AN EXTENSION OF THE METHOD OF TREFFTZ FOR FINDING LOCAL BOUNDS ON THE SOLUTIONS OF BOUNDARY VALUE PROBLEMS, AND ON THEIR DERIVATIVES*

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1. Introduction. Methods for the approximate solution of the boundary value problems of mathematical physics are of little practical use unless there is reason to believe that the difference between the approximate and the exact solutions is sufficiently small for the purpose in mind. On the other hand, this difference, or error, is generally difficult to estimate. It is, therefore, a little surprising that there should exist a method, originated by E. Trefftz, which is capable of giving not merely an estimate, but upper and lower bounds on the solutions, and their derivatives, of the boundary value problems of physics. The method has been developed further by many writers. The purpose of the present paper is to encompass the previous work in a general scheme. In doing this, the writer borrows much from his predecessors; the debt is acknowledged by the bibliography at the end of this paper.

Before going into the general, and consequently abstract, theory of this method, we shall illustrate it by using it in the field of electrostatics. Let u, u_1, u_2 be functions of the rectangular coordinates x, y. Let D be a region, not necessarily simply-connected, in the x, y plane and let γ_1 and γ_2 be portions of the boundary of D such that $\gamma_1 + \gamma_2$ is the entire boundary, γ. Then, the general two-dimensional boundary value problem of electrostatics may be indicated by the equations:

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + f = 0 \qquad \text{in} \qquad D, \tag{1}$$

$$u_1 \frac{\partial x}{\partial n} + u_2 \frac{\partial y}{\partial n} + g = 0 \qquad \text{on} \qquad \gamma_1, \tag{2}$$

$$u = k \qquad \text{on} \qquad \gamma_2, \tag{3}$$

$$u_1 = \frac{\partial u}{\partial x}, \qquad u_2 = \frac{\partial u}{\partial y} \qquad \text{in} \qquad D + \bar{\gamma}, \tag{4}$$

where ds is the differential of the arc-length along γ. If u, u_1, u_2 denote the functions or γ_2 as the entire boundary, thus eliminating equation (3) or (2), respectively. Also, the function f, which is 4π times the charge density, may be identically zero without invalidating the theory.

The preceding problem, which may be labeled Problem I, is known to be equivalent to the following variational problem which may be called Problem I′. Let u', u'_1, u'_2 be continuous functions satisfying equations (3) and (4). Then, Problem I′ consists of

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finding the particular set of functions of this type which minimize the energy integral given by

$$J_1[u'] = \iint_D \left\{\frac{1}{2}(u_1'^2 + u_2'^2) - fu'\right\} dx\, dy + \int_{\gamma_1} gu'\, ds \tag{5}$$

where ds is the differential of the arc-length along γ. If u, u_1, u_2 denote the functions satisfying (1) to (4), then the statement that Problems I and I′ are equivalent amounts to saying that

$$\text{Min } J_1[u'] = J_1[u]. \tag{6}$$

Moreover, by an application of Green's theorem, we obtain

$$J_1[u] = \frac{1}{2}\left\{-\iint_D fu\, dx\, dy + \int_{\gamma_1} gu\, ds + \int_{\gamma_2} k\left(u_1 \frac{\partial x}{\partial n} + u_2 \frac{\partial y}{\partial n}\right) ds\right\}. \tag{7}$$

Equation (6) is nothing else but the familiar Rayleigh-Ritz method, for it states that for any functions u', u_1', u_2' of the class previously defined, the energy is greater than the energy due to the true potential. In order to obtain a lower bound for $J_1[u]$, we shall have recourse to a method similar to the Rayleigh-Ritz. Let us consider continuous functions u_1'', u_2'' which satisfy equations (1) and (2). Note that therefore u_1'' and u_2'' are not necessarily derivatives of a function u''. For this class of functions, let us consider the problem, which we shall call I″, of extremizing the integral $K_1[u'']$ defined by

$$K_1[u''] = -\frac{1}{2}\iint_D (u_1''^2 + u_2''^2)\, dx\, dy + \int_{\gamma_2} k\left(u_1'' \frac{\partial x}{\partial n} + u_2'' \frac{\partial y}{\partial n}\right) ds. \tag{8}$$

$K_1[u'']$ is obtained by forming the expression $2J_1[u] - J_1[u']$ and by then substituting for the functions u, u', etc., the functions u_1'', u_2''.

With regard to Problem I″, the following theorem can be proved.

Theorem:

(*a*) $K_1[u'']$ attains a maximum value;

(b) Max $K_1[u'']$ = Min $J_1[u']$;

(c) Max $K_1[u'']$ is attained for the same functions u, u_1, u_2 which minimize $J_1[u']$.

A discussion of the physical meaning of this theorem would be in order at this point.

Let us consider, first, Problem I′. In this problem, the class of allowable trial functions consists of functions which satisfy the same boundary condition as the potential of a charge distribution given by the function f. In calculating the energy integral, $J_1[u']$, one uses the derivatives of u' as the components of the field strength. However, no attempt is made to have these derivatives satisfy the conditions imposed on the field strength (equations 1-2). Thus, the trial functions of Problem I′ satisfy the conditions imposed on the potential, but not those imposed on the field strength.

In contrast to this, the trial functions allowed in Problem I″ satisfy the conditions imposed on the field strength but not, except for the solution itself, those on the potential. Thus, in Problem I′, one attempts to approximate the potential whereas in Problem I″, one attempts to approximate the field strength.

