NON-UNIFORM SUPERSONIC FLOW*

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1. Introduction. The forces which act on an aerofoil in a non-uniform main stream have formed the subject of a number of papers (refs. 1-5). In all these investigations, incompressible flow only is considered. References 1-4 are concerned with the effect of non-uniformity of a two-dimensional main stream, while reference 5 deals with the effect of non-uniformity in spanwise direction. Much of the work mentioned was done in connection with wind tunnel problems, but that application may be less important in supersonic flow. However there are other cases of non-uniform supersonic flow which may be of practical interest, such as the flow around jet vanes.[1] In any case, the physical significance of the problem appears to warrant its investigation.

In the present paper, we shall analyse the problem of accelerated supersonic flow for the two-dimensional case on the basis of linearised theory. According to that theory, the effect of any small curvature in the main flow would be regarded as equivalent to a curvature of the aerofoil in opposite direction and there is therefore no need to consider the problem further. To be sure, acceleration will be associated with curvature somewhere in the main stream, but in order to consider the problem of acceleration in isolation one may suppose that the aerofoil has been placed in a plane of symmetry of the main flow so that the velocity component of the main flow in a direction normal to that plane vanishes throughout.

The acceleration of the main flow can be measured conveniently in terms of the variation of the static pressure. We shall derive closed analytical expressions for the pressure distribution around the aerofoil in the particular case of linear variation of the static pressure in the main stream. Simplified formulae will be given for sufficiently high Mach numbers. These do not apply to Mach numbers near unity, but in that region the entire theory, like the theory of uniform flow, will be less reliable in any case.

2. Linearization. As stated in the introduction, we assume that the aerofoil is placed, approximately, in a horizontal plane of symmetry of the flow, and we take the x-axis along the direction of the main stream in that plane, and the y-axis in a direction normal to it. Let $U(x,y)$, $V(x,y)$ be the velocity components of the main stream, and $U(x,y) + u(x,y)$, $V(x,y) + v(x,y)$ the velocity components in the presence of the aerofoil. Both the main flow and the flow induced by the aerofoil are supposed to be irrotational (at least within the approximation adopted) so that

$$\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0, \tag{2.1}$$

and

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{2.2}$$

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[1]This remark is due to a referee to whom I am indebted also for some further suggestions.

We infer the existence of an induced velocity potential $\phi(x,y)$ such that

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}. \tag{2.3}$$

In addition, the equation of compressible flow must be satisfied both in the presence, and in the absence, of the aerofoil so that

$$\left(1 - \frac{U^2}{a^2}\right)\frac{\partial U}{\partial x} + \left(1 - \frac{V^2}{a^2}\right)\frac{\partial V}{\partial y} - \frac{2UV}{a^2}\frac{\partial U}{\partial y} = 0, \tag{2.4}$$

and

$$\left(1 - \frac{(U+u)^2}{a^2}\right)\frac{\partial(U+u)}{\partial x} + \left(1 - \frac{(V+v)^2}{a^2}\right)\frac{\partial(V+v)}{\partial y} - 2\frac{(U+u)(V+v)}{a^2}\frac{\partial(U+u)}{\partial y} = 0 \tag{2.5}$$

where $a^2 = dp/d\rho$ is the square of the velocity of sound, p being the pressure and ρ the density. Strictly speaking, a is a function of $(U+u)^2 + (V+v)^2$, but we shall assume that the proportionate changes of pressure and density are small so that a may be regarded as constant.

Expanding (2.5) and taking into account (2.4) we obtain

$$\left(\left(1 - \frac{U^2}{a^2}\right) - \left(\frac{2Uu}{a^2} + \frac{u^2}{a^2}\right)\right)\frac{\partial U}{\partial x} - \left(\frac{2Uu}{a^2} + \frac{u^2}{a^2}\right)\frac{\partial u}{\partial x} + \left(1 - \frac{V^2}{a^2}\right)\frac{\partial v}{\partial y} - \left(\frac{2Vv}{a^2} + \frac{v^2}{a^2}\right)\left(\frac{\partial V}{\partial y} + \frac{\partial v}{\partial y}\right) - \frac{2UV}{a^2}\frac{\partial u}{\partial y} - \left(\frac{2Uv}{a^2} + \frac{2uV}{a^2} + \frac{2uv}{a^2}\right)\left(\frac{\partial U}{\partial y} + \frac{\partial u}{\partial y}\right) = 0 \tag{2.6}$$

