QUARTERLY

OF

APPLIED MATHEMATICS

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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\surd$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t\, (a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354-372 (1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

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STATISTICAL THEORY IN RESEARCH

By R. L. Anderson, University of North Carolina, and T. A. Bancroft, Iowa State College. In press

Divided into two parts, this text first presents basic statistical theory, including the elementary principles of probability, population and sampling distribution theory with moment-generating functions, orthogonal linear forms, and the theory of estimation and tests of significance. A chapter on the theory behind the uses of Chi-square is included. Part II presents the theory of least squares and its use in the analysis of actual experimental data, including multiple regression, experimental design and variance component models.

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This new text presents a mathematical foundation in the principal branches of advanced mathematics which are used throughout physics and engineering. Clarity and understanding are primary considerations, although a reasonable degree of mathematical rigor has been achieved. The text shows how the fundamental formulations simplify to special cases which usually form the starting point in solutions of problems in physics and engineering.

LOGIC FOR MATHEMATICIANS

By J. Barkley Rosser, Cornell University. *International Series in Pure and Applied Mathematics.* In press

A comprehensive treatment of symbolic logic, this book offers an exposition of those principles of logic which are used by the mathematician in his daily work. The logic is developed simply from first principles. Then the formal concepts are related directly to the fundamentals of thinking in all fields of mathematics up through the theory of cardinal and ordinal arithmetic.

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BOOK REVIEWS

Partial differential equations in physics. By Arnold Sommerfeld. Academic Press, Inc., New York, 1949.

This book is essentially concerned with the treatment of the classical linear equations of mathematical physics. It opens with the expansion of a function in a trigonometric series and the extension of this notion to the Fourier Integral and generalized Fourier Series. The second chapter distinguishes the elliptic, hyperbolic, and parabolic partial differential equations and discusses the analytic character of the solutions of such equations. It also introduces the Green's function and Riemann function and the associated methods of integration. Chapter III is confined to heat conduction problems and Chapter IV treats a large variety of boundary value problems having spherical or cylindrical symmetry.

Chapter V deals with various eigenvalue problems and the final chapter is concerned with electromagnetic radiation. The book contains an excellent choice of very instructive problems and should be invaluable in teaching the technique of solving the classical type boundary value problems.

G. F. Carrier

Differential equations. By H. B. Phillips. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1951. viii + 149 pp. $3.00.

This is a text-book on the theory and application of the usual types of ordinary differential equations with elementary solutions. Types included are: first order with variables separable, other first order types, special types of second order equations, and equations with constant coefficients. Both methods of solution and development of the equations for physical systems are treated. Applications are taken from several fields with emphasis on mechanics. The discussion of variable mass problems is particularly clear. This book seems particularly suited for a first course in differential equations for science and engineering students.

E. H. Lee

Lectures on classical differential geometry. By Dirk J. Struik. Addison-Wesley Press, Inc., Cambridge 42, Mass., 1950. viii + 221 pp. $6.00.

This text, written by an outstanding geometer, contains an excellent vector treatment of the classical theory of Metric Differential Geometry in Euclidean three-space. The author's style is clear and concise and offers the reader a deep geometric insight into the subject. In addition, the text contains: (1) a very good collection of problems and numerous illustrative examples; (2) more than one hundred excellent figures; (3) many interesting historical notes.

In the first section of the text, the author discusses the theory of space curves and considers such topics as: curvature and torsion; contact; the Frenet-Serret formulas; the fundamental theorem; some results on ovals in differential geometry in the large. Secondly, the imbedding theory of surfaces in three space is discussed. Here, the stress is on curvature properties such as: Euler's theorem; the Dupin indicatrix, asymptotic directions; the fundamental theorem of surface theory. Next, the intrinsic properties of surfaces (geodesic curvature, geodesic coordinates, etc.) are considered. Finally, the following special topics are briefly discussed: conformal maps, minimal and ruled surfaces.

N. Coburn

Integral transforms in mathematical physics. By C. J. Tranter. Methuen & Co., Ltd., London, and John Wiley & Sons, Inc., New York, 1951. ix + 118 pp. $1.50.

This member of the Methuen Monograph Series contains a concise account of the application to physical problems of the Laplace, Fourier, Hankel, and Mellin transforms. After defining the transforms and their inversion formuli, the use of each is illustrated by relatively conventional example problems. The techniques of finding asymptotic and power series developments of solutions defined by not easily inverted transforms are discussed and Filon's numerical inversion technique is presented.

Finally, the author presents a chapter on the combined use of the finite transform and the relaxation technique.

G. F. Carrier

An introduction to electron optics. By L. Jacob. Methuen & Co., Ltd., London, and John Wiley & Sons, Inc., New York, 1951. x + 150 pp. $2.00.