To prove the theorem, we start from the identity

$$\int_{\gamma} \left\{u'\left(u_1'' \frac{\partial x}{\partial n} + u_2'' \frac{\partial y}{\partial n}\right)\right\} ds = \iint_D \left\{\frac{\partial}{\partial x}(u'u_1'') + \frac{\partial}{\partial y}(u'u_2'')\right\} dx\, dy. \tag{9}$$

By use of equations (1) to (4), this can be transformed into

$$-\int_{\gamma_1} gu'\, ds + \int_{\gamma_2} k\left(u_1'' \frac{\partial x}{\partial n} + u_2'' \frac{\partial y}{\partial n}\right) ds = -\iint_D fu'\, dx\, dy$$

$$+ \iint_D (u_1'u_1'' + u_2'u_2'')\, dx\, dy. \tag{10}$$

But

$$u_1'u_1'' + u_2'u_2'' = \tfrac{1}{2}\{(u_1'^2 + u_2'^2) + (u_1''^2 + u_2''^2) - (u_1' - u_1'')^2 - (u_2' - u_2'')^2\}. \tag{11}$$

Substituting (11) in (10) and comparing the result with equations (5) and (8), one obtains

$$J_1[u'] - K_1[u''] = \frac{1}{2}\iint_D \{(u_1' - u_1'')^2 + (u_2' - u_2'')^2\}\, dx\, dy. \tag{12}$$

The right-hand side of (12) is never negative and vanishes if and only if

$$u_1' \equiv \frac{\partial u'}{\partial x} = u_1'', \qquad u_2' \equiv \frac{\partial u'}{\partial y} = u_2''. \tag{13}$$

The rest of the proof is easy, for:

(a) from (12), it follows that Min $J_1[u'] \geqq K_1[u'']$, and hence $K_1[u'']$ is bounded above;
(b) since Min $J_1[u'] = J_1[u]$ and Max $K_1[u''] = K_1[u]$ and since $u_1\ u_1$, u_2 are allowable trial functions in both Problems I′ and I″, it follows that in this case (13) will be satisfied and consequently by (12), that $J_1[u] = K_1[u]$;
(c) this part follows from what was said above.

This theorem enables us to bound the energy from above and below. In order to bound the potential function, therefore, we need to be able to express the value of the potential function at a given point as a function of certain energies. We start by considering the expression for the potential involving Green's function. This is

$$u(x, y) = -\iint_D \left\{f(\xi, \eta)\left(\frac{1}{2\pi}\log r + R\right)\right\} d\xi\, d\eta$$

$$+ \int_{\gamma_1} \left\{g\left(\frac{1}{2\pi}\log r + R\right)\right\} ds + \int_{\gamma_2} \left\{k \frac{\partial}{\partial n}\left(\frac{1}{2\pi}\log r + R\right)\right\} ds \tag{14}$$

where $r = [(x - \xi)^2 + (y - n)^2]^{1/2}$ and R is a function to be discussed subsequently. Since f, g and k are known functions, we can evaluate $\sigma(x,y)$ where σ is defined by

$$\sigma(x, y) = -\frac{1}{2\pi}\iint_D f(\xi, \eta) \log r\, d\xi\, d\eta + \frac{1}{2\pi}\int_{\gamma_1} g \log r\, ds + \frac{1}{2\pi}\int_{\gamma_2} k \frac{\partial}{\partial n}(\log r)\, ds. \tag{15}$$

Hence,

$$u(x, y) - \sigma(x, y) = -\iint_D f(\xi, \eta)R \, d\xi \, d\eta + \int_{\gamma_1} gR \, ds + \int_{\gamma_2} k \frac{\partial R}{\partial n} \, ds. \tag{16}$$

Now, R is the regular part of the Green's function and so is a solution of the boundary value problem denoted by the equations

$$\frac{\partial}{\partial x} R_1 + \frac{\partial}{\partial y} R_2 = 0 \qquad \text{in} \quad D, \tag{17}$$

$$R_1 \frac{\partial x}{\partial n} + R_2 \frac{\partial y}{\partial n} + \frac{1}{2\pi} \frac{\partial}{\partial n} (\log r) = 0 \qquad \text{on} \quad \gamma_1 , \tag{18}$$

$$R_1 = -\frac{1}{2\pi} \log r \qquad \text{on} \quad \gamma_2 \tag{19}$$

$$R_1 = \frac{\partial R}{\partial x}, \qquad R_2 = \frac{\partial R}{\partial y} \qquad \text{in} \quad D + \gamma. \tag{20}$$

But this problem is of exactly the same type as Problem I and the corresponding variational integrals, which we shall call $J_2[R']$ and $K_2[R'']$ may be obtained by setting $f = 0$, $g = (1/2\pi) \, (\partial/\partial n) \, (\log r)$, $k = -1/2\pi \log r$ in equations (5) and (8). Similarly, we have

$$\text{Min } J_2[R'] = \text{Max } K_2[R''] = \frac{1}{4\pi} \left\{ \int_{\gamma_1} R \frac{\partial}{\partial n} (\log r) \, ds - \int_{\gamma_2} \log r \frac{\partial R}{\partial n} \, ds \right\}. \tag{21}$$

Let $q = u - \alpha R$, where α is a parameter whose value remains to be determined. Then, q is a solution of the boundary value problem denoted by the equations

$$\frac{\partial}{\partial x} q_1 + \frac{\partial}{\partial y} q_2 + f = 0 \qquad \text{in} \quad D, \tag{22}$$

$$q_1 \frac{\partial x}{\partial n} + q_2 \frac{\partial y}{\partial n} + g - \frac{\alpha}{2\pi} \frac{\partial}{\partial n} (\log r) = 0 \qquad \text{on} \quad \gamma_1 , \tag{23}$$

$$q = k + \frac{\alpha}{2\pi} \log r \qquad \text{on} \quad \gamma_2 , \tag{24}$$

$$q_1 = \frac{\partial q}{\partial x}, \qquad q_2 = \frac{\partial q}{\partial y} \qquad \text{in} \quad D + \gamma. \tag{25}$$