We suppose that U is of the same order as a, so that U/a is of the order unity, but that U/a differs from unity numerically to the extent that $1 - U^2/a^2$ also is of order unity. Furthermore, we suppose that V, u, v are small compared with U and a. Regarding (2.6) as a linear form of the derivatives $\partial u/\partial x, \cdots, \partial V/\partial y$ and omitting from the coefficients terms which are small compared with other terms still included, we obtain

$$\left(1 - \frac{U^2}{a^2}\right)\frac{\partial u}{\partial x} - \frac{2Uu}{a^2}\frac{\partial U}{\partial x} + \frac{\partial v}{\partial y} - \frac{2Vv + v^2}{a^2}\frac{\partial V}{\partial y} - \frac{2UV}{a^2}\frac{\partial u}{\partial y} - \frac{2Uv}{a^2}\left(\frac{\partial U}{\partial y} + \frac{\partial u}{\partial y}\right) = 0.$$

Moreover, although that assumption may break down locally, we suppose that the relative orders of magnitude laid down above persist for the derivatives, so that we may omit the terms

$$\frac{2Vv + v^2}{a^2}\frac{\partial V}{\partial y}, \qquad \frac{2UV}{a^2}\frac{\partial u}{\partial y}, \qquad \text{and} \qquad \frac{2Uv}{a^2}\frac{\partial u}{\partial y}.$$

Then

$$\left(1 - \frac{U^2}{a^2}\right)\frac{\partial u}{\partial x} - \frac{2Uu}{a^2}\frac{\partial U}{\partial x} + \frac{\partial v}{\partial y} - \frac{2Uv}{a^2}\frac{\partial U}{\partial y} = 0. \tag{2.7}$$

Finally, we suppose that the variation of U in the direction of the y-axis is small compared with the variation of U along the chord, $\partial U/\partial y \ll \partial U/\partial x$ so that we may regard U as a function of x only. Since $\partial U/\partial y = \partial V/\partial x = 0$ in the plane of symmetry, this condition must be satisfied at least in the neighbourhood of the aerofoil. In consequence, we shall neglect $(2Uv/a^2)\partial U/\partial y$ while retaining $(2Uu/a^2)\partial U/\partial x$. Equation (2.7) then becomes

$$\left(1 - \frac{U^2}{a^2}\right)\frac{\partial u}{\partial x} - \frac{2U}{a^2}\frac{dU}{dx}u + \frac{\partial v}{\partial y} = 0.$$

Or, in terms of the induced velocity potential, ϕ,

$$\left(\frac{U^2}{a^2} - 1\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{2U}{a^2}\frac{dU}{dx}\frac{\partial \phi}{\partial x} - \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{2.8}$$

Let $\alpha = \alpha(x)$ be the local incidence at a point on the surface of the aerofoil, taken as positive when the slope dy/dx is positive. Then the boundary condition is

$$V + v = (U + u) \tan \alpha(x).$$

We shall assume, as usual, that this condition is satisfied at the projection of the point on the x-axis so that $V = 0$. Hence, for small α,

$$v = U(x)\alpha(x) \tag{2.9}$$

at the aerofoil.

To calculate the pressure, we have Bernoulli's equation,

$$\int \frac{dp}{\rho} + \frac{1}{2}[(U + u)^2 + (V + v)^2] = H \tag{2.10}$$

where H is constant throughout the medium. Or, neglecting terms of second order of smallness,

$$\int \frac{dp}{\rho} + \frac{1}{2}(U^2 + 2Uu) = H. \tag{2.11}$$

Let p_0, ρ_0, U_0, u_0 be pressure, density, and longitudinal velocity components at an arbitrary but fixed point. The proportionate changes of p and ρ are small, by assumption, so that if

$$\rho = \rho_0(1 + s),$$

then $p = k\rho^\gamma \doteqdot k\rho_0^\gamma(1 + \gamma s) = p_0(1 + \gamma s)$, $dp = p_0\gamma\, ds$, and so

$$\int_{p_0}^{p} \frac{dp}{\rho} = \frac{p_0\gamma}{\rho_0}\int_0^s \frac{ds}{1 + s} \doteqdot \frac{p_0\gamma s}{\rho_0} = \frac{p_0(1 + \gamma s) - p_0}{\rho_0} = \frac{p - p_0}{\rho_0}.$$

Accordingly, (2.11) may be rewritten in the form

$$p - p_0 = -\frac{\rho_0}{2}[(U^2 - U_0^2) + 2(Uu - U_0u_0)]. \tag{2.12}$$

It should be observed that we cannot now neglect Uu compared with U^2 since the difference $Uu - U_0u_0$ is not in general small compared with $U^2 - U_0^2$.

