$$A = (1 - M^2)^{1/2}\mathfrak{A} \tag{6.2}$$

where the representative chord shown in Fig. 3 is taken as the mean aerodynamic chord, $(dC_L/d\alpha)$ is the three dimensional lift curve slope, corrected for compressibility, and $\mathfrak{A}$ is the aspect ratio.

If the results of the foregoing section are examined, it is found that the dominant term was $\ln k$, and now is effectively replaced by $-\ln A$. It appears, therefore, that the two dimensional results are directly applicable only to very large aspect ratio wings, although the effects that we have discussed may be of some considerable importance in many applications. We remark that, in view of Reissner's results for incompressible flow, we should expect the discrepancies between the two and three dimensional results to be less pronounced for larger values of the reduced frequency, so that it should not be inferred from our results that the two-dimensional work of Possio is necessarily inapplicable in typical flutter problems.

RETARDED POTENTIALS OF SUPERSONIC FLOW*

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Introduction. This paper deals with a space filled with ideal compressible fluid moving at velocities greater than sonic velocity. A formula will be derived for the velocity potential at any point in space when the conditions on the disturbing surfaces are given. The analysis is based on the linearized partial differential equation; therefore the disturbances must be small.

Use is made of the concept of the finite part of an infinite integral. This concept was first defined by Hadamard (Reference 1). It is used to overcome the difficulty that arises due to the elementary solution becoming infinite on the machcone emanating from the origin of the source. *Hyperbolic vector operators and definitions.* In reference (2) a vector operator was used in treating source and vortex distributions in the linearized theory of steady supersonic flow. This and other operators are defined below.

The hyperbolic gradient operator is defined as

$$\nabla h = -\mathbf{i}\beta^2 \frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{1}$$

where β is a constant, and **i**, **j**, and **k** are unit vectors in the x, y, z directions respectively.

The divergence of the hyperbolic gradient operator is defined as

$$\nabla^2 h = -\beta^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2}$$

The vector $\mathbf{n}^1$ is defined by the equation below

$$\mathbf{n}' = -\mathbf{i}\beta^2\nu_1 + \mathbf{j}\nu_2 + \mathbf{k}\nu_3 \tag{3}$$

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where ν_1, ν_2, and ν_3 are the direction cosines of the normal to the surface to which it is associated.

The divergence theorem can be expressed as

$$\oint \mathbf{E}\cdot\mathbf{n}\, da = \int \nabla\cdot\mathbf{E}\, dv$$

where **E** is a vector function and da and dv are elements of area and volume respectively. The normal to the element of area, da, is denoted by the vector **n**. The divergence theorem may also be expressed as

$$\oint \mathbf{W}\cdot\mathbf{n}'\, da = \int \nabla\mathbf{h}\cdot\mathbf{W}\, dv \tag{4}$$

since

$$\mathbf{E}\cdot\mathbf{n} = \mathbf{W}\cdot\mathbf{n}'$$

and

$$\nabla\cdot\mathbf{E} = \nabla\mathbf{h}\cdot\mathbf{W}$$

if

$$\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z = -\mathbf{i}\beta^2 W_x + \mathbf{j}W_y + \mathbf{k}W_z$$

The symbol f before an integral sign will denote the finite part of the integral.

Let **E** be a vector and ϕ and ψ be scalar functions of x, y, z. By direct expansion the following identities can be proved:

$$\nabla\mathbf{h}\psi\cdot\nabla\phi = \nabla\mathbf{h}\phi\cdot\nabla\psi \tag{5}$$

$$\nabla\mathbf{h}\cdot\psi\mathbf{E} = \psi\nabla\mathbf{h}\cdot\mathbf{E} + \mathbf{E}\cdot\nabla\mathbf{h}\psi \tag{6}$$

