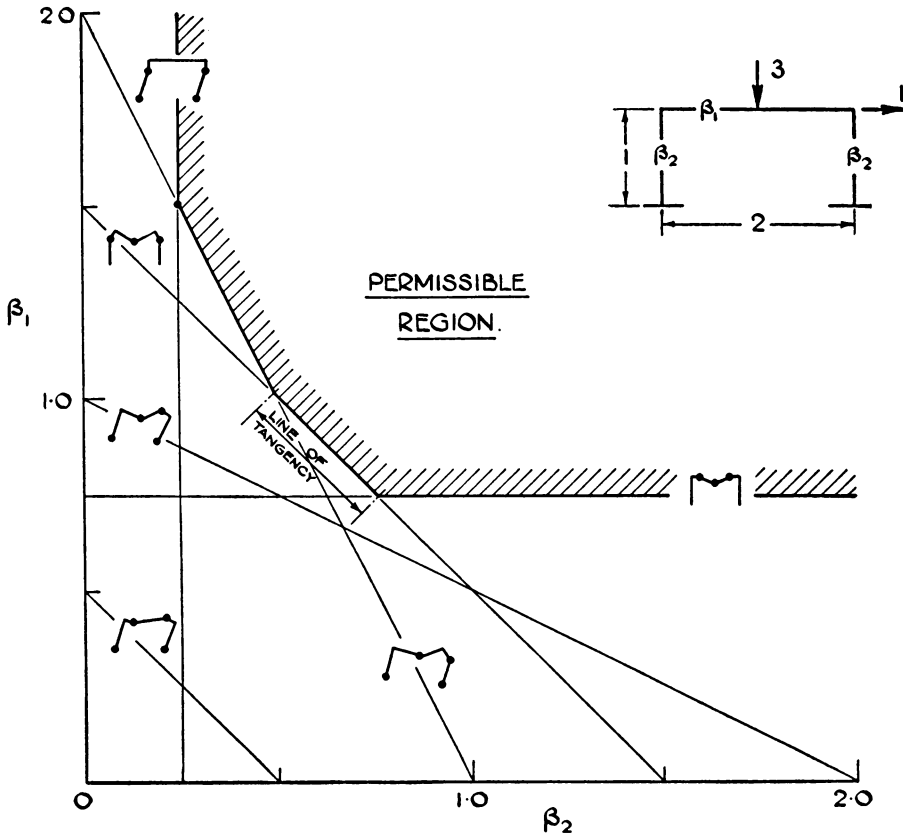


FIG. 2.

i. e. the points within or on the boundary of the permissible subspace represent structures which will bear the given loads.

The following theorem lays severe restrictions on the possible forms of the permissible subspace.

Theorem. If β_v and β_z are any two points within or on the boundary of the permissible subspace, then the straight line joining these two points lies wholly within the permissible subspace.

Proof. β_v and β_z are external to all flats. If in proceeding from β_v to β_z in a straight line a flat X is crossed, a region is entered which is not external to flat X . Thus X must be recrossed at least once more before the line proceeds to β_z , for β_z is external to flat X . A straight line, however, cannot cross a flat more than once, and thus the straight line $\beta_v \beta_z$ cannot cross any flat, and so the theorem is true.

The statement embodied in this theorem is the mathematical definition of convexity, and so, since $\beta_i \geq 0$ and $\alpha_i \geq 0$ it can be shown that the permissible subspace is a convex set contained in the positive sector. Further, by employing the fact that $\alpha > l_i > 0$ it can be proved that there is always one, and only one, position of the weight flat in which it is tangent to the permissible subspace. Hence there is always one, and only one, minimal solution to a given problem. The solution may of course allow a range of values for the β_i , in which case all the designs of the range have the same weight, and all are minimal solutions.