At a point upstream of the wing $u_0 = 0$, and so

$$p - p_0 = -\frac{\rho_0}{2}[U^2 - U_0^2 + 2Uu]. \tag{2.13}$$

When the aerofoil is absent,

$$p - p_0 = -\frac{\rho_0}{2}[U^2 - U_0^2],$$

so that the static pressure is a function of x only, within the approximation adopted here. More particularly, we shall assume that the main stream static pressure is a linear function of x. Assuming that p_0, ρ_0, U_0 correspond to the origin, we then have

$$p = p_0 + qx.$$

Hence, using (2.13),

$$U^2 = U_0^2 + \frac{2}{\rho_0}(p - p_0) = U_0^2 - \frac{2q}{\rho_0}x = U_0^2 + \lambda x, \qquad \lambda = -\frac{2q}{\rho_0}$$

Thus, $\lambda > 0$, $q < 0$ for accelerated flow and $\lambda < 0$, $q > 0$ for decelerated flow.

3. Accelerated flow. Let $M = U/a$ be the Mach number of the flow, and $\beta = \sqrt{M^2 - 1}$, so that M and β are functions of x only. Then (2.8) may be rewritten as

$$\beta^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{d}{dx}(\beta^2)\frac{\partial \phi}{\partial x} - \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{3.1}$$

Taking first the case of accelerated flow, $\lambda > 0$, we introduce a new variable r by

$$r = \frac{2a^2}{\lambda}\beta = \frac{2a^2}{\lambda}\sqrt{\beta_0^2 + \frac{\lambda}{a^2}x}, \qquad r > 0. \tag{3.2}$$

Then

$$x = \frac{\lambda r^2}{4a^2} - \frac{a^2}{\lambda}\beta_0^2,$$

and

$$\frac{dr}{dx} = \frac{1}{\sqrt{\beta_0^2 + \frac{\lambda}{a^2}x}} = \frac{2a^2}{\lambda r}.$$

Hence

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\frac{dr}{dx} = \frac{2a^2}{\lambda r}\frac{\partial \phi}{\partial r}.$$

Also

$$\frac{d}{dx}\beta^2 = \frac{d}{dx}\left(\frac{U_0^2 + \lambda x}{a^2} - 1\right) = \frac{\lambda}{a^2},$$

and so

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{2a^2}{\lambda r}\frac{\partial}{\partial r}\left(\frac{2a^2}{\lambda r}\frac{\partial \phi}{\partial r}\right) = \frac{4a^4}{\lambda^2 r^2}\frac{\partial^2 \phi}{\partial r^2} - \frac{4a^4}{\lambda^2 r^3}\frac{\partial \phi}{\partial r} = \frac{1}{\beta^2}\left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r}\right).$$

Substituting in (3.1)

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{3.3}$$

Now consider the partial differential equation

$$\frac{\partial^2\phi}{\partial\xi^2} - \frac{\partial^2\phi}{\partial\zeta^2} - \frac{\partial^2\phi}{\partial y^2} = 0. \tag{3.4}$$

This may be regarded (amongst other things) as the linearised equation of steady supersonic flow for a Mach number $\sqrt{2}$, where the main stream is directed along the ξ-axis (compare refs. 6, 7, 8). Particular solutions of (3.4) are obtained by means of source distributions over the ξ, ζ-plane, $\xi \geq 0$, thus,

$$\phi(\xi, y, \zeta) = \frac{1}{2\pi}\iint \frac{\sigma(\xi_0\,,\,\zeta_0)\,d\xi_0\,d\zeta_0}{\sqrt{(\xi-\xi_0)^2 - y^2 - (\zeta-\zeta_0)^2}} \tag{3.5}$$

where the area of integration on the right hand side is given by $(\xi - \xi_0)^2 - y^2 - (\zeta - \zeta_0)^2 > 0$, $\xi - \xi_0 > 0$. The source strength $\sigma(\xi_0, \zeta_0)$ is connected with the normal derivative of π at the ξ, ζ-plane by the relation

$$\left(\frac{\partial\phi}{\partial y}\right)_{y=+0,\,\xi_0>0} = -\frac{1}{2}\,\sigma(\xi_0\,,\,\zeta_0). \tag{3.6}$$