If

$$R = \left| [(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2(z-\zeta)^2]^{1/2} \right|$$

then

$$\nabla\left(\frac{1}{R}\right)\cdot\nabla hR = \frac{\beta^2}{R^2} \tag{7}$$

Theoretical development. In supersonic flow an occurrence at a point $Q(\xi, \eta, \zeta)$ does not produce immediate effects at a distant point $P(x, y, z)$ but only during two later intervals when its influence is passing P. The state of the field at the point P at time t is determined by states existing at points Q at the two earlier times (Reference 3)

$$t_1 = t - \frac{(x-\xi)M}{\beta^2 c} + \frac{R}{\beta^2 c} \tag{8}$$

$$t_2 = t - \frac{(x-\xi)M}{\beta^2 c} - \frac{R}{\beta^2 c} \tag{9}$$

where M is the stream velocity divided by the sonic velocity, c, and β^2 is given by

$$\beta^2 = M^2 - 1$$

The value of the potential at point P, $\phi(x, y, z, t)$, is associated with the values of the disturbance at points Q at times

$$t - \frac{(x-\xi)M}{\beta^2 c} + \frac{R}{\beta^2 c} \qquad \text{and} \qquad t - \frac{(x-\xi)M}{\beta^2 c} - \frac{R}{\beta^2 c}$$

In the following discussion the notation $\phi(x, y, z, t)$ will be used to denote the value of the potential at P at time t. $\phi(x, y, z, t)$ will be divided into two functions $\phi_1(x, y, z, t)$ and $\phi_2(x, y, z, t)$. $\phi_1(x, y, z, t)$ denotes the contributions to the potential at P due to disturbances at points Q at times t_1, and $\phi_2(x, y, z, t)$ denotes the contributions to the potential at P due to disturbances at points Q at times t_2. $\phi_1(\xi, \eta, \zeta, t_1)$ and $\phi_2(\xi, \eta, \zeta, t_2)$ will be used to denote the values of the potential at points Q at times t_1 and t_2 respectively.

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

is the gradient of ϕ at P obtained by differentiating ϕ with t constant.

$$\nabla\phi_1 = \mathbf{i}\frac{\partial\phi_1}{\partial\xi} + \mathbf{j}\frac{\partial\phi_1}{\partial\eta} + \mathbf{k}\frac{\partial\phi_1}{\partial\zeta}$$

is the gradient of the potential at points Q at times t_1, and

$$\nabla\phi_2 = \mathbf{i}\frac{\partial\phi_2}{\partial\xi} + \mathbf{j}\frac{\partial\phi_2}{\partial\eta} + \mathbf{k}\frac{\partial\phi_2}{\partial\zeta}$$

is the gradient at points Q at times t_2. $\nabla_t\phi_1$ denotes the gradient at Q with t constant. $\nabla_t\mathbf{h}\phi_1$ is the hyperbolic gradient at Q with t constant.

The potential must satisfy the equation

$$-\beta^2\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{2V}{c^2}\frac{\partial^2\phi}{\partial x\,\partial t} = 0$$

The above equation may be written

$$\nabla^2 h\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{2V}{c^2}\frac{\partial^2\phi}{\partial x\,\partial t} = 0 \tag{10}$$

Equation (10) is a special case of one of the types of equations treated in reference 1. The problem is to find a function, ϕ, that will satisfy given boundary conditions and the above partial differential equation in space.

The hyperbolic divergence with t constant of the vector

$$\mathbf{U}_1 \equiv \frac{1}{R}\nabla\phi_1 - \left(\phi_1 - \frac{R}{\beta^2 c}\frac{\partial\phi_1}{\partial t_1}\right)\nabla\frac{1}{R} + \mathbf{i}\frac{M}{\beta^2 c}\frac{\partial\phi_1}{\partial t_1} \tag{11}$$

is

$$\nabla_t h\cdot\mathbf{U}_1 = \nabla^2 h\phi_1 - \frac{2V}{c^2}\frac{\partial^2\phi_1}{\partial x\,\partial t_1} - \frac{1}{c^2}\frac{\partial^2\phi_1}{\partial t_1^2} \tag{12}$$

