

First, it is clear from (1) that $x''(t)$ is non-positive if and only if $x(t)$ is non-negative. It follows therefore from (2) that the (t, x) -graph represents a *convex* arch over the segment $[a, b]$ of the t -axis. Needless to say, the line $t = c$, where c denotes the abscissa belonging to the maximum (for $a \leq t \leq b$) of the ordinate, need not be a line of symmetry of the arch. But it is clear from the convexity of the latter that the triangle having the base $[a, b]$ and the vertex $t = c, x = x(c)$ is situated beneath the arch. In particular, this triangle has a smaller area than the convex region surrounded by the segment $[a, b]$ and the arch. In view of (3), this means that

$$A > \frac{1}{2}(b - a)M. \quad (5)$$

On the other hand, by a well-known inequality which goes back to Liapounoff,¹

$$\int_a^b \omega^2(t) dt > 4/(b - a). \quad (6)$$

Clearly, (6) and (5) together imply (4).

It is also clear that the inequality (5) turns into an inequality when the convex arch degenerates into two sides of the triangle considered above. On the other hand, the 4 occurring in (6) is the best possible absolute constant,² and the optimal nature of this 4 can also be concluded by confining the convex arch to be arbitrarily close to the triangular form.³ Accordingly, the *best* absolute constants, $\frac{1}{2}$ and 4, occurring in (5) and (6) respectively, can be approximated by the *same* sequence of examples. This proves the statement italicized after (4).

¹See G. Borg, Amer. Jour. Math. **71**, 67-70 (1949). For a proof embedding (6) into a general theory, cf. A. Wintner, *ibid.* **73**, 368-380 (1951). For refinements of (6), cf. P. Hartman and A. Wintner, *ibid.* **73** 885-890 (1951).

²E. R. van Kampen and A. Wintner, *ibid.* **59**, 270-274 (1937).

³G. Borg, *loc. cit.*

A FORMULA FOR THE NORMALIZATION CONSTANT IN EIGEN VALUE PROBLEMS*

BY GOTTFRIED GUDERLEY

Given the differential equation

$$y'' + \lambda r(x)y = 0 \quad (1)$$

with suitable homogeneous boundary conditions at the end of an interval which extends from $x = a$ to $x = b$. In (1) λ denotes an eigen value. During the following derivation λ is considered as arbitrary and only the boundary condition at one end of the interval, say point a , will be imposed, so the functions y need not be eigen functions. If $y = y_1$

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and $y = y_{II}$ are two such solutions for $\lambda = \lambda_I$ and $\lambda = \lambda_{II}$ respectively, then one has the relation

$$\int_a^b r(x)y_I y_{II} dx = \left. \frac{y'_{II}y_I - y'_I y_{II}}{\lambda_I - \lambda_{II}} \right|_a^b$$

(for a derivation see Ref. 1). Considering y as a function of x and λ , one obtains by the limiting process $\lambda_{II} \rightarrow \lambda_I$, $y_{II} \rightarrow y_I$, that

$$\int_a^b r(x)y^2 dx = \left(\frac{\partial y}{\partial x} \cdot \frac{\partial y}{\partial \lambda} - \frac{\partial^2 y}{\partial x \partial \lambda} \cdot y \right) \Big|_a^b$$

If now the function y is an eigen function and its dependence upon λ is explicitly known, this formula yields the normalization constant $\int_a^b r(x)y^2 dx$ immediately. Examples are the Jacobi polynomials or the functions of Ref. 2. In either case the functions y are expressed by hypergeometric series, the endpoints of the interval are singular points. The formula may even be useful in problems for which the eigen functions are determined by numerical integration for different values of λ and subsequent interpolation.

REFERENCES

1. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Interscience Publishers, New York, Vol. 1, Chap. V, Sec. 3.
2. G. Guderley, *Two-dimensional flow patterns with a free stream Mach number close to one*, A. F. Technical Report No. 6285.

A NOTE ON THE HODOGRAPH TRANSFORMATION*

By A. R. MANWELL (*University College Swansea*)

The hodograph transformation [1, 2] for plane compressible flow is too well known to need any discussion here, whilst its singularities have been very fully investigated in [3, 4, 5]. Briefly, it has been found that in cases where plane potential flow is impossible, the continuation of the solution in the hodograph plane may be regular but cannot be mapped into the physical solution. In any given hodograph solution there is no difficulty in deciding if a solution is mappable, for the condition of the vanishing of the Jacobian of the transformation is just that the level lines of the stream function in hodograph space should touch the fixed characteristics in the hodograph space. This criterion of course requires the calculation of all the level lines. Beyond this not much is known, but it has also been shown by various writers, particularly Friedrichs [6], that limit lines cannot appear initially in a certain sense at points inside the solutions. The purpose of this note is to show that the somewhat abstract results of Friedrichs, may be included in the simpler and rather more general statement, that for a wide class of hodograph solutions, *if the boundary streamline can be mapped then so can the whole field.*

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