Again, this problem is of the same type as Problem I and the corresponding variational integrals which we shall call $J_3[q']$ and $K_3[q'']$ may be found by making the substitutions $g - \alpha/2\pi \, \partial/\partial n \, (\log r)$ for g, $k + \alpha/2\pi \log r$ for k. Hence, we have for $J_3[q] = K_3[q]$, the expression

$$\text{Min } J_3[q'] = \text{Max } K_3[q''] = \frac{1}{2} \left\{ -\iint_D fq \, dx \, dy \right.$$

$$\left. + \int_{\gamma_1} \left(g - \frac{\alpha}{2\pi} \frac{\partial}{\partial n} \log r \right) q \, ds + \int_{\gamma_2} \left(k + \frac{\alpha}{2\pi} \log r \right) \frac{\partial q}{\partial n} \, ds \right\}. \tag{26}$$

Now, equation (25) can be put into the following form

$$\begin{aligned}\text{Min } J_3[q'] = \text{Max } K_3[q''] &= \frac{1}{2}\left\{-\iint_D fu\,dx\,dy + \int_{\gamma_1} gu\,ds + \int_{\gamma_2} k\frac{\partial u}{\partial n}\,ds\right\} \\ &+ \frac{\alpha}{2}\left\{\iint_D fR\,dx\,dy - \int_{\gamma_1} gR\,ds - \int_{\gamma_2} k\frac{\partial R}{\partial n}\,ds\right\} \\ &+ \frac{\alpha^2}{2}\left\{\frac{1}{2\pi}\int_{\gamma_1} R\frac{\partial}{\partial n}(\log r)\,ds - \frac{1}{2\pi}\int_{\gamma_2}\log r\frac{\partial R}{\partial n}\,ds\right\} \\ &+ \frac{\alpha}{2}\left\{-\frac{1}{2\pi}\int_{\gamma_1} u\frac{\partial}{\partial n}(\log r)\,ds + \frac{1}{2\pi}\int_{\gamma_2}\frac{\partial u}{\partial n}\log r\,ds\right\}. \end{aligned} \tag{27}$$

Let

$$\begin{aligned} d_1 &= \text{Min } J_1[u'] = \text{Max } K_1[u''], \\ d_2 &= \text{Min } J_2[R'] = \text{Max } K_2[R''], \\ d_3 &= \text{Min } J_3[q'] = \text{Max } K_3[q'']. \end{aligned} \tag{28}$$

Then, by use of equations (7), (16), and (21), equation (27) becomes

$$d_3 = d_1 + \alpha^2 d_2 - \frac{\alpha}{2}\left\{u - \sigma + \frac{1}{2\pi}\int_{\gamma_1} u\frac{\partial}{\partial n}(\log r)\,ds - \frac{1}{2\pi}\int_{\gamma_2}\log r\frac{\partial u}{\partial n}\,ds\right\}. \tag{29}$$

But, by Green's identity, we can write

$$\begin{aligned}\frac{1}{2\pi}\int_{\gamma_1} u\frac{\partial}{\partial n}(\log r)\,ds - \frac{1}{2\pi}\int_{\gamma_2}\frac{\partial u}{\partial n}\log r\,ds &= -\iint_D\left\{\frac{\partial u}{\partial x}\frac{\partial R}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial R}{\partial y}\right\}dx\,dy \\ &= \iint_D R\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)dx\,dy + \int_{\gamma_1} gR\,ds + \int_{\gamma_2} k\frac{\partial R}{\partial n}\,ds \\ &= -\iint_D fR\,dx\,dy + \int_{\gamma_1} gR\,ds + \int_{\gamma_2} k\frac{\partial R}{\partial n}\,ds \\ &= u - \sigma. \end{aligned} \tag{30}$$

Hence, (29) reads

$$d_3 = d_1 + \alpha^2 d_2 - \alpha(u - \sigma) \tag{31}$$

or

$$\alpha(u - \sigma) = d_1 + \alpha^2 d_2 - d_3\,. \tag{32}$$

(32) gives us the value of the solution in terms of a known quantity σ, a parameter α, and the minima, d_1 d_2 , d_3 of three variational problems.

Since the quantities d_1 d_2 , d_3 , are bounded above and below, we can also bound $\mu - \sigma$. Thus

$$K_1[u''] + \alpha^2 K_2[R''] - J_3[q''] \leqq \alpha(u - \sigma) \leqq J_1[u'] + \alpha^2 J_2[R'] - K_3[q'']. \tag{33}$$

Furthermore, since $q' = u' - \alpha R'$, $q'' = u'' - \alpha R''$, we can write

$$J_3[q'] = J_1[u'] + \alpha^2 J_2[R'] - 2\alpha b',$$

$$K_3[q''] = K_1[u''] + \alpha^2 K_2[R''] - 2\alpha b'', \tag{34}$$

where

$$b' = \frac{1}{2}\left\{\iint_D (u_1' R_1' + u_2' R_2')\, dx\, dy - \iint_D fR'\, dx\, dy \right.$$

$$\left. + \int_{\gamma_1} \left(gR' - \frac{1}{2\pi} u' \frac{\partial}{\partial n} [\log r]\right) ds\right\}, \tag{35}$$

$$b'' = -\frac{1}{2}\left\{\iint_D (u_1'' R_1'' + u_2'' R_2'')\, dx\, dy - \frac{1}{2\pi}\int_{\gamma_2} \log r \left(u_1'' \frac{\partial x}{\partial n} + u_2'' \frac{\partial y}{\partial n}\right) ds \right.$$

$$\left. + \int_{\gamma_2} k\left(R_1'' \frac{\partial x}{\partial n} + R_2'' \frac{\partial y}{\partial n}\right) ds\right\}. \tag{36}$$