In (3.4), introduce new variables r, ψ by means of

$$\xi = r\cosh\psi, \qquad \eta = r\sinh\psi. \tag{3.7}$$

(3.4) then becomes

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\phi}{\partial\psi^2} - \frac{\partial^2\phi}{\partial y^2} = 0, \tag{3.8}$$

while (3.5) may be written as

$$\phi(r\cosh\psi, y, r\sinh\psi)$$
$$= \frac{1}{2\pi}\iint \frac{\sigma(r_0\cosh\psi_0\,,\,r_0\sinh\psi_0)r_0\,dr_0\,d\psi_0}{\sqrt{(r\cosh\psi - r_0\cosh\psi_0)^2 - y^2 - (r\sinh\psi - r_0\sinh\psi_0)^2}},$$

or

$$\phi(r\cosh\psi, y, r\sinh\psi) = \frac{1}{2\pi}\iint \frac{\sigma(r_0\cosh\psi_0\,,\,r_0\sinh\psi_0)r_0\,dr_0\,d\psi_0}{\sqrt{r^2 + r_0^2 - 2rr_0\cosh(\psi-\psi_0) - y^2}} \tag{3.9}$$

where the area of integration is given by

$$r\cosh\psi - r_0\cosh\psi_0 > 0, \qquad r^2 + r_0^2 - 2rr_0\cosh(\psi-\psi_0) - y^2 > 0.$$

Suppose now that σ is independent of ψ_0, $\sigma = \mathrm{f}(r_0)$.
Then (3.9) becomes

$$\phi(r\cosh\psi, y, r\sinh\psi)$$
$$= \frac{1}{2\pi}\int \mathrm{f}(r_0)r_0\,dr_0 \int \frac{d\psi_0}{\sqrt{r^2 + r_0^2 - 2rr_0\cosh(\psi-\psi_0) - y^2}} \tag{3.10}$$

We denote the integral with respect to ψ_0 on the right hand side by J. Substituting $\omega = \psi_0 - \psi$ in that integral, we obtain

$$J = \int \frac{d\omega}{\sqrt{r^2 + r_0^2 - 2rr_0\cosh\omega - y^2}} \tag{3.11}$$

where the range of integration is given by

$$r^2 + r_0^2 - 2rr_0 \cosh \omega - y^2 > 0$$

i. e.,

$$\cosh \omega \leq \frac{r^2 + r_0^2 - y^2}{2rr_0}. \tag{3.12}$$

Hence J is independent of ψ, $J = J(r,y,r_0)$. If follows that the function ϕ, which is given by (3.10), i. e., by

$$\phi = \frac{1}{2\pi} \int J(r, y, r_0) f(r_0) r_0 \, dr_0 \tag{3.13}$$

is independent of ψ. Now all integrals (3.9) satisfy (3.8), and so ϕ as given by (3.13) is a solution of (3.3). Moreover, by (3.6),

$$\left(\frac{\partial \phi}{\partial y}\right)_{y=+0, r_0>0} = -\frac{1}{2} f(r_0). \tag{3.14}$$

This suggests that we may solve (2.8), for the case of uniformly accelerated flow and for the boundary condition (2.9), by a suitable choice of $f(r_0)$. However, in addition to satisfying (2.9), the physically correct solution of (2.8) must vanish ahead of the Mach lines (characteristic curves) which emanate from the leading edge of the aerofoil in a down-stream direction. The equation of the characteristic curves of (2.8) is

$$\left(\frac{U^2}{a^2} - 1\right) dy^2 - dx^2 = 0 \tag{3.15}$$

so that the characteristic curves which emanate from the leading edge in downstream direction are given by

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{U^2}{a^2} - 1\right)^{-1/2} \quad \text{for} \quad y > 0, \\ \frac{dy}{dx} &= -\left(\frac{U^2}{a^2} - 1\right)^{-1/2} \quad \text{for} \quad y < 0. \end{aligned} \tag{3.16}$$

The corresponding curves in the r, y-plane as given by (3.2) are characteristic curves of (3.3). They have the equations

$$y = \pm r + \text{const}. \tag{3.17}$$

In particular, if the leading edge of the aerofoil is at the origin of coordinates in the x, y-plane, then the corresponding value of r is $r = (2a^2/\lambda)\beta_0$ and so the two Mach lines in question are given by

$$y = \pm(r - r') \qquad \text{where} \qquad r' = \frac{2a^2}{\lambda} \beta_0 . \tag{3.18}$$

To satisfy the condition that ϕ vanishes upstream of these curves we only have to assume $f(r_0) = 0$ for $r_0 < r'$ or

$$\phi = \frac{1}{2\pi} \int_{r'}^{r} J(r, y, r_0) f(r_0) r_0 \, dr_0 . \tag{3.19}$$

To carry out the integration in (3.11) we introduce the variable

$$t = \cosh \frac{\omega}{2}, \qquad d\omega = \frac{2\,dt}{\sqrt{t^2 - 1}}. \tag{3.20}$$

