Noting the structure of p and q; e.g., p_1 and p_2, in terms of f and g, we state an obvious corollary of Theorem 3.

Corollary. *Let* $|f^{(n)}(t)| < \epsilon$, $|g^{(n)}(t)| < \epsilon$ for $n = 1, 2, 3, 4$. *Then for* ϵ *sufficiently small, all the characteristic values of* (2) *are real.*

It is also clear from the structure of p and q that less restrictive, although perhaps more complicated, conditions on f and g than those in the hypothesis of the above corollary will yield the same conclusion.

The author wishes to thank Professors W. Feller and W. R. Sears for suggesting the problem and for helpful suggestions toward its solution.

References

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ON AN EQUATION OCCURRING IN THE HARMONIC ANALYSIS OF VISCOUS FLUID FLOW*

By RICHARD BELLMAN (*Stanford University*)

1. Introduction. It was shown by J. Kampé de Feriet[1] that the Fourier transform

$$z(w_1, w_2, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int \zeta(x, y) \exp[-i(w_1 x + w_2 y)]\, dx\, dy \tag{1}$$

of the vorticity, $\zeta(x, y)$, associated with the two-dimensional flow of an incompressible fluid extending over the entire (x,y)-plane, under mild conditions, satisfies the non-linear integro-differential equation

$$\begin{aligned} \frac{\partial}{\partial t} z(w_1, w_2, t) &= -v(w_1^2 + w_2^2) z(w_1, w_2, t) \\ &+ 2 \int_{-\infty}^{\infty} \int \left(\frac{\theta_1 w_2 - \theta_2 w_1}{\theta_1^2 + \theta_2^2} \right) z(\theta_1, \theta_2, t) \bar{z}(\theta_1 + w_1, \theta_2 + w_2, t)\, d\theta_1\, d\theta_2, \end{aligned} \tag{2a}$$

and the boundary condition

$$z(w_1, w_2, 0) = \phi(w_1, w_2). \tag{2b}$$

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[1]J. Kampé de Feriet, *Harmonic analysis of the two-dimensional flow of an incompressible viscous fluid*, Q. Appl. Math. **6**, 1-13 (1948).

It seems quite difficult to determine the properties of the solution of 2a for general ϕ. Consequently, it may be of interest to indicate the following theorem which is an analogue of the well-known result of Poincaré and Liapounoff[2] concerning the stability of solutions of non-linear ordinary differential equations.

THEOREM: *If* $\mathrm{Max}_w \mid \phi(w_1, w_2) \mid$ *is sufficiently small, there is a solution to* (2a) *and* (2b) *which is unique, and satisfies the inequality*

$$\mid z(w_1, w_2, t) \mid \leq \frac{8 \operatorname{Max} \mid \phi(w_1, w_2) \mid}{[1 + v(w_1^2 + w_2^2)t]^2}, \tag{3}$$

for all w_1, w_2 *and* $t \geq 0$.

From this we conclude that the solution $z = 0$ *of* (2a) *is stable.*

By the expression "sufficiently small" we mean that there exists a constant $c = c(v)$ with the property that $\operatorname{Max} \mid \phi \mid \leq c$ suffices to establish (3). The constant c may be determined from the analysis below. However, we feel that the value of c obtained in the course of our proof has no particular significance. At the expense of decreasing c we can replace the exponent 2 on the right side of (3) by any arbitrary n.

While the general method, namely that of applying the technique of successive approximations, is standard, the details are not as simple as might be believed upon first viewing the equation. It might be expected that in place of (3) one could assert

$$\mid z(w_1, w_2, t) \mid \leq c_1 \operatorname{Max} \mid \phi(w_1, w_2) \mid \exp[-v(w_1^2 + w_2^2)t], \tag{4}$$

provided that, as above, $\mathrm{Max}_w \mid \phi(w_1, w_2) \mid$ is sufficiently small. This result seems difficult to prove, and it is quite possible that it is not true.