Let $a = J_1[u'] - K_1[u'']$ and $c = J_2[R'] - K_2[R'']$. Then, the inequality (33) may be written

$$-a - \alpha^2 c + 2\alpha b' \leqq \alpha(u - \sigma) \leqq a + \alpha^2 c + 2\alpha b'' \tag{37}$$

which can be broken up into

$$a + \alpha[(u - \sigma) - 2b'] + \alpha^2 c \geqq 0, \tag{38}$$

$$a + \alpha[2b'' - (u - \sigma)] + \alpha^2 c \geqq 0. \tag{39}$$

Since (38) and (39) must hold for arbitrary values of α, it follows that their discriminants must never be positive. Hence,

$$4ac - [(u - \sigma) - 2b']^2 \geqq 0, \tag{40}$$

$$4ac - [2b'' - (u - \sigma)]^2 \geqq 0, \tag{41}$$

or

$$2(b' - \sqrt{ac}) \leqq u - \sigma \leqq 2(b' + \sqrt{ac}), \tag{42}$$

$$2(b'' - \sqrt{ac}) \leqq u - \sigma \leqq 2(b'' + \sqrt{ac}). \tag{43}$$

Furthermore, both (42) and (43) must hold simultaneously. Thus, if $b' \geqq b''$, we have

$$2(b' - \sqrt{ac}) \leqq u - \sigma \leqq 2(b'' + \sqrt{ac}) \tag{44}$$

and if $b'' \geqq b'$, (44) holds with b' and b'' interchanged. In either case, we can define a "best" approximation to u which we shall call u^* by

$$u^* = \sigma + b' + b'' \tag{45}$$

and, it is clear that

$$| u - u^* | \leqq 2\sqrt{ac} - | b' - b'' | \tag{46}$$

(46) gives the bound on the "error" for the best approximation,

Examining equation (14), one notices that one can obtain a formula giving $\partial u/\partial x$ and $\partial u/\partial y$ by straightforward differentiation. Since in this formula f, g, and k are functions of ξ, η and not of x, y, it is only r and R that is differentiated. Hence, one needs to set up variational problems for $\partial R/\partial x$ and $\partial R/\partial y$, but not for $\partial u/\partial x$ or $\partial u/\partial y$. Thus, Problems I′ and I″ remain unchanged. Problems II′ and II″ are obtained by differentiation of equations (17) to (20). Problems III′ and III″ are set up in the same manner as previously for a function $q = u - \alpha\, \partial R/\partial x$ or $q = u - \alpha\, \partial R/\partial y$ and, finally, one obtains equations entirely similar to (45) and (46).

2. Notation. We shall now try to generalize this method so that upper and lower bounds on the solutions and their derivatives of a certain class of boundary value problems may be obtained. The class of boundary value problems which will be treated consist of all those having the same solution as a positive definite, quadratic variational problem. We shall here restrict ourselves to the boundary value problems of systems of one or more second order partial differential equations. However, the plate equation can also be treated by the same method.

Let φ_i and $\varphi_{ik}(i = 1 \cdots m, k = 1 \cdots n)$ be functions of the independent variables x_k, and let $E'[\varphi]$ represent a homogeneous quadratic form in the φ_i and φ_{ik} with coefficients which are functions of the x_k. $E'[\varphi]$ will be either positive definite or semi-definite in the algebraic sense, that is, $E'[\varphi] \geqq 0$ for arbitrary functions φ_i, φ_{ik}. The quantities $E'_i[\varphi]$ and $E'_{ik}[\varphi]$ are defined by

$$E'_i[\varphi] = \frac{\partial E'[\varphi]}{\partial \varphi_i}, \qquad E'_{ik}[\varphi] = \frac{\partial E'[\varphi]}{\partial \varphi_{ik}}. \tag{1}$$

Using the equations above, two other expressions, $L_i[\varphi]$ and $M_i[\varphi]$ are defined by

$$L_i[\varphi] = E'_{ik,k}[\varphi] - E'_i[\varphi], \qquad M_i[\varphi] = E'_{ik}[\varphi]x_{k,n} \tag{2}$$

where

$$E'_{ik,k}[\varphi] = \frac{\partial}{\partial x_k} E'_{ik}[\varphi], \qquad x_{k,n} = \frac{\partial x_k}{\partial n}. \tag{3}$$

In these equations, n stands for the outward drawn normal to a domain D, and the summation convention is used.

By $E[\varphi]$, we mean the integral of $E[\varphi]$ over a domain D. $E[\varphi, \psi]$ and $E'[\varphi, \psi]$ are the bilinear expressions corresponding to $E[\varphi]$ and $E'[\varphi]$. We also define other integrals H and B by

$$H[f, \varphi] = \int \underset{D}{\cdots} \int f\varphi \, dV, \tag{4}$$

$$B[g, \varphi] = \int \underset{\gamma}{\cdots} \int g\varphi \, dA, \tag{5}$$

where γ is the boundary of D and dV and dA are the appropriate volume and surface elements. $B_1[g, \varphi]$ and $B_2[g, \varphi]$ designate the integral $B[g, \varphi]$ extended over a portion γ_1 or γ_2 of the boundary, and the case $\gamma_1 = 0$ or $\gamma_2 = 0$ is not excluded. In any case, we shall always have $\gamma_1 + \gamma_2 = \gamma$.

3. Some lemmas. In this section, we state without proof some simple lemmas which will be used frequently in the sequel. Although simple, these lemmas form the foundation on which the theory of the Trefftz method rests.