The details of taking the hyperbolic divergence with t constant of the vector $\mathbf{U}_1$, are given in the appendix. Likewise, the hyperbolic divergence with t constant of the vector

$$\mathbf{U}_2 \equiv \frac{1}{R}\nabla\phi_2 - \left(\phi_2 + \frac{R}{\beta^2 c}\frac{\partial\phi_2}{\partial t_2}\right)\nabla\frac{1}{R} + \mathbf{i}\frac{M}{\beta^2 cR}\frac{\partial\phi_2}{\partial t_2} \tag{13}$$

is

$$\nabla_t h \cdot \mathbf{U}_2 = \nabla^2 h\phi_2 - \frac{2V}{c^2}\frac{\partial^2 \phi_2}{\partial \xi \, \partial t_2} - \frac{1}{c^2}\frac{\partial^2 \phi_2}{\partial t_2^2} \tag{14}$$

Since

$$\phi(\xi, \eta, \zeta, t) = \phi_1(\xi, \eta, \zeta, t_1) + \phi_2(\xi, \eta, \zeta, t_2)$$

then it follows that

$$\nabla_t^2 h\phi = \nabla_t \mathbf{h} \cdot \mathbf{U}_1 + \nabla_t \mathbf{h} \cdot \mathbf{U}_2 \tag{15}$$

Let

$$\mathbf{U} = \mathbf{U}_1 + \mathbf{U}_2$$

then by equation (4)

$$\oint \mathbf{U} \cdot \mathbf{n}' \, da = \int \nabla \mathbf{h} \cdot \mathbf{U} \, dv \tag{16}$$

or

$$\oint \mathbf{U} \cdot \mathbf{n}' \, da = \int \left[\nabla^2 h\phi_1 - \frac{2V}{c^2}\frac{\partial^2 \phi_1}{\partial \xi \, \partial t_1} - \frac{1}{c^2}\frac{\partial^2 \phi_1}{\partial t_1^2} + \nabla^2 h\phi_2 - \frac{2V}{c^2}\frac{\partial^2 \phi_2}{\partial \xi \, \partial t_2} - \frac{1}{c^2}\frac{\partial^2 \phi_2}{\partial t_2^2} \right] dv \tag{17}$$

For a supersonic flow field equation (17) becomes

$$\oint \left\{ \frac{1}{R} \nabla \phi - \left(\phi - \frac{R}{\beta^2 c}\frac{\partial \phi_1}{\partial t_1} + \frac{R}{\beta^2 c}\frac{\partial \phi_2}{\partial t_2} \right) \nabla \frac{1}{R} + \mathbf{i} \frac{M}{\beta^2 c R} \left(\frac{\partial \phi_1}{\partial t_1} + \frac{\partial \phi_2}{\partial t_2} \right) \right\} \cdot \mathbf{n}' \, da = 0 \tag{18}$$

The surface of integration of equation (18) will be taken as the surface of a region bounded by the forecone from point P, a surface given by

$$\xi = x - \epsilon$$

where ϵ is small, and an arbitrary surface inclosed in the forward machcone from point P.

The integral over the machcone is zero due to the concept of the finite part (Reference 4).

Since ϵ is small then

$$t_1 \approx t_2 \approx t$$

The surface integral over the area

$$\xi = x - \epsilon$$

reduces to a time independent problem in the limit as ϵ approaches zero. Thus this surface integral in the limit as ϵ approaches zero becomes

$$\lim_{\epsilon \to 0} {}^{f}\!\!\int \left[\frac{1}{R}\frac{\partial \phi}{\partial n'} - \phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) \right] da = 2\pi\phi(x, y, z, t) \tag{19}$$

The details of the above integration may be found in reference (4).

From equations (18) and (19) the integral over the surface being considered is given by

$$2\pi\phi(x, y, z, t) + \int_{S_1} \left[\frac{1}{R}\nabla\phi - \left(\phi - \frac{R}{\beta^2 c}\frac{\partial\phi_1}{\partial t_1} + \frac{R}{\beta^2 c}\frac{\partial\phi_2}{\partial t_2}\right)\nabla\frac{1}{R} + \mathbf{i}\frac{M}{\beta^2 cR}\left(\frac{\partial\phi_1}{\partial t_1} + \frac{\partial\phi_2}{\partial t_2}\right)\right]\cdot\mathbf{n}'\,da = 0 \tag{20}$$

where S_1 , is the arbitrary surface inclosed in the forward machcone from point P.