4. A method of solution. It would seem that any method of solution must involve either a total, or a partial, guided exploration of the field of the positive sector in the above analogue. The former mode of attack is the one employed by J. Heyman in his inequalities method, and the economists also favour this line of attack. Any such method, however, suffers from the serious disadvantage that all the constraining inequalities $\alpha_i, \beta_i \geq 1$, i.e., all the conceivable collapse mechanisms of a frame, have to be accounted for in algebraic terms. E.g. In the example of Fig. 1(a) all the conceivable mechanisms could be accounted for by the formula

$$\delta_1^*(3\beta_1 - 3) + \delta_2^*(3\beta_2 - 1) + \delta_3^*(\beta_1 \sim \beta_2) \geq 0,$$

where the δ_i^* 's are Kronecker deltas which can have values of one or zero. If all the possible combinations $\delta_i^* = 0$ or 1 are taken, all the constraining inequalities will be accounted for. The task is thus not impossible, but prohibitively laborious in any but the most simple cases.

The method to be described here involves a partial exploration of the field, which discovers the constraining inequalities as it requires them.

Suppose Fig. 3 represents in diagrammatic form a section of the permissible region, and suppose point *P* represents an assumed form of the frame β_i^* . If the weight of this

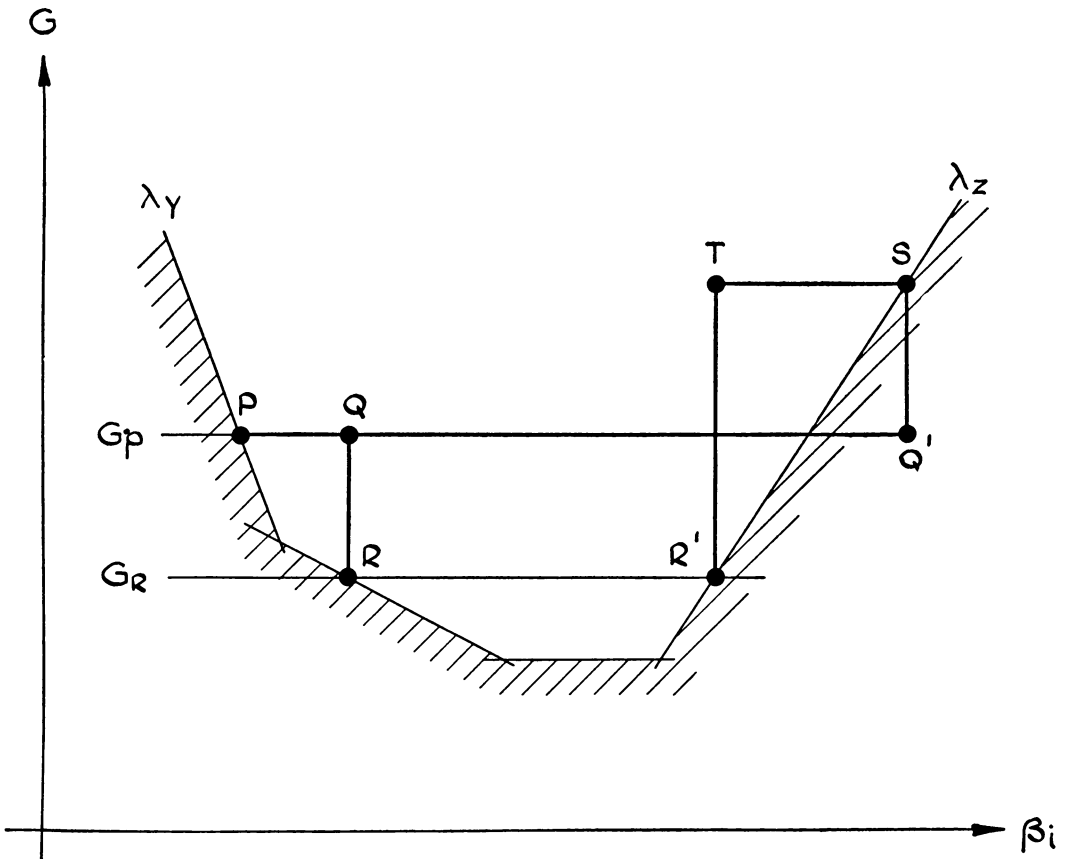
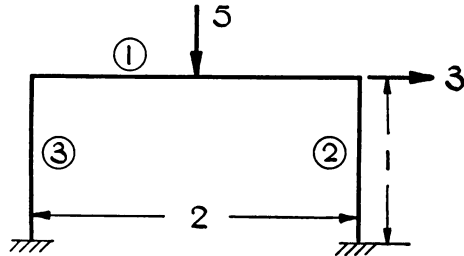


FIG. 3.