The author has compressed a great deal of material, including a large number of diagrams, into this brief survey of what is now a subject of considerable magnitude, both on the experimental and theoretical sides. The book contains thirteen chapters, list of general reference books, bibliography chapter by chapter, and index. The subject matter falls into two roughly equal parts, the first eight chapters dealing with optical concepts and analogies, electrostatic lenses (their image-forming properties and aberrations), and phase-focussing, while the last five chapters deal with magnetic lenses and the deflection and properties of beams.

The purpose of the series, as set out on the jacket, is "to supply readers of average scientific attainment with a compact statement of the modern position in each subject." Such a hypothetical reader should undoubtedly get from the book a good general impression of what the problems of electron optics are, and much information of a more detailed nature, which he can supplement by looking up the papers cited in the bibliography. Nevertheless, the proper achievement of the worthy purpose of the series demands (in the opinion of the reviewer) much more forethought in arrangement and care in detail than are to be found in this book—more in fact than are necessarily demanded in a book intended for specialists, since the specialist has greater powers of digestion and correction than the hypothetical reader "of average scientific attainment."

It is safe to say that such a reader gets nothing but confusion from formulae in which the notation is insufficiently explained or which are marred by a general carelessness, and his plight is still worse when this carelessness is also present in the descriptive writing. Here are some examples:

p. 4: "where s_1, s_2 is the length of path" should read "where s_1, s_2, etc. are the lengths of the paths."

p. 14: $\mathbf{E}ds$ is used for a scalar product, without explanation.

p. 15: A minus sign is omitted at least five times in equations (IV.4).

pp. 15, 16: The unsophisticated reader will be puzzled to read that the curl of **E** is an important concept associated with the field, and a few lines further that **E** possesses no curl.

pp. 16, 17: The usual symbol for partial differentiation is used unnecessarily for ordinary derivatives, and changed to d suddenly.

pp. 37–39: The 3-page chapter on the trajectory equation deals only with motion in a plane of symmetry; the reader will be in a state of doubt as to whether this is the general case.

p. 58: The sentence "the beam angle has begun to exceed the value for which the sine condition $\sin\theta = \theta$ holds" would be vastly improved by the addition of the word "approximately"; but even then the words "the sine condition" make one wonder what the author had in mind.

These things are in a sense very trivial, but not so for a reader of average scientific attainment who is struggling to understand a difficult subject. Like other subjects in mathematical physics, electron optics has a wide range, from description of experiments on the one hand to mathematical theory on the other, and no author is to be blamed for failing to do justice to the whole range in a single short book. But the very difficulty (or impossibility) of the task is all the more reason for avoiding gaucheries of the type indicated above.

J. L. Synge

The classical theory of fields. By L. Landau and E. Lifshitz. Translated from the Russian by Morton Hamermesh. Addison-Wesley Press, Inc., Cambridge 42, Mass., 1951. ix + 354 pp. $7.50.

This book constitutes an attempt to treat the theory of electromagnetic and gravitational fields in a systematic manner. The authors restrict themselves to classical fields; no mention is made of the quantum theory of fields, although there is a reference to the fact that a limitation is placed on certain aspects of classical electrodynamics by quantum effects.

By way of an introduction to the theory of the electromagnetic field (which is presented principally in the four-dimensional notation of the special theory of relativity), the first two chapters are devoted to the principle of relativity and relativistic mechanics. These two chapters are written very clearly, but present only a review of those parts of special relativity which are required for the remainder of the book; they are far too condensed for a student not already familiar with the physical concepts underlying special relativity theory.

The following seven chapters deal with classical electromagnetic field theory. The topics considered in the first two of these chapters include the motion of charged particles in uniform fields, transformations and invariants of the fields, Maxwell's equations and the energy-momentum tensor of the field. Chapter 5 includes a discussion of constant fields and dipole, multipole and magnetic moments. It also touches on the field of a moving charge, but this is rather the subject matter of Chapter 8. Chapter 7, entitled propagation of light, deals with various aspects of geometrical optics and diffraction. Finally, as far as electromagnetic field theory is concerned, Chapters 6 and 9 deal with electromagnetic waves and radiation of such waves respectively.

The theory of electromagnetic fields is limited to vacuum and point charges, the problems of continuous media being completely omitted. It is given in both three and four-dimensional notation, with particular emphasis on the latter when there are definite advantages to be gained from its use.

In the eyes of the reviewer there is one regrettable feature in this part of the book, namely, the purely formal manner in which the theory is developed with its consequent neglect of the physics.

The last two chapters of the book treat particles in a gravitational field and the gravitational field equations. While an attempt is made to discuss the concepts underlying general relativity theory before launching into the mathematical formalism, it is the opinion of the reviewer that the authors pass off the subtle questions of the meaning of space and time in general relativity much too glibly.