2. Proof of theorem. From (2) we obtain, assuming for the moment that the solution exists,

$$\begin{aligned} &z(w_1, w_2, t) \\ &\quad = \phi(w_1, w_2) \exp[-v(w_1^2 + w_2^2)t] + \int_0^t \exp[-v(w_1^2 + w_2^2)(t - t_1)] J(z)\, dt_1, \end{aligned} \tag{1}$$

where we have set

$$J(z) = 2 \int_{-\infty}^{\infty} \int \frac{(\theta_1 w_2 - \theta_2 w_1)}{w_1^2 + w_2^2} \bar{z}(\theta_1 + w_1, \theta_2 + w_2, t) z(\theta_1, \theta_2, t)\, d\theta_1\, d\theta_2 .$$

This equation is solved by the method of successive approximations, by means of the algorithm

$$\begin{aligned} z_0 &= \phi(w_1, w_2) \exp[-v(w_1^2 + w_2^2)t] \\ z_{n+1} &= z_0 + \int_0^t \exp[-v(w_1^2 + w_2^2)(t - t_1)] J(z_n)\, dt_1, \qquad n = 0, 1, \cdots . \end{aligned} \tag{3}$$

The first step of the proof consists of showing that the sequence $\{z_n\}$ is uniformly bounded by an appropriate function of w_1, w_2 and t, namely

$$\mid z_n \mid \leq \frac{8\alpha}{[1 + v(w_1^2 + w_2^2)t]^2}, \qquad n = 0, 1, \cdots ,$$

[2]R. Bellman, *On the boundedness of solutions of non-linear differential and difference equations*, Trans. Amer. Math. Soc. **62**, 357-86 (1947).

for all w_1, w_2, t, where we have set, for the sake of convenience,

$$\alpha = \operatorname*{Max}_{w} | \phi(w_1, w_2) |. \tag{5}$$

Throughout we shall use, without further mention, the following simple inequalities,

$$\begin{aligned} &\text{(a)} \quad e^{-x} \leq 1/(1 + x), \qquad x \geq 0 \\ &\text{(b)} \quad a/b \leq (1 + ax)/(1 + bx) \leq 1, \qquad b \geq a \geq 0, \qquad x \geq 0. \end{aligned} \tag{6}$$

We turn now to the proof of (4). The result is clear for $n = 0$, since

$$\begin{aligned} | z_0 | &\leq \operatorname*{Max}_{w} | \phi | \exp [-v(w_1^2 + w_2^2)t] \leq \frac{\alpha}{[1 + v/2(w_1 + w_2)t]^2} \\ &\leq 4\alpha/[1 + v(w_1 + w_2)t]^2. \end{aligned} \tag{7}$$

To treat the general case, we proceed by induction, assuming that (4) holds for $n = 0, 1, \cdots, N$, and then proving it for $N + 1$. We first require upper bounds for $J(z_N)$. Introducing polar coordinates,

$$\begin{aligned} \theta_1 &= R \cos \psi, \qquad w_1 = r \cos \theta \\ \theta_2 &= R \sin \psi, \qquad w_2 = r \sin \theta, \end{aligned} \tag{8}$$

$$J(z_N) = 2r \int_0^\infty \int_0^{2\pi} \sin (\psi - \theta) z_N(\theta_1, \theta_2, t) \bar{z}_N(\theta_1 + w_1, \theta_2 + w_2, t)\, dR\, d\psi. \tag{9}$$

Applying our inductive hypothesis,

$$\begin{aligned} | J(z_N) | &\leq 128\alpha^2 r \int_0^\infty \int_0^{2\pi} \left[\frac{1}{(1 + vR^2t)^2 \cdot [1 + vt(R^2 + 2Rr \cos (\theta - \psi) + r^2)]^2} \right] dR\, d\psi \\ &\leq c_1 \alpha^2 r \int_0^\infty \left[\frac{dR}{(1 + vR^2t)^2 [1 + v(R - r)^2 t]^2} \right], \end{aligned} \tag{10}$$

where $c_1 = 256\pi$.

This last integral is now split into three parts,

$$c_1 r \int_0^\infty = c_1 r \int_0^{r/2} + c_1 r \int_{r/2}^{2r} + c_1 r \int_{2r}^\infty = J_1 + J_2 + J_3, \tag{11}$$

which we discuss separately. We have, since $r - R \geq r/2$ for $0 \leq R \leq r/2$,

$$\begin{aligned} J_1 &\leq \frac{c_1 r}{(1 + vr^2t/4)^2} \int_0^{r/2} \frac{dR}{(1 + vr^2t)^2} < \frac{16c_1 r}{(1 + vr^2t)^2} \int_0^\infty \frac{dR}{(1 + vR^2t)^2} \\ &< \frac{c_2 r}{t^{1/2}} \frac{1}{(1 + vr^2t)^2}, \end{aligned} \tag{12}$$

where

$$c_2 = 16c_1 \int_0^\infty ds/(1 + vs^2)^2.$$

Turning to J_2, we obtain

$$J_2 \leq \frac{3c_1}{2} r^2 \frac{1}{(1 + vr^2t/4)^2} \leq \frac{24c_1r^2}{(1 + vr^2t)^2} . \tag{13}$$