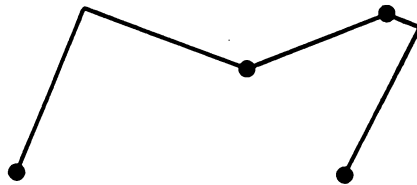
form is G_P and the collapse mechanism is Y , P is a point on the intersection of the two flats

$$G = l_i \beta_i^*, \quad \lambda_Y = \alpha_{Y,i} \beta_i^* = 1$$

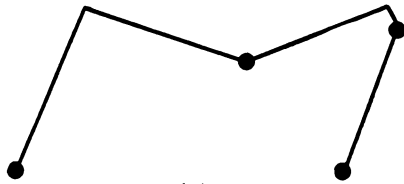
Now if β_i^* is given a small displacement $\delta \beta_i^*$ in the flat G_P in such a sense that $|\lambda_Y|$ is increased, P moves along G_P in the figure to the point Q . A recalculation of the collapse



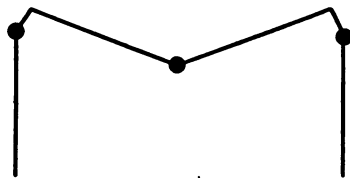
(a)



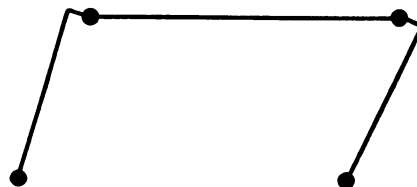
(b)



(c)



(d)



(e)

FIG. 4.

mechanism and load factor will then move Q down to R . If $\delta \beta^*$ is too large, however, the path $P Q' S$ may result. But, in this case, a further operation $S T R'$, will still result in a lowering of the weight flat from G_P to G_R . If the operation $P Q R$ or the whole operation $P Q S, S T R'$ is referred to as a step, the method consists in making a series of such random steps until the minimum is found.

Example. It is required to design the frame of Fig. 4(a) when it is subjected to the loads shown. Let the ringed numbers designate the members of the portal.

Assume as a starting point that the sections of the members are in the proportions

$$\beta_1 = 1, \quad \beta_2 = 1, \quad \beta_3 = 1$$

The collapse mechanism can be determined as that of Fig. 4(b), [5], and for unity load factor the sections are

$$\beta_1 = \frac{4}{3}, \quad \beta_2 = \frac{4}{3}, \quad \beta_3 = \frac{4}{3}$$

This is the point P referred to above, and it lies on the intersection of the two planes

$$G_P = 2\beta_1 + \beta_2 + \beta_3,$$

$$\lambda_Y = \frac{1}{2}\beta_1 + \frac{1}{8}\beta_2 + \frac{1}{8}\beta_3 = 1$$

Now a displacement $\delta \beta^*$ is required which will keep G_P constant and will increase $|\lambda_Y|$. There are various ways in which this can be done and what these various procedures imply can be seen in Fig. 5. Any displacement takes P to P' , PP' lying in the G_P plane.