There is a definite need for a book that covers electromagnetic and gravitational fields and points out the similarities and differences of the problems encountered in each case. This book is a good step in this direction for the reader who is at home with the basic concepts and physical principles of electrodynamics and special and general relativity.

L. C. Maximon

The algebra of vectors and matrices. By Thomas L. Wade. Addison-Wesley Press, Inc., Cambridge, 1951. ix + 189 pp. $4.50.

This is a clearly written text, on an elementary-intermediate level, dealing with the algebra of vectors and matrices. In particular, the author's treatment of matrix algebra is illustrated by many examples and should provide a good introduction to this subject. Although the point of view of the author is that of the mathematician, there are a sufficient number of interesting results in matrix theory to make this a useful book for the engineer.

The contents of the book are divided into two sections: Vector Algebra and Matrix Algebra. In the first section, the author treats the scalar and vector product and linear dependence of vectors in two, three and n-dimensional Euclidean space. The section on Matrix Algebra covers: addition and multiplication of matrices; groups of linear transformations; the characteristic equation and the Cayley-Hamilton theorem; the rank of a matrix and linear dependence; reductions of matrices, and of bilinear and quadratic forms to canonical form.

N. Coburn

Universal mechanics and Hamilton's quaternions. By Otto F. Fischer. The Axion Institute, Stockholm, 1951. vi + 356 pp. $10.00.

The quaternion retired from active participation in mathematical physics over fifty years ago, withered by the caustic wit of Oliver Heaviside (Electromagnetic Theory, Chap. III), and its reappearances since have been few and far between. As the title of this book implies, the author believes in the power of the quaternion. His Preface begins as follows:

"This is a book written by a civil engineer on universal mechanics with an attempt to introduce a certain order in its mathematical structure by means of the calculus of Hamilton's Quaternions."

"The term 'universal mechanics' refers to the mathematics of ordinary physics of motions, elasticity, hydrodynamics, aerodynamics, electromagnetism, together with relativistic and cosmic physics as well as quantum mechanics. With Eddington we may divide it into three main groups of atomic physics, ordinary 'molar' physics and cosmic physics."

"In attacking a subject of such magnitude with a new tool it is difficult to follow the regular roads. I have called the book a 'cavalcade' indicating a rather erratic path of approach. To begin with, the quaternions are applied in a rather haphazard manner to a series of examples, a wandering together with the reader through many 'countries' in which I have found new beauties and exhilarating outlooks. Gradually, however, new lines of structure begin to take shape, growing more and more clear until contours of mighty trunk lines seem to appear."

The book contains four chapters with the following headings: I. Extracts of ordinary vector mechanics. II. Simple quaternions in mechanics. III. Quadric quaternions and Eddington's E-numbers. IV. Further developments and ideas. Chapter II takes up more than half the book, and some idea of its variety will be given by the following samples from its 66 headings: Quaternation or rigid rotation. Maxwell's equations and Minkowski's space. Electromechanical parallelisms. Möbius' null-system and Plücker's line coordinates. On the theory of the general magneton. Eddington's electromagnetic energy-momentum tensor. On gravitation and restricted relativity. Einstein's theory of gravitation. Einstein's theory of 1950.

The book is rotaprinted. This does not seriously detract from the legibility of the letterpress, although it is a little too small for full reading comfort. But the formulae are cramped, being written in by hand on the original typescript in spaces inadequate to receive them. The practice of marking vectors with a straight underline and quaternions with a wavy underline is good, but the placing of a suffix directly under an underlined symbol is objectionable because it gives the appearance of a fraction. One particularly unfortunate slip must be mentioned; in the formula (2.1) on p. 39, defining a quaternion, the quaternion is equated to a vector.

This book will not, I fear, fulfil the purpose of bringing the quaternion into common use. For the merit of the quaternion lies in its algebraic simplicity, and a direct clear statement of fundamental properties, such as that given in L. Brand's Vector and Tensor Analysis (Wiley, New York, 1947), is better propaganda for the quaternion than pages of complicated formulae in which the basic simplicities are lost. However, one should not judge a book in terms of its propaganda value—that is a reaction induced in the reader by over-enthusiasm on the part of the author and by a hangover from the old days of quaternionic controversy. The book is not, as the author remarks, one to be read straight through. It contains a mine of formulae, ranging over a wide extent of physics, and it is in fact a book to keep on one's shelves and consult in connection with special interests. There is an adequate index and a list of references.

J. L. Synge

Table of the reciprocal of the gamma function for complex argument. By J. P. Stanley and M. V. Wilkes. Computation Centre, University of Toronto, 1950. 101 pp. $4.50.

The reciprocal of the Gamma function $\Gamma(Z)$ is calculated in the rectangular domain $-.5 \leq \operatorname{Re} Z \leq .5$, $0 \leq \operatorname{Im} Z \leq 1$, in increments of .01. The functional values are given to seven decimal places.

G. F. Carrier