Lemma I.

$$E'[\varphi, \psi] = \tfrac{1}{2}\{E'_{ik}[\varphi]\psi_{ik} + E'_i[\varphi]\psi_i\}.$$

Lemma 2. Let $\psi_{i,k}$ denote the derivative of ψ_i with respect to x_k. Then,

$$E'[\varphi, \psi] = \tfrac{1}{2}\{-\psi_i L_i[\varphi] + (\psi_{ik} - \psi_{i,k})E'_{ik}[\varphi] + (\psi_i E'_{ik}[\varphi])_{,k}\}.$$

Corollary.

$$E[\varphi, \psi] = \tfrac{1}{2}\{-H[L_i[\varphi], \psi_i] + H[E'_{ik}[\varphi], \psi_{ik} - \psi_{i,k}] + B[M_i[\varphi], \psi_i]\}.$$

Lemma 3. The equations $E'[\varphi] = \eta_i$, $E_{ik}[\varphi] = \eta_{ik}$ always possess solutions under the condition that if any pair of the $E'_i[\varphi]$ or $E'_{ik}[\varphi]$ are identically equal, then only one of the corresponding equations should be used.

Lemma 4. Let ζ_i, ζ_{ik} be a set of continuous functions satisfying the conditions

$$L_i[\zeta] = 0 \quad \text{in} \quad D, \qquad M_i[\zeta] = 0 \quad \text{on} \quad \gamma_1,$$

If $H[E'_{ik}[\zeta], v] = 0$ for all such sets of functions, then $v = 0$.

4. The complementary principle. Let a set of functions u_i, u_{ij} exist such that they satisfy the conditions:

$$L_i[u] = f_i \qquad \text{in} \qquad D, \tag{1}$$

$$M_i[u] = -g_i \qquad \text{on} \qquad \gamma_1, \tag{2}$$

$$u_i = k_i \qquad \text{on} \qquad \gamma_2, \tag{3}$$

$$u_{ij} = u_{i,j} \equiv \frac{\partial u_i}{\partial x_j} \qquad \text{in} \qquad D + \gamma. \tag{4}$$

The problem of finding these functions u_i, u_{ij} is the fundamental boundary value problem which we shall try to solve. We shall call this Problem I.

Let V' be the class of all continuous* functions u'_i, u'_{ij} satisfying equations (3) and (4). Let the functional $J_1[u']$ be defined on the class U' by

$$J_1[u'] = E[u'] + H[f_i, u'_i] + B_1[g_i, u'_i]. \tag{5}$$

The problem of finding those functions u'_i, u'_{ij} for which $J_1[u']$ attains its minimum is a variational problem which we shall call Problem I′. By the Dirichlet principle, we know that Problem I and Problem I′ are equivalent, i.e., they have the same solution.

Let U'' be the class of all continuous functions u''_i, u'_{ij} satisfying equations (1) and (2), and let the functional $K_1[u'']$ be defined on U'' by

$$K_1[u''] = -E[u''] + B_2[M_i[u''], k_i]. \tag{6}$$

The problem of finding those functions u''_i, u''_{ij} for which $K_1[u'']$ attains a stationary value is a variational problem which we shall call Problem I″.

The complementary principle is expressed by the following theorem.

Theorem I.

(a) Problem I″ is equivalent to Problems I and I′.

*For the sake of simplicity, we shall consider only continuous functions. However, less restrictive conditions could be imposed, e.g., piecewise continuity.

(b) $K_1[u'']$ attains a maximum value.
(c) Max $K_1[u''] =$ Min $J_1[u']$

Proof. (a) Let v_i and v_{ik} be functions of the class U'' and let ζ_i and ζ_{ik} be a variation. Thus, the ζ's satisfy the equations

$$L_i[\zeta] = 0 \quad \text{in} \quad D, \qquad M_i[\zeta] = 0 \quad \text{on} \quad \gamma_1 . \tag{7}$$

If v_i and v_{ik} are to provide a stationary value for $K_1[u'']$, it is necessary that they satisfy the condition

$$-2E[v, \zeta] + B_2[M_i[\zeta], k_i] = 0 \tag{8}$$

for all ζ. By the corollary to Lemma 2, this condition becomes

$$H[E'_{ik}[\zeta], v_{i,k} - v_{ik}] + B_2[M_i[\zeta], k_i - v_i] = 0. \tag{9}$$

It is clear that we can choose ζ's such that $M_i[\zeta] = 0$ on γ_2. Hence, by Lemma 4, applied to (9), the conclusion is

$$v_{i,k} = v_{ik} . \tag{10}$$

Equation (9) now reduces to

$$B_2[M_i[\zeta], k_i - v_i] = 0. \tag{11}$$

It can be shown that ζ's can be found such that the $M_i[\zeta]$ are equal to arbitrary continuous functions. Hence,

$$v_i = k_i \qquad \text{on} \qquad \gamma_2 . \tag{12}$$

Thus, the v_i and v_{ik} satisfy all the conditions of Problem I. This proves Part (a) of the theorem.

(b) Since the v_i and v_{ik} are solutions of Problem I, and since we assume that solutions of Problem I exist, we know that the v_i and v_{ik} exist, and we can identify them with the u_i and u_{ik}. Hence, $K_1[u'']$ actually attains a stationary value. Because of the fact that $E'[u'']$ is never negative, this stationary value must be a maximum.