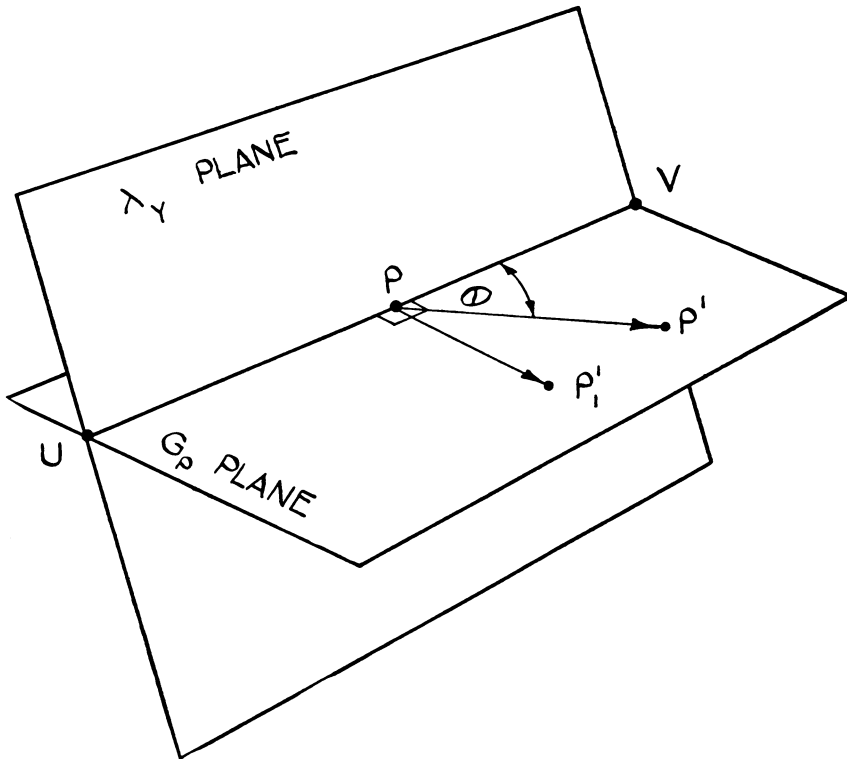


FIG. 5.

The particular displacement chosen will specify the angle θ that PP' makes with the line of intersection of the planes UV . The method of giving displacements could be systematised by specifying that PP' had to be at right angles to UV as PP'_1 is drawn, and a succession of such defined steps will usually be fewer in number than a series of random ones. Also, by further restricting the nature of a step the process can be guaranteed to be a converging one. But, the extra calculation required to ensure a particular type of step invariably consumes much more time than the extra steps necessary in a random search. A series of random steps is thus recommended, and the example will be solved by such a method.

If we write the weight and load factor expressions as follows:

$$G = l_i \beta_i, \quad \lambda_Y = \alpha_Y \beta_i$$

where Y is the relevant mechanism at any stage, then the only rules required in making a random step are:

(i) values β with a large quotient α_Y/l are to be increased, those with a small one are to be decreased.

(ii) The increments added or subtracted to or from the values β are to be in the ratio of the lengths l so that the weight is kept constant.

It is also advisable not to alter more than two variables in any step.

Proceeding by this method then, it is patent that β_1 has to be increased and β_3 decreased. Hence,

Step 1. Increase β_1 by $\frac{1}{3}$ decrease β_3 by $\frac{2}{3}$ then

$$\beta_1 = \frac{5}{3}, \quad \beta_2 = \frac{4}{3}, \quad \beta_3 = \frac{2}{3}$$

A check shows that the mechanism is now that of Fig. 4(c) which gives the load factor expression as

$$\lambda = \frac{1}{4} \beta_1 + \frac{3}{8} \beta_2 + \frac{1}{8} \beta_3 = 1$$

Accordingly, the values β remain as they are.

Step 2. The above load factor expression indicates that β_2 has to be increased and β_3 decreased. Therefore, increase β_2 by $\frac{1}{3}$ decrease β_3 by $\frac{1}{3}$ then

$$\beta_1 = \frac{5}{3}, \quad \beta_2 = \frac{5}{3}, \quad \beta_3 = \frac{1}{3}.$$

These values give the mechanism as that of Fig. 4(d) which has the load factor expression

$$\lambda = \frac{2}{5} \beta_1 + \frac{1}{5} \beta_2 + \frac{1}{5} \beta_3 = \frac{16}{15},$$

and hence for unity load factor

$$\beta_1 = \frac{25}{16}, \quad \beta_2 = \frac{25}{16}, \quad \beta_3 = \frac{5}{16}.$$