Then

$$\begin{aligned} r^2 + r_0^2 - 2rr_0 \cosh \omega - y^2 &= (r + r_0)^2 - y^2 - 4rr_0 \cosh^2 \frac{\omega}{2} \\ &= ((r + r_0)^2 - y^2)\left(1 - k^2 \cosh^2 \frac{\omega}{2}\right) \end{aligned}$$

where

$$k^2 = \frac{4rr_0}{(r + r_0)^2 - y^2}. \tag{3.21}$$

We observe that, for $r > 0$, $r_0 > 0$,

$$(r + r_0)^2 \geq 4rr_0$$

so that, given r, r_0, the right hand side of (3.21) is positive but not greater than 1 for sufficiently small $| y |$, and we may assume $0 < k \leq 1$. For such y, (3.11) becomes

$$\begin{aligned} J &= \frac{4}{\sqrt{(r + r_0)^2 - y^2}} \int_1^{1/k} \frac{dt}{\sqrt{(t^2 - 1)(1 - k^2t^2)}} \\ &= \frac{4}{\sqrt{(r + r_0)^2 - y^2}} K'(k) = \frac{4}{\sqrt{(r + r_0)^2 - y^2}} K(k') \end{aligned} \tag{3.22}$$

where K, K' are the complete elliptic integrals of the first kind, and the complementary modulus, k', is given by

$$k'^2 = 1 - \frac{4rr_0}{(r + r_0)^2 - y^2} = \frac{(r - r_0)^2 - y^2}{(r + r_0)^2 - y^2}.$$

In particular, for $y = 0$,

$$J = \frac{4}{r + r_0} K\left(\frac{r - r_0}{r + r_0}\right) = \frac{4}{r + r_0} K\left(\frac{1 - (r_0/r)}{1 + (r_0/r)}\right).$$

But

$$K\left(\frac{1 - (r_0/r)}{1 + (r_0/r)}\right) = \frac{1}{2}\left(1 + \frac{r_0}{r}\right)K'\left(\frac{r_0}{r}\right) = \frac{r + r_0}{2r} K'\left(\frac{r_0}{r}\right)$$

by Landen's transformation, so that, for $y = 0$,

$$J = \frac{2}{r} K'\left(\frac{r_0}{r}\right). \tag{3.23}$$

Substituting this expression in (3.19), we obtain

$$\phi = \frac{1}{\pi} \int_{r'}^{r} f(r_0) \frac{r_0}{r} K'\left(\frac{r_0}{r}\right) dr_0 \tag{3.24}$$

The function $f(r)$ is given by (2.7) and (3.14),

$$f(r) = 2U\alpha. \tag{3.25}$$

Now

$$r = \frac{2a^2}{\lambda}\sqrt{\frac{U^2}{a^2} - 1} = \frac{2a^2}{\lambda}\sqrt{M^2 - 1}, \qquad \text{by (3.2)}$$

and, for any modulus k, $K'(k) = K(\sqrt{1 - k^2})$, so that (3.24) may be replaced by

$$\phi(x, 0) = \frac{4a^3}{\pi\lambda}\int_{M_0}^{M} \frac{\alpha}{\sqrt{M^2 - 1}} K\left(\sqrt{\frac{M^2 - m^2}{M^2 - 1}}\right) m^2\, dm \tag{3.26}$$

where

$$M_0 = U_0/a, \qquad M = U(x)/a.$$

4. Approximate treatment. The elliptic integral K in (3.26) can be expressed in terms of the hypergeometric function F, as follows

$$K\left(\sqrt{\frac{M^2 - m^2}{M^2 - 1}}\right) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{M^2 - m^2}{M^2 - 1}\right), \tag{4.1}$$

so that

$$K\left(\sqrt{\frac{M^2 - m^2}{M^2 - 1}}\right) = \frac{\pi}{2}\left(1 + \frac{1}{4}\frac{M^2 - m^2}{M^2 - 1}\right), \tag{4.2}$$

except for higher powers of $(M^2 - m^2)/(M^2 - 1)$. With this approximation

$$\phi = \frac{2a^3}{\lambda}\left\{\frac{1}{\sqrt{M^2 - 1}}\int_{M_0}^{M} \frac{5\alpha}{4} m^2\, dm + \frac{1}{(M^2 - 1)^{3/2}}\int_{M_0}^{M} \alpha(m^2 - m^4)\, dm\right\} \tag{4.3}$$

and so

$$\begin{aligned} u &= -\frac{\partial\phi}{\partial x} = -\frac{\partial\phi}{\partial M}\frac{dM}{dx} \\ &= -\frac{U(x)\alpha(x)}{\sqrt{M^2 - 1}} + \frac{a}{\sqrt{M^2 - 1}}\left\{\frac{5}{4(M^2 - 1)}\int_{M_0}^{M} \alpha m^2\, dm \right. \\ &\qquad \left. + \frac{3}{4(M^2 - 1)^2}\int_{M_0}^{M} \alpha(m^2 - m^4)\, dm\right\} \end{aligned} \tag{4.4}$$