(c) By the corollary to Lemma 2,

$$E[u] = \tfrac{1}{2}\{-H[f_i , u_i] + B_1[g_i , u_i] + B_2[M_i[u], k_i]\}. \tag{13}$$

Then,

$$K_1[u] = \text{Max } K_1[u''] = \tfrac{1}{2}\{H[f_i , u_i] + B_1[g_i , u_i] + B_2[M_i[u], k_i]\}, \tag{14}$$

and also

$$J_1[u] = \text{Min } J_1[u'] = \tfrac{1}{2}\{H[f_i , u_i] + B_1[g_i , u_i] + B_2[M_i[u], k_i]\}. \tag{15}$$

Comparing (14) and (15), we have

$$\text{Max } K_1[u''] = \text{Min } J_1[u']. \tag{16}$$

This completes the proof of the theorem.

5. The auxiliary problems. Let F_{ij} denote the components of the fundamental part of Green's matrix and R_{ij}, the components of the regular part. It is assumed that the

F_{ij} are known functions. Then, for a given value of j, say $j = k$, the functions R_{ik} solve the boundary value problem indicated by the following equations:

$$\mathrm{L}_i[R_k] = 0 \quad \text{in} \quad D, \tag{1}$$

$$M_i[R_k] = -M_i[F_k] \quad \text{on} \quad \gamma_1, \tag{2}$$

$$R_{ik} = -F_{ik} \quad \text{on} \quad \gamma_2, \tag{3}$$

$$R_{ikj} = R_{ik,j} \quad \text{in} \quad D + \gamma. \tag{4}$$

This is a problem of exactly the same type as was treated in the preceding sections. Let us call it Problem II.

Problem II′ consists of finding the functions which minimize the functional $J_2[R_k']$ defined by

$$J_2[R_k'] = E[R_k'] + B_1[M_i[F_k], R_{ik}'] \tag{5}$$

among all continuous functions R_{ik}', R_{ikj}' which satisfy conditions (3) and (4). Problem II′ consists of finding the functions which maximize the functional $K_2[R_k'']$ defined by

$$K_2[R_k''] = -E[R_k''] - B_2[M_i[R_k''], F_{ik}] \tag{6}$$

among all continuous functions R_{ikj}'', R_{ikj}'' satisfying conditions (1) and (2).

In the same way, we can consider the problems which have as their solution the functions, $Q_{ik} = u_i - \alpha R_{ik}$, where α is a parameter. The boundary value problem which we label Problem III is indicated by the equations

$$L_i[Q_k] = f_i \quad \text{in} \quad D, \tag{7}$$

$$M_i[Q_k] = -g_i + \alpha M_i[F_k] \quad \text{on} \quad \gamma_1, \tag{8}$$

$$Q_{ik} = k_i + \alpha F_{ik} \quad \text{on} \quad \gamma_2, \tag{9}$$

$$Q_{ikj} = Q_{ik,j} \quad \text{in} \quad D + \gamma. \tag{10}$$

Problem III′ consists of finding the functions which minimize the functional $J_3[Q_k']$ defined by

$$J_3[Q_k'] = E[Q_k'] + H[f_i, Q_{ik}'] + B_1[g_i - \alpha M_i[F_k], Q_{ik}'] \tag{11}$$

among all continuous functions satisfying (9) and (10). Problem III″ consists of finding those functions which maximize the functional $K_3[Q_k'']$ defined by

$$K_3[Q_k''] = -E[Q_k''] + B_2[M_i[Q_k''], k_i + \alpha F_{ik}] \tag{12}$$

among all continuous functions satisfying (7) and (8).

6. Bounds on the solution. If u_k denotes the value of the function $u_k(x,y)$ at the pole of the Green's matrix, then it is well known that

$$u_k - \sigma_k = H[f_i, R_{ik}] + B_1[g_i, R_{ik}] + B_2[M_i[R_k], k_i], \tag{1}$$

where

$$\sigma_k = H[f_i, F_{ik}] + B_1[g_i, F_{ik}] + B_2[M_i[F_k], k_i]. \tag{2}$$

Our object at this point is to express $u_k - \sigma_k$ in terms of the stationary values of the variational problems set up in the two preceding sections.

Consider the stationary values achieved in Problems III′ and III″. By the complementary principle, these stationary values are equal. Since $Q_{ik} = u_i - \alpha R_{ik}$, we can write

$$\begin{aligned} J_3[Q_k] = K_3[Q_k] = E[u] - 2\alpha E[u, R_k] + \alpha^2 E[R_k] \\ + H[f_i, u_i] - \alpha H[f_i, R_{ik}] + B_1[g_i, u_i] \\ - \alpha B_1[M_i[F_k], u_i] - \alpha B_1[g_i, R_{ik}] + \alpha^2 B_1[M_i[F_k], R_{ik}] \end{aligned} \tag{3}$$

which by considering equations (4.5) and (5.5) can be transformed into

$$\begin{aligned} J_3[Q_k] = J_1[u] + \alpha^2 J_2[R_k] - \alpha\{2E[u, R_k] + H[f_i, R_{ik}] \\ + B_1[M_i[F_k], u_i] + B_1[g_i, R_{ik}]\}. \end{aligned} \tag{4}$$

By the corollary to Lemma 2, 2 $E[u,R_k]$ can be transformed. This gives, taking $\varphi_i = R_{ik}$, $\psi_i = u_i$

$$2E[u, R_k] = B[M_i[R_k], u_i] = -B_1[M_i[F_k], u_i] + B_2[M_i[R_k], k_i]. \tag{5}$$