It is now seen that the coefficients of the load factor expression are in the same ratio to one another as those in the weight expression:

$$G = 2\beta_1 + \beta_2 + \beta_3, \quad \lambda = \frac{2}{5}\beta_1 + \frac{1}{5}\beta_2 + \frac{1}{5}\beta_3.$$

No further steps are thus possible and the operations have reached the minimal solution. This has occurred in this particular problem because there existed a load factor plane,

$$\lambda = \frac{2}{5}\beta_1 + \frac{1}{5}\beta_2 + \frac{1}{5}\beta_3 = 1$$

which was parallel to the weight plane, and was not an impossible one. Clearly in the tangential position, the weight flat will always adjoin those possible load factor flats which are least inclined to the weight flat, and if there exists a possible load factor flat which is parallel to the weight flat, the minimal solution is the range of values of β_i which lie on this flat within the permissible subspace. The interesting conclusion may be drawn from these facts that for a frame with a particular geometry there are certain mechanisms which the minimal solution will favour, e.g. In the case of a portal with the relative proportions of that above, the mechanisms of Figs. 4(d) and (e) give load factor flats which are parallel to the weight flat. So, if the loads will "allow" either of these two mechanisms, that mechanism will be the minimal collapse mode. If both of these are impossible, the minimal solution will probably be the line of intersection of the planes relevant to mechanisms of the type shown in Fig. 4(c) and (b), i.e. the section of the beam of the portal would equal the section of the right hand stanchion. The plastic hinge required at the upper right hand joint may therefore form either in the beam or in the stanchion, that is, the frame has two alternative modes of failure. In general if there are n independent variables β_i , a minimal solution will be a corner of the permissible subspace where n flats meet. A minimal solution, not necessarily the only solution, will therefore have n alternative mechanisms of collapse. This implies that the members meeting at a joint must be proportioned so that the joint can fail in either of two ways, as above, and/or that some members have to be proportioned so that their failure or non-failure makes no difference to the value of the load factor.

Conclusion. This paper, it is hoped, has clarified to some extent the nature of the problem of minimum weight design, when the theory of such design is based on the assumptions of the theory of plastic collapse, and on the assumption that the weight of a member is proportional to the product of its length and fully plastic moment. This last assumption does not seriously limit the validity of the above theory, for the true expression for the weight of a member merely converts the load factor flats above, into slightly curved surfaces; the general features of the problem are not altered at all.

A more serious limitation on the above theory is the assumption, taken over from the theory of plastic collapse, that instability effects can be ignored. For it sometimes happens that the loading conditions of a frame will force the theory to choose long slender columns which will obviously buckle. This occurs when a rectangular frame has little or no sidesway loads, in ordinary cases quite reasonable sections are usually demanded. This danger can be offset, however, by stipulating certain minimum values for the β_i of the stanchions and setting the theory to work with these additional restraining conditions. This is easily done for instance in the method of random steps

above. Another advantage of the method is that the calculator can stop his calculations at any stage and still have a design which is adequately strong and lighter than his original guess. The drawback of the method is, of course, the laborious and uncertain nature of the calculations. However, the calculations involved in any other method to date have to be programmed for calculating machines, and there is no reason why the method of random steps could not also be similarly programmed.

BIBLIOGRAPHY

1. J. F. Baker, *The design of steel frames*, The Structural Engineer, **27**, (1949).
2. H. J. Greenberg and W. Prager, *On limit design of beams and frames*, Proc. A. S. C. E. **77**, Separate No. 59 (1951).
3. J. Heyman, *Plastic design of beams and plane frames for Minimum material consumption*, Q. Appl. Maths. **8**, 373 (1951).
4. T. C. Koopmans, *Activity analysis of production and allocation*, John Wiley & Sons, 1951.
5. P. S. Symonds and B. G. Neal, *The rapid calculation of the plastic collapse load for a framed structure*, Inst. Civ. Eng., Structural Paper No. 29.