Assume now that α is constant in the interval $0 < x < c$. For points in that interval, the expression in the curly brackets on the right hand side of (4.4) equals, approximately,

$$\left(\frac{5\alpha M^2}{4(M^2 - 1)} + \frac{3\alpha(M^2 - M^4)}{4(M^2 - 1)^2}\right)(M - M_0) = \frac{1}{2}\alpha\frac{M^2}{M^2 - 1}(M - M_0).$$

Hence

$$u = -\frac{U(x)\alpha}{\sqrt{M^2 - 1}}(1 - h) \tag{4.5}$$

where

$$h = \frac{M(M - M_0)}{2(M^2 - 1)}. \tag{4.6}$$

We observe that $-U(x)\,\alpha/\sqrt{M^2 - 1}$ would be the value of u if the theory of uniform flow applied at the point under consideration, so that h may be regarded as a correction term. The approximate formula (4.5) applies only provided $M - M_0$ is small compared with M and M_0, and provided these Mach numbers are not too close to 1. For example, if $M_0 = 1.4$, $M = 1.6$, then $h = 0.10$.

An alternative expression for u is obtained by writing (4.5) in the form

$$u = -a\alpha\left(\frac{M}{\sqrt{M^2 - 1}} - \frac{M^2}{2(M^2 - 1)^{3/2}}(M - M_0)\right),$$

and expanding the right hand side at M_0. Then

$$u = -a\alpha\left(\frac{M_0}{\sqrt{M_0^2 - 1}} - \frac{2 + M_0^2}{2(M_0^2 - 1)^{3/2}}(M - M_0)\right) \tag{4.7}$$

except for higher powers of $M - M_0$.

The following expressions for the pressure are then obtained by substituting (4.5) and (4.7), respectively, in (2.13).

$$p = p_0 - \frac{\rho_0}{2}(U^2 - U_0^2) + \rho_0 \frac{U^2\alpha}{\sqrt{M^2 - 1}}(1 - h) \tag{4.8}$$

and

$$p = p_0 - \frac{\rho_0}{2}(U^2 - U_0^2) + \rho_0 \frac{U_0^2\alpha}{\sqrt{M_0^2 - 1}} + \rho_0 U_0 a\alpha \frac{M_0^2 - 4}{2(M_0^2 - 1)^{3/2}}(M - M_0). \tag{4.9}$$

It will be seen from (4.8) and (4.9) that for given constant α, p depends only on the local Mach number and on the leading edge Mach number and (free stream) pressure but not on the absolute rate of acceleration. In other words, p is the same for points x and $x' = \mu\, x$ on two aerofoils at incidence α if the main stream velocity and free stream pressure at the leading edge are the same in both cases, while the rate of change of the square of the main stream velocity, $d/dx(U^2)$, along the second aerofoil equals $1/\mu$ times the rate of change of the square of the main stream velocity along the first aerofoil. The same conclusion is reached if we express the pressure in terms of the more exact formula (3.26) for the velocity potential.

Assume now that the local incidence at the top surface is α for $0 < x < c/2$ and $-\alpha$ for $c/2 < x < c$. For $0 < x < c/2$ the pressure is then still given by (4.8) or (4.9), while for $c/2 < x < c$ we now have instead of (4.5),

$$u = \frac{U(x)\alpha}{\sqrt{M^2 - 1}}(1 - h') \tag{4.10}$$

where

$$h' = \frac{M[(M - M_1) - (M_1 - M_0)]}{2(M^2 - 1)}, \tag{4.11}$$

M_1 being the Mach number at $x = c/2$, $M = U(c/2)/a$.
Similarly, (4.7) is replaced by

$$u = a\alpha\left(\frac{M_0}{\sqrt{M_0^2 - 1}} - \frac{M_0^2 + 2}{2(M_0^2 - 1)^{3/2}}(M - M_0) + \frac{M_0^2}{(M_0^2 - 1)^{3/2}}(M_1 - M_0)\right). \tag{4.12}$$