Since S_1 is an arbitrary surface, equation (20) can be applied to a region such as is shown in figure (1). The potential or the derivative of the potential with respect to $\mathbf{n}'$ or both may be discontinuous across the surfaces S_0 . (See figure 1.)

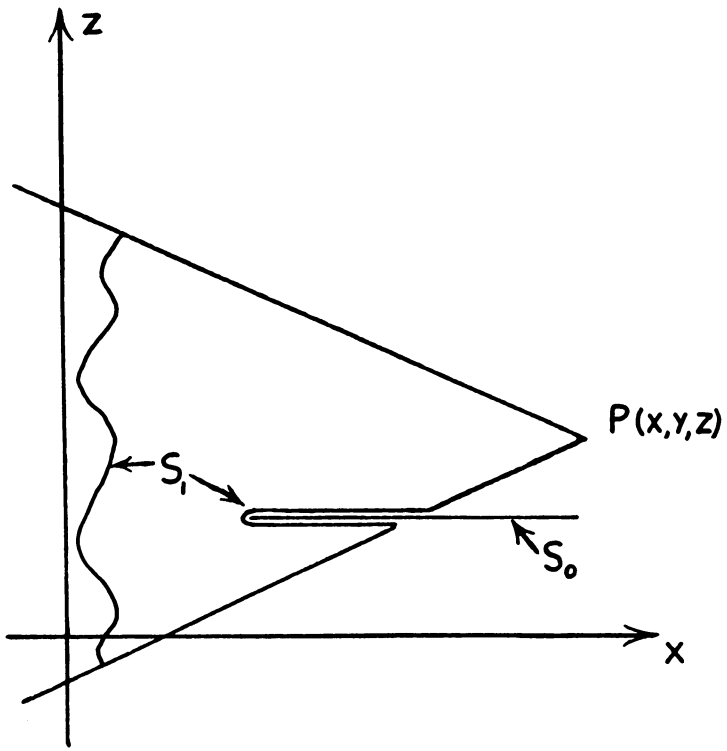

FIG. 1.

For most problems dealing with linearized supersonic flow ϕ and $\partial\phi/\partial n'$ are zero at a finite distance upstream. In these cases the potential at any point is given by

$$\phi(x, y, z, t) = -\frac{1}{2\pi}{}'\!\!\int_{S_0} \left[\frac{1}{R}\nabla\Delta\phi - \left(\Delta\phi - \frac{R}{\beta^2 c}\frac{\partial\Delta\phi_1}{\partial t_1} + \frac{R}{\beta^2 c}\frac{\partial\Delta\phi_2}{\partial t_2}\right)\nabla\frac{1}{R} + \mathbf{i}\frac{M}{\beta^2 cR}\left(\frac{\partial\Delta\phi_1}{\partial t_1} + \frac{\partial\Delta\phi_2}{\partial t_2}\right)\right]\cdot n'\,da \tag{21}$$

where $\Delta\phi$ represents the potential difference across the surface S_0. Equation (21) may be written

$$\phi(x, y, z, t) = -\frac{1}{2\pi}\,{}'\!\!\int_{S_0}\left[\frac{1}{R}\frac{\partial\Delta\phi}{\partial n'} - \frac{\partial}{\partial n'}\left\{\frac{\Delta\phi_1(t_1) + \Delta\phi_2(t_2)}{R}\right\}\right] da \tag{22}$$

Provided that in the differentiation in the last term of equation (22) $\Delta\phi_1$ and $\Delta\phi_2$ are treated as being functions of t_1 and t_2 only. Equation (21) or (22) corresponds to the Kirchhoff formula, the difference being that the Kirchhoff formula applies to stationary mediums where equation (21) or (22) applies to mediums moving at velocities greater than sonic velocity.