Finally, since $R - r \geq r$ for $2r \leq R < \infty$,

$$J_3 \leq \frac{c_1M}{(1 + vr^2t)^2} \int_{2r}^{\infty} \frac{dR}{(1 + vR^2t)^2} < \frac{c_3r}{t^{1/2}} \frac{1}{(1 + vr^2t)^2} \tag{14}$$

where

$$c_3 = c_1 \int_0^{\infty} ds/(1 + vs^2)^2.$$

Collecting the results,

$$\begin{aligned} | J(z_N) | &\leq \alpha^2(J_1 + J_2 + J_3) \\ &\leq \alpha^2\left[\frac{(c_2 + c_3)r}{t^{1/2}} \frac{1}{(1 + vr^2t)^2} + \frac{24c_1r^2}{(1 + vr^2t)^2}\right]. \end{aligned} \tag{15}$$

Applying these inequalities to (3), the result is

$$\begin{aligned} | z_{N+1} | &\leq | z_0 | + \int_0^t \exp [-vr^2(t - t_1)] \, | J(z_N) | \, dt_1 \\ &\leq \frac{4\alpha}{(1 + vr^2t)^2} + (c_2 + c_3)\alpha^2 r \int_0^t \frac{\exp [-vr^2(t - t_1)] \, dt_1}{(1 + vr^2t_1)^2(t_1)^{1/2}} \\ &\quad + 24c_1\alpha^2r^2 \int_0^t \frac{\exp [-vr^2(t - t_1)] \, dt_1}{(1 + vr^2t_1)^2} . \end{aligned} \tag{16}$$

The first integral may be written

$$r \int_0^t = r \int_0^{t/2} + r \int_{t/2}^t = I_1 + I_2 . \tag{17}$$

Then

$$\begin{aligned} I_1 &= r \int_0^{t/2} \frac{\exp [-vr^2(t - t_1)]}{(1 + vr^2t_1)^2} \frac{dt_1}{(t_1)^{1/2}} \leq \exp [-vr^2t/2] r \int_0^{t/2} \frac{dt_1}{(1 + vr^2t_1)^2(t_1)^{1/2}} \\ &\leq \frac{1}{(1 + vr^2t/4)^2} r \int_0^{\infty} \frac{dt_1}{(1 + vr^2t_1)^2(t_1)^{1/2}} \leq \frac{c_4}{(1 + vr^2t)^2} , \end{aligned} \tag{18}$$

where

$$c_4 = 16 \int_0^{\infty} ds/(1 + vs)^2 s^{1/2}.$$

The second integral leads to

$$I_2 = r \int_{t/2}^{t} \frac{\exp[-vr^2(t - t_1)]\, dt_1}{(1 + vr^2 t_1)^2 (t_1)^{1/2}} \leq \frac{r}{(1 + vr^2 t/2)^2} \int_{t/2}^{t} \exp[-vr^2(t - t_1)] \frac{dt_1}{(t_1)^{1/2}}$$

$$\leq \frac{4r}{(1 + vr^2 t)^2} \int_0^{t/2} \frac{\exp[-vr^2 t_1]\, dt_1}{(t - t_1)^{1/2}} \leq \frac{4r}{(1 + vr^2 t)^2} \int_0^{\infty} \frac{\exp[-vr^2 t_1]\, dt_1}{(t_1)^{1/2}} \tag{19}$$

$$\leq \frac{c_5}{(1 + vr^2 t)^2},$$

where

$$c_5 = 4 \int_0^{\infty} \frac{e^{-vs}\, ds}{s^{1/2}}.$$

The second integral in (16) is broken up in like fashion into I_3, the integral over $[0, t/2]$, and I_4, the integral over $[t/2, t]$. The first integral satisfies the inequality

$$I_3 \leq r^2 \exp[-vr^2 t/2] \int_0^{\infty} \frac{dt_1}{(1 + vr^2 t_1)^2} \leq \frac{c_6}{(1 + vr^2 t)^2}, \tag{20}$$