Consequently, we have

$$\begin{aligned} 2E[u, R_k] + H[f_i, R_{ik}] + B_1[M_i[F_k], u_i] + B_1[g_i, R_{ik}] \\ = H[f_i, R_{ik}] + B_1[g_i, R_{ik}] + B_2[M_i[R_k], k_i] \end{aligned} \tag{6}$$

which by equation (1) is equal to $u_k - \sigma_k$. Hence, (4) becomes

$$\alpha(u_k - \sigma_k) = J_1[u] + \alpha^2 J_2[R_k] - J_3[Q_k]. \tag{7}$$

Let

$$\begin{aligned} a &= J_1[u'] - K_1[u''] \geqq 0, \\ b_k' &= J_3[u', R_k'], \\ b_k'' &= K_3[u_i'', R_k''], \\ c_k &= J_2[R_k'] - K_2[R_k''] \geqq 0, \end{aligned} \tag{8}$$

where

$$2J_3[u', R_k'] = 2E[u', R_k'] + H[f_i, R_{ik}'] + B_1[M_i[F_k], u_i'] + B_1[g_i, R_{ik}'], \tag{9}$$

and

$$2K_3[u'', R_k''] = -2E[u'', R_k''] - B_2[M_i[u''], F_{ik}] + B_2[M_i[R_k''], k_i]. \tag{10}$$

Then, since each of the quantities on the right-hand side of equation (7) have upper and lower bounds, e.g., $K_1[u''] \leqq J_1[u] \leqq J_1[u']$ (7) can be turned into the following two inequalities.

$$a + \{(u_k - \sigma_k) - 2b_k'\}\alpha + c_k\alpha^2 \geqq 0, \tag{11}$$

$$a + \{2b_k'' - (u_k - \sigma_k)\}\alpha + c_k\alpha^2 \geqq 0. \tag{12}$$

Since the discriminants of the quadratic forms on the left of these inequalities must be non-positive, we arrive at two further inequalities.

$$2(b_k' - \sqrt{ac_k}) \leqq u_k - \sigma_k \leqq 2(b_k' + \sqrt{ac_k}). \tag{13}$$

$$2(b_k'' - \sqrt{ac_k}) \leqq u_k - \sigma_k \leqq 2(b_k'' + \sqrt{ac_k}). \tag{14}$$

Since both inequalities must hold simultaneously, we have, if $b_k' \leqq b_k''$,

$$2(b_k'' - \sqrt{ac_k}) \leqq u_k - \sigma_k \leqq 2(b_k' + \sqrt{ac_k}), \tag{15}$$

and if $b_k' \geqq b_k''$, the same inequality with b_k' and b_k'' interchanged.

It follows from (15) that the best approximation, u_k^* to u_k is given by

$$u_k^* = \sigma_k + b_k' + b_k''. \tag{16}$$

For this choice of u^*, we have that the "error", namely, $| u_k - u_k^* |$, is bounded by

$$| u_k - u_k^* | \leqq 2\sqrt{ac_k} - | b_k' - b_k'' |. \tag{17}$$

It is interesting to see the meaning of the quantities, b_k' and b_k'', on the right-hand side of (17). By applying the corollary to Lemma 2 to equation (9), we get

$$\begin{aligned} 2b_k' = 2J_3[u', R_k'] = H[f_i, R_{ik}'] + B_1[g_i, R_{ik}'] \\ + B_2[M_i[R_k'], k_i] - H[L_i[R_k'], u_i'] \\ + B_1[M_i[F_i] + M_i[R_k'], u_i']. \end{aligned} \tag{18}$$

The first three terms represent an approximation to $u_k - \sigma_k$ in terms of the functions R_{ik}'. The last two terms would vanish if the R_{ik}' were actually the regular part of the Green's matrix. Hence, $2b_k'$ is an approximation to $u_k - \sigma_k$. A similar interpretation can be given for $2b_k''$. Thus, u_k^* is actually the sum of an "exact term", σ_k, and the average of two approximations to the remainder, one from above and one from below.

7. Bounds on the derivatives. By differentiation of equation (6.1) with respect to the x_j coordinate of the pole, we get the relationship

$$u_{k,j} - \sigma_{k,j} = H[f_i, R_{ik,j}] + B_1[g_i, R_{ik,j}] + B_2[M_i[R_{k,j}], k_i]. \tag{1}$$

The boundary value problem solved by the functions $R_{ik,j}$ (k,j fixed indices) can be derived from Problem II by differentiation of equations (5.1) to (5.4). This gives

$$L_i[R_{k,j}] = 0 \qquad \text{in} \qquad D, \tag{2}$$

$$M_i[R_{k,j}] = -M_i[F_{k,j}] \qquad \text{on} \qquad \gamma_1, \tag{3}$$

$$R_{ik,j} = -F_{ik,j} \qquad \text{on} \qquad \gamma_2, \tag{4}$$

$$R_{ikl j} = R_{ik,lj} \qquad \text{in} \qquad D + \gamma. \tag{5}$$

Thus, the functions $R_{ik,j}$ satisfy conditions analogous to the conditions of Problem II. Then, just as we did previously, we can define upper and lower variational problems. The same can be done for the functions $Q_{ik,j} = u_i - \alpha R_{ik,j}$. The theory carries through in exactly the same way and enables us to find a best approximation $u_{k,j}^*$ and bounds on the quantity $| u_{k,j} - u_{k,j}^* |$