Accordingly, the two expressions for the pressure are now

$$p = p_0 - \frac{\rho_0}{2}(U^2 - U_0^2) - \rho_0 \frac{U^2\alpha}{\sqrt{M^2 - 1}}(1 - h') \tag{4.13}$$

and

$$p = p_0 - \frac{\rho_0}{2}(U^2 - U_0^2) - \rho_0 \frac{U_0^2\alpha}{\sqrt{M_0^2 - 1}} - \rho_0 U_0^2 \alpha \frac{M_0^2 - 4}{2M_0(M_0^2 - 1)^{3/2}}(M - M_0)$$
$$- \rho_0 U_0^2 \alpha \frac{M_0}{(M_0^2 - 1)^{3/2}}(M_1 - M_0). \tag{4.14}$$

5. Decelerated flow. For decelerated flow, $\lambda < 0$, we put

$$r = -\frac{2a^2}{\lambda}\beta = -\frac{2a^2}{\lambda}\sqrt{\beta_0^2 + \frac{\lambda}{c^2}x}, \qquad r > 0. \tag{5.1}$$

Then

$$x = \frac{\lambda}{4a^2}r^2 - \frac{a^2}{\lambda}\beta_0^2, \qquad \frac{dr}{dx} = \frac{2a^2}{\lambda r}$$

as before. Thus the differential equation (3.3) still governs the flow, and we may adopt the same procedure as in section 3. However, r now decreases with increasing x and so the characteristic lines in the plane which correspond to the Mach lines (3.16) in the physical plane are

$$\begin{aligned} y &= r' - r \qquad \text{for} \quad y > 0, \\ y &= -(r' - r) \qquad \text{for} \quad y < 0, \end{aligned} \tag{5.2}$$

where $r' = -2a^2/\lambda\, \beta_0$. Accordingly, we now have instead of (3.19),

$$\phi = \frac{1}{2\pi}\int_r^{r'} J(r, y, r_0)f(r_0)r_0\, dr_0 \tag{5.3}$$

where $J(r,y,r_0)$ is still defined by (3.11). Carrying out the integration in (3.11), we have to take into account that r is now smaller than or equal to r_0. We then obtain, for $y = 0$,

$$J = \frac{2}{r_0}K'\left(\frac{r}{r_0}\right) \tag{5.4}$$

and so

$$\begin{aligned} \phi &= \frac{2}{\pi}\int_r^{r'} U\alpha K'\left(\frac{r}{r_0}\right) dr_0 \\ &= -\frac{4a^3}{\pi\lambda}\int_M^{M_0} \alpha K\left(\sqrt{\frac{m^2 - M^2}{m^2 - 1}}\right)\frac{m^2\, dm}{\sqrt{m^2 - 1}}. \end{aligned} \tag{5.5}$$

Using the approximation

$$K\left(\sqrt{\frac{m^2 - M^2}{m^2 - 1}}\right) = \frac{\pi}{2}\left(1 + \frac{1}{4}\frac{m^2 - M^2}{m^2 - 1}\right), \tag{5.6}$$

and adopting the same procedure as in section 3, we find the same approximate formulae for the longitudinal induced velocity and the pressure as given in section 3 for accelerated flow.

6. Lift and drag. Consider now a thin flat aerofoil of chord length c at incidence (Note that according to our definition, α is positive when dy/dx is positive.) Then the pressure is given by (4.8) or (4.9) for points on the top surface and, by familiar considerations of symmetry, by

$$p = p_0 - \frac{\rho_0}{2}(U^2 - U_0^2) - \rho_0 \frac{U^2\alpha}{\sqrt{M^2 - 1}}(1 - h) \tag{6.1}$$

or

$$p = p_0 - \rho_0(U^2 - U_0^2) - \rho_0 \frac{U_0^2\alpha}{\sqrt{M_0^2 - 1}} - \rho_0 U_0^2 \alpha \frac{M_0^2 - 4}{2M_0(M_0^2 - 1)^{3/2}}(M - M_0) \tag{6.2}$$

for points on the bottom surface of the aerofoil. Hence the pressure difference between top and bottom surfaces is

$$\Delta p = \frac{2\rho_0 U\alpha}{\sqrt{M^2 - 1}}(1 - h), \tag{6.3}$$

or

$$\Delta p = 2\rho_0 \frac{U_0^2\alpha}{\sqrt{M_0^2 - 1}} + \rho_0 U_0^2\alpha \frac{M_0^2 - 4}{M_0(M_0^2 - 1)^{3/2}}(M - M_0). \tag{6.4}$$