where

$$c_6 = 16 \int_0^{\infty} ds/(1 + vs)^2,$$

while the second satisfies the inequality

$$I_4 \leq c_7/(1 + vr^2 t)^2, \tag{21}$$

with $c_7 = 1/v$. Collating these results, we obtain

$$\begin{aligned} |z_{N+1}| &\leq \frac{4\alpha}{(1 + vr^2 t)^2} + \frac{\alpha^2}{(1 + vr^2 t)^2} [(c_2 + c_3)(c_4 + c_5) + 24c_1(c_6 + c_7)] \\ &\leq \frac{8\alpha}{(1 + vr^2 t)^2}, \end{aligned} \tag{22}$$

provided that

$$\alpha \leq 2/[(c_2 + c_3)(c_4 + c_5) + 24c_1(c_6 + c_7)]. \tag{23}$$

This completes the induction.

We must now show that z_N converges to a solution of the original functional equation. In the usual manner, this is accomplished by demonstrating the uniform convergence of the series $\sum_{n=0}^{\infty} (z_{n+1} - z_n)$. From (2) we obtain

$$|z_1 - z_0| \leq 2 \int_0^t \exp[-v(w_1^2 + w_2^2)(t - t_1)]\, |J(z_0)|\, dt_1 \leq \frac{c_8 \alpha^2}{(1 + vr^2 t)^2}. \tag{24}$$

It now follows by induction, using the same procedure as above, that there exists a constant c_9 such that

$$|z_{N+1} - z_N| \leq \frac{(c_9 \alpha)^{N+1}}{(1 + vr^2 t)^2}. \tag{25}$$

Hence if

$$\alpha \leq \text{Min } [1/c_9 \, , \, 1/(c_4 + c_5 + c_6 + c_7)], \tag{26}$$

we have uniform convergence of z_N to a function z over the entire $(w_1 \, , \, w_2)$ plane and the infinite t-interval, $0 \leq t \leq \infty$. It follows from the uniform convergence that we can pass to the limit as $N \to \infty$ under the integral sign in (3), obtaining (1). Differentiation of (1) yields the original equation.

The uniqueness is now established in the standard fashion.

BOOK REVIEWS

Table of the Bessel functions $Y_0(z)$ and $Y_1(z)$ for complex arguments. Prepared by the Computation Laboratory, National Bureau of Standards. Columbia University Press, New York, 1950. xi + 427 pp. $7.50.

This volume supplements the earlier volume of tables of $J_0(z)$ and $J_1(z)$ for complex arguments [see Q. of Appl. Math. **2**, 276 (1944) and **6**, 95 (1948)]. The main tables give $Y_0(\rho e^{i\varphi})$ and $Y_1(\rho e^{i\varphi})$ to ten decimal places for $\rho = 0(.01)10$ and $\varphi = 0(5°)90°$. Auxiliary tables give $Y_0(\rho e^{i\varphi}) - (2/\pi)J_0(\rho e^{i\varphi}) \log \rho$ and $Y_1(\rho e^{i\varphi}) - (2/\pi)J_1(\rho e^{i\varphi}) \log \rho + (2/\pi\rho)e^{-i\varphi}$, the complex zeros of Bessel functions, and five-point Lagrangian interpolation coefficients.

W. Prager

The inelastic behavior of engineering materials and structures. By Alfred M. Freudenthal. John Wiley & Sons, Inc., New York and Chapman & Hall, Limited, London, 1950. xvi +587 pp. $7.50.

The amazing scope of the book and its detailed coverage of so many facets of inelastic action bear eloquent testimony to the author's wide-spread reading and his own research. Quantum statistics, conventional metallurgy, mathematical theories of plasticity, visco-elasticity, stress analysis solutions, and design criteria for metals and concrete are all presented from a unified and extremely interesting point of view. The reader is made to feel equally familiar with electron clouds, simple and complex mechanical models of the behavior of real materials, Brownian motion, limit design, and testing machines.

The only objection to be noted is that little indication is given at the highly controversial nature of the field. Opinions are often stated as facts. For example, this reviewer believes that much of the material on thermodynamics and the mechanical equation of state is based on demonstrably over-simple and probably incorrect assumptions about the dissipated work. However, read with an open and skeptical mind the book is invaluable.

D. C. Drucker

Electromagnetic fields. Theory and application. Volume I: Mapping of Fields. By Ernst Weber. John Wiley & Sons, Inc., New York and Chapman & Hall Limited, London, 1950. xiv + 590 pp. $10.00.

The author has divided electromagnetic theory into static electric and magnetic fields on one hand