It is possible to represent the potential at point P by other expressions. Equation (18) can be applied to a region bounded by the forward machcone from point P, the same arbitrary surface in the forward machcone as shown in figure (1), and a surface upstream at infinity inclosed in the forward machcone from point P. The potential in this region will be denoted by ϕ'. The result of applying equation (18) to the region defined above is

$$\begin{aligned}{}'\!\!\int_{S_1}\Bigg[\frac{1}{R}\nabla\phi' - \left(\phi' - \frac{R}{\beta^2 c}\frac{\partial\phi_1'}{\partial t_1} + \frac{R}{\beta^2 c}\frac{\partial\phi_2'}{\partial t_2}\right)\nabla\frac{1}{R}\\ + \mathbf{i}\frac{M}{\beta^2 cR}\left(\frac{\partial\phi_1'}{\partial t_1} + \frac{\partial\phi_2'}{\partial t_2}\right)\Bigg]\cdot\mathbf{n}''\, da = 0\end{aligned} \tag{23}$$

Equations (20) and (23) can be written as

$$\phi(x, y, z, t) = -\frac{1}{2\pi}\,{}'\!\!\int_{S_1}\left\{\frac{1}{R}\frac{\partial}{\partial\mathbf{n}'}\phi - \frac{\partial}{\partial\mathbf{n}'}\left[\frac{\phi_1(t_1) + \phi_2(t_2)}{R}\right]\right\} da \tag{24}$$

$$0 = -\frac{1}{2\pi}\,{}'\!\!\int_{S_1}\left\{\frac{1}{R}\frac{\partial\phi'}{\partial n''} - \frac{\partial}{\partial n''}\left[\frac{\phi_1'(t_1) + \phi_2'(t_2)}{R}\right]\right\} da \tag{25}$$

where in the differentiation with respect to $\mathbf{n}'$ or $\mathbf{n}''$, $\phi_1(t_1)\phi_2(t_2)$, $\phi_1'(t_1)$ and $\phi_2'(t_2)$ are treated as being functions of t_1 and t_2 only. The result of adding equations (24) and (25) is

$$\begin{aligned}\phi(x, y, z, t) = -\frac{1}{2\pi}\,{}'\!\!\int_{S_1}\Bigg\{\frac{1}{R}\left(\frac{\partial\phi}{\partial n'} + \frac{\partial\phi'}{\partial n''}\right)\\ - \frac{\partial}{\partial n'}\left[\frac{\phi_1(t_1) + \phi_2(t_2) - \phi_1'(t_1) - \phi_2'(t_2)}{R}\right]\Bigg\} da\end{aligned} \tag{26}$$

Since ϕ' is an arbitrary solution of equation (10) then it may be chosen so that either $\partial\phi/\delta n' + \partial\phi'/\partial n''$ or

$$\frac{\partial}{\partial n'}\left[\frac{\phi_1(t_1) + \phi_2(t_2) - \phi_1'(t_1) - \phi_2'(t_2)}{R}\right]$$

is zero; therefore $\phi(x, y, z, t)$ may be given by

$$\phi(x, y, z, t) = -\frac{1}{2\pi}\int_{S_1}\frac{1}{R}\left(\frac{\partial\phi}{\partial n'} + \frac{\partial\phi'}{\partial n''}\right) da \tag{27}$$

or

$$\phi(x, y, z, t) = \frac{1}{2\pi} {}^{f}\!\!\int_{S_1} \frac{\partial}{\partial n'} \left[\frac{\phi_1(t_1) + \phi_2(t_2) - \phi_1'(t_1) - \phi_2'(t_2)}{R} \right] da \tag{28}$$

depending on whether ϕ' is chosen so that $\partial\phi/\partial n' + \partial\phi'/\partial n''$ or

$$\frac{\partial}{\partial n'} \left[\frac{\phi_1(t_1) + \phi_2(t_2) - \phi_1'(t_1) - \phi_2'(t_2)}{R} \right]$$

is zero. It should be remembered that in the differentiation with respect to $n'\phi_1(t_1)$, $\phi_2(t_2)$, $\phi_1'(t_1)$ and $\phi_2'(t_2)$ are treated as being functions of t_1 and t_2 only.