Using (6.4), we obtain for the resultant lift

$$L = -\int_0^c \Delta p\, dx = -\frac{1}{2}\rho_0 U_0^2 c \frac{4\alpha}{\sqrt{M_0^2 - 1}} - \rho_0 U_0^2 \alpha \frac{M_0^2 - 4}{M_0(M_0^2 - 1)^{3/2}} \int_0^c (M - M_0)\, dx$$

$$= -\frac{1}{2}\rho_0 U_0^2 c \frac{4\alpha}{\sqrt{M_0^2 - 1}} - \rho_0 U_0^2 \alpha c \frac{M_0^2 - 4}{M_0(M_0^2 - 1)^{3/2}} \frac{2M_2 + M_0}{3(M_2 + M_0)}(M_2 - M_0)$$

where M_2 is the free stream Mach number at the trailing edge. Replacing $(2M_2 + M_0)/3(M_2 + M_0)$ in the last term by $1/2$, in harmony with the approximations adopted so far, we obtain

$$L = -\frac{1}{2}\rho_0 U_0^2 c \frac{4\alpha}{\sqrt{M_0^2 - 1}}\left(1 + \frac{M_0 - 4}{4M_0(M_0^2 - 1)}(M_2 - M_0)\right). \tag{6.5}$$

Thus

$$C_L = -\frac{4\alpha}{\sqrt{M_0^2 - 1}}(1 + \eta) \tag{6.6}$$

where

$$\eta = \frac{M_0^2 - 4}{4M_0(M_0^2 - 1)} (M_2 - M_0). \tag{6.7}$$

The correction term, η, changes sign for $M_0 = 2$, and it is easy to show that for larger values of M_0, $M_0 > 2$, η remains negligible for reasonably small values of $M_2 - M_0$, $| M_2 - M_0 | \leq 0.2$, say.

The wave drag corresponding to L is

$$D = -L\alpha. \tag{6.8}$$

Next, we assume that the top surface of the aerofoil is at incidence α for $0 < x < c/2$, and at incidence $-\alpha$ for $c/2 < x < c$, while the bottom surface is at incidence $-\alpha$ for $0 < x < c/2$ and at incidence α for $c/2 < x < c$. This is the case of a symmetrical double wedge (diamond-shaped) aerofoil. The pressure at both top and bottom surfaces is then given by (4.8) or (4.9) and by (4.13) or (4.14) for the front and rear halves of the aerofoil respectively. The wave drag of the aerofoil therefore is

$$\begin{aligned} D &= 2\left(\int_0^{c/2} p\alpha\, dx - \int_{c/2}^{c} p\alpha\, dx\right) \\ &= \rho_0(U^2 - U_0^2)\frac{c\alpha}{4} + \frac{1}{2}\rho_0 U_0^2 c \frac{4\alpha^2}{\sqrt{M_0^2 - 1}}\left(1 + \frac{M_0^2 - 4}{4M_0(M_0^2 - 1)}(M_2 - M_0)\right) \\ &\qquad + \rho_0 U_0^2 c \frac{\alpha^2}{2} \frac{M_0}{(M_0^2 - 1)^{3/2}}(M_1 - M_0) \end{aligned}$$

where U is the free stream velocity at the trailing edge, $U = M_2\, a$. Now

$$\frac{M_1 - M_0}{M_2 - M_0} \doteq \frac{M_1^2 - M_0^2}{M_2^2 - M_0^2} = \frac{1}{2}.$$

Hence

$$D = \frac{1}{2}\rho_0 U_0^2 c\left[\frac{\alpha}{2}\left(\frac{M_2^2}{M_0^2} - 1\right) + \frac{4\alpha^2}{\sqrt{M_0^2 - 1}}\left(1 + \frac{3M_0^2 - 8}{8M_0(M_0^2 - 1)}(M_2 - M_0)\right)\right]. \tag{6.9}$$

and so

$$C_D = \frac{\alpha}{2}\left(\frac{M_2^2}{M_0^2} - 1\right) + \frac{4\alpha^2}{\sqrt{M_0^2 - 1}}\left(1 + \frac{3M_0^2 - 8}{8M_0(M_0^2 - 1)}(M_2 - M_0)\right). \tag{6.10}$$

We observe that in this formula α may also be interpreted as the maximum thickness-chord-ratio of the aerofoil. More precisely, the latter is $\tau = \tan \alpha$.

For a positive angle of attack the lift on this aerofoil is the same as for a flat aerofoil, while the wave drag is the sum of the drag at zero incidence as given by (6.9) and of the drag associated with the incidence (6.8).

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