8. The boundary value problems of elasticity. As an illustration of the application of this method, we consider the equations of elasticity. We identify the u_i as the displacements. If e_{ij} represents a component of strain and S_{ij} a component of stress, then these quantities are defined by

$$e_{ij} = \tfrac{1}{2}(u_{ij} + u_{ji}), \tag{1}$$

$$S_{ij} = c_{ijkl}e_{kl}\,, \tag{2}$$

where the c_{ijkl} are symmetric. The boundary value problem is given by the equations

$$L_i[u] \equiv S_{ij,j} = f_i \quad \text{in} \quad D, \tag{3}$$

$$M_i[u] \equiv S_{ij}x_{j,n} = -g_i \quad \text{on} \quad \gamma_1\,, \tag{4}$$

$$u_i = k_i \quad \text{on} \quad \gamma_2\,, \tag{5}$$

$$u_{ij} = u_{i,j} \quad \text{in} \quad D + \gamma. \tag{6}$$

The quantity $E[u]$ is defined by

$$E[u] = \frac{1}{2}\iiint\limits_D e_{ij}S_{ij}\,dV. \tag{7}$$

The quantities a, b'_k, b''_k, c_k are given by

$$a = \frac{1}{2}\iiint\limits_D e'_{ij}S'_{ij}\,dV + \iiint\limits_D f_iu'_i\,dV + \iint\limits_{\gamma_1} g_iu'_i\,dA + \frac{1}{2}\iiint\limits_D e''_{ij}S''_{ij}\,dV - \iint\limits_{\gamma_2} S''_{ij}x_{j,n}k_i\,dA \tag{8}$$

$$2b'_k = \frac{1}{2}\iiint\limits_D \{e'_{ij}S_{ij}[R'_k] + e_{ij}[R'_k]S'_{ij}\}\,dV + \iiint\limits_D f_iR'_{ik}\,dV + \iint\limits_{\gamma_1} g_iR'_{ik}\,dA + \iint\limits_{\gamma_1} S_{ij}[F_k]x_{j,n}u'_i\,dA, \tag{9}$$

$$2b''_k = -\frac{1}{2}\iiint\limits_D \{e''_{ij}S_{ij}[R''_k] + e_{ij}[R''_k]S''_{ij}\}\,dV + \iint\limits_{\gamma_2} k_iS_{ij}[R''_k]x_{j,n}\,dA - \iint\limits_{\gamma_2} F_{ik}S''_{ij}x_{j,n}\,dA, \tag{10}$$

$$c_k = \frac{1}{2}\iiint\limits_D \{e_{ij}[R'_k]S_{ij}[R'_k]\}\,dV + \iint\limits_{\gamma_1} S_{ij}[F_k]x_{j,n}R'_{ik}\,dA + \frac{1}{2}\iiint\limits_D \{e_{ij}[R''_k]S_{ij}[R''_k]\}\,dV + \iint\limits_{\gamma_2} S_{ij}[R''_k]x_{j,n}F_{ik}\,dA, \tag{11}$$

where e'_{ij}, S'_{ij} mean that these quantities are defined by equations (1) and (2) replacing u_{ij} by u'_{ij}. Expressions such as $e_{ij}[R'_k]$, etc., have a similar meaning. It then follows that the best approximation to u_k is u^*_k where

$$u^*_k = \sigma_k + b'_k + b''_k \tag{12}$$

and, the error is bounded by

$$| u_k - u^*_k | \leqq 2\sqrt{ac_k} - | b'_k - b''_k |. \tag{13}$$

The stresses and strains can then be bounded by applying the same method to the derivatives of the u_i.

9. Boundary value problems of single, second order, elliptic equations. It is well known that any linear, self-adjoint, elliptic partial differential equation can be put in the form

$$\Delta u - du = f. \tag{1}$$

Uniqueness of the solution is guaranteed if we take $d(x,y) > 0$. The boundary value problems associated with this equation can be written

$$L[u] \equiv \frac{\partial}{\partial x} u_1 + \frac{\partial}{\partial y} u_2 - du = f \qquad \text{in} \qquad D, \tag{2}$$

$$M[u] \equiv u_1 x_n + u_2 y_n = -g \qquad \text{on} \qquad \gamma_1, \tag{3}$$

$$u = k \qquad \text{on} \qquad \gamma_2, \tag{4}$$

$$u_1 = \frac{\partial u}{\partial x}, \qquad u_2 = \frac{\partial u}{\partial y} \qquad \text{in} \qquad D + \gamma. \tag{5}$$

$E[u]$ is defined by

$$E[u] = \frac{1}{2} \iint_D (u_1^2 + u_2^2 + du^2)\, dx\, dy. \tag{6}$$

In this case, the quantities a, b', b'', c are given by

$$a = \frac{1}{2} \iint_D (u_1'^2 + u_2'^2 + du'^2)\, dx\, dy + \iint_D fu'\, dx\, dy$$

$$+ \int_{\gamma_1} gu'\, ds + \frac{1}{2} \iint_D (u_1''^2 + u_2''^2 + du''^2)\, dx\, dy - \int_{\gamma_2} k(u_1'' x_n + u_2'' y_n)\, ds, \tag{7}$$

$$2b' = \iint_D (u_1' R_1' + u_2' R_2' + du'\, R')\, dx\, dy$$

$$+ \iint_D fR'\, dx\, dy + \int_{\gamma_1} gR'\, ds + \int_{\gamma_1} (F_1 x_n + F_2 y_n) u'\, ds, \tag{8}$$

$$2b'' = -\iint_D (u_1''R_1'' + u_2''R_2'' + du''\, R'')\, dx\, dy$$
$$+ \int_{\gamma_2} k(R_1''x_n + R_2''y_n)\, ds - \int_{\gamma_2} F(u_1''x_n + u_2''y_n)\, ds, \tag{9}$$

$$c = \frac{1}{2}\iint_D (R_1'^2 + R_2'^2 + dR'^2)\, dx\, dy$$
$$+ \int_{\gamma_1} (F_1x_n + F_2y_n)R'\, ds$$
$$+ \frac{1}{2}\iint_D (R_1''^2 + R_2''^2 + dR''^2)\, dx\, dy + \int_{\gamma_2} (R_1''x_n + R_2''y_n)F\, ds. \tag{10}$$

Then, just as before, we have

$$u^* = \sigma + b' + b'', \tag{11}$$

$$|\, u - u^* \,| \leqq 2\sqrt{ac} - |\, b' - b'' \,| \tag{12}$$

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