APPENDIX

The evaluation of $\nabla_t \mathbf{h} \cdot \mathbf{U}_1$. The valuation of $\nabla_t \mathbf{h} \cdot \mathbf{U}_1$ requires the use of equations (5), (6), (7), (8) and (9). From equation (6) $\nabla_t \mathbf{h} \cdot \mathbf{U}_1$ can be expressed as

$$\nabla_t \mathbf{h} \cdot \mathbf{U}_1 = \nabla_t \mathbf{h} \cdot \frac{1}{R} \nabla \phi_1 - \nabla_t \mathbf{h} \cdot \phi_1 \nabla \frac{1}{R} + \nabla_t \mathbf{h} \cdot \frac{R}{\beta^2 c} \frac{\partial \phi_1}{\partial t_1} \nabla \frac{1}{R} + \nabla_t \mathbf{h} \cdot \mathbf{i} \frac{M}{\beta^2 c R} \frac{\partial \phi_1}{\partial t_1} \tag{A-1}$$

The first term of equation (A-1) may be expanded to give

$$\nabla_t \mathbf{h} \cdot \frac{1}{R} \nabla \phi_1 = \frac{1}{R} \nabla^2 h \phi_1 - \frac{M}{cR} \frac{\partial^2 \phi_1}{\partial \xi\, \partial t_1} + \frac{1}{\beta^2 c R} \nabla \mathbf{h} R \cdot \nabla \frac{\partial \phi_1}{\partial t_1} + \nabla \phi_1 \cdot \nabla \mathbf{h} \frac{1}{R}$$

The second term in equation (A-1) may be expanded to give

$$-\nabla_t \mathbf{h} \cdot \phi_1 \nabla \frac{1}{R} = -\nabla \frac{1}{R} \cdot \nabla \mathbf{h} \phi_1 + \nabla \frac{1}{R} \cdot \mathbf{i} \frac{M}{c} \frac{\partial \phi_1}{\partial t_1} - \frac{1}{R^2 c} \frac{\partial \phi_1}{\partial t_1}$$

The third term in equation (A-1) can be written as

$$\nabla_t h \cdot \frac{R}{\beta^2 c} \frac{\partial \phi_1}{\partial t_1} \nabla \frac{1}{R} = \frac{1}{cR^2} \frac{\partial \phi_1}{\partial t_1} + \frac{R}{\beta^2 c} \nabla \frac{1}{R} \cdot \nabla \mathbf{h} \frac{\partial \phi_1}{\partial t_1} - \frac{RM}{\beta^2 c^2} \nabla \frac{1}{R} \cdot \mathbf{i} \frac{\partial^2 \phi_1}{\partial t_1^2} + \frac{1}{\beta^2 c^2 R} \frac{\partial^2 \phi_1}{\partial t_1^2}$$

The last term in equation (A-1) can be expanded to give

$$\nabla_t \mathbf{h} \cdot \mathbf{i} \frac{M}{\beta^2 c R} \frac{\partial \phi_1}{\partial t_1} = -\frac{M}{c} \frac{\partial \phi_1}{\partial t_1} \frac{\partial}{\partial \zeta} \left(\frac{1}{R} \right) - \frac{M}{cR} \frac{\partial^2 \phi_1}{\partial \xi\, \partial t_1} - \frac{M^2}{\beta^2 c^2 R} \frac{\partial^2 \phi_1}{\partial t_1^2} - \frac{M}{\beta^2 c^2 R} \frac{\partial^2 \phi_1}{\partial t_1^2} \frac{\partial R}{\partial \xi}$$

From the above results equation (A-1) can now be written as

$$\nabla_t \mathbf{h} \cdot \mathbf{U}_1 = \frac{1}{R} \nabla^2 h \phi_1 - \frac{2V}{Rc^2} \frac{\partial^2 \phi_1}{\partial \xi\, \partial t_1} - \frac{1}{Rc^2} \frac{\partial^2 \phi_1}{\partial t_1^2}$$

which was to